Palestine Journal of Mathematics

Vol 15(Special Issue I)(2026) , 84–95

Treatise on a Bessel-type integral transform

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MSC 2010 Classifications: Primary 33C60; Secondary 42A20.

Keywords and phrases: Modified Bessel-type integral transform; Lipschitz function; Turán type inequality; Hermite-Hadamard inequality; functions of matrix arguments

The authors would like to thank the reviewers and editor for their constructive comments and valuable suggestions that improved the quality of our paper.

Abstract The prime focus of this paper is the modified Bessel-type integral transform and its kernel introduced by Gleaske et.al. [A modified Bessel-type integral transform and its compositions with fractional calculus operators on spaces $F_{p,\mu}$ and $F'_{p,\mu}$. Journal of Computational and Applied Mathematics, 118, 151-168 (2000)]. Analytic properties like, continuity, integrabilty, convexity and various inequalites including Turán type are investigated for the kernel function. To enable the applicabilty of the modified Bessel-type transform, its compostion with various generalized special functions are obtained along with the computable series representation of its kernel. Moreover, the kernel function has been extended to matrix arguments in both real and complex cases.

1 Introduction

In 1965, Krätzel [17] introduced a Bessel-type integral transform $\left(\mathcal{L}_\nu^{(n)} f\right)(x)$, defined as

$$\left(\mathcal{L}_\nu^{(n)} f\right)(x) := \int_0^{+\infty} \lambda_\nu^{(n)}(xt) f(t)\mathrm{d}t, \tag{1.1}$$

where $x \in \mathbb{R}_+ = (0, \infty)$, and the kernel function $\lambda_\nu^{(n)}(\cdot)$ is defined by

$$\lambda_\nu^{(n)}(u) := \frac{(2\pi)^{(n-1)/2}\sqrt{n}}{\Gamma(\nu + 1 - \frac{1}{n})} \left(\frac{u}{n}\right)^{\nu n} \int_1^{+\infty} (t^n - 1)^{\nu - 1/n} \exp(-ut)\mathrm{d}t, \tag{1.2}$$

with $u \in \mathbb{R}_+$, $\nu > \frac{1}{n} - 1$, $n \in \mathbb{N} = \{1, 2, 3, \dots\}$. From [6, 7.12(23)] with $z = t$, it is easy to see that

$$\lambda_\nu^{(2)}(t) = 2\left(\frac{t}{2}\right)^\nu \mathcal{K}_\nu(t), \tag{1.3}$$

where $\mathcal{K}_\nu(\cdot)$ is the Macdonald function or the modified Bessel function of the third kind, see [6, 7.2.2]. When $n = 1$ and $n = 2$, $\left(\mathcal{L}_\nu^{(n)} f\right)(x)$ simplifies to the Laplace transform and the Meijer transform, as mentioned in [17]. Krätzel extensively examined the inversion formulas, convolution theorems, operational rules, differentiation relations, and the association with differential operators of $\mathcal{L}_\nu^{(n)}$, see [18, 19, 20, 21].

In 1998, Glaeske and Kilbas [8] studied the transform (1.1) in Banach spaces $L_{\gamma,\tau}(0, \infty)$, consisting of the complex-valued functions g with norm

$$\|g\|_{\gamma,\tau} = \left\{\int_0^{+\infty} \frac{1}{t}|t^\gamma g(t)|^\tau \mathrm{d}t\right\}^{\frac{1}{\tau}} < \infty,$$

where $1 \leq \tau < \infty$, $\gamma \in \mathbb{R} = (-\infty, +\infty)$. It was also proved that the transform (1.1) can be represented as a particular type of H-transform, for more details of H-transform, see [13]. In particular, the paper covers the boundedness of the function, the representations in terms of special functions, and the range of the transform using the $L_{\gamma,\tau}-$theory of integral transforms. In 2000, Glaeske et al. [7] introduced a modified Bessel-type integral transform $\left(\mathcal{L}_{\nu,\sigma}^{(\beta)} f\right)(x)$, which is defined as

$$\left(\mathcal{L}_{\nu,\sigma}^{(\beta)} f\right)(x) := \int_0^{+\infty} \lambda_{\nu,\sigma}^{(\beta)}(xt) f(t) \mathrm{d}t, \tag{1.4}$$

where $x \in \mathbb{R}_+$, and the kernel function $\lambda_{\nu,\sigma}^{(\beta)}(\cdot)$ is given by

$$\lambda_{\nu,\sigma}^{(\beta)}(z) := \frac{\beta}{\Gamma(\nu + 1 - \beta^{-1})} \int_1^{\infty} (t^\beta - 1)^{\nu - \beta^{-1}} t^\sigma \exp(-zt) \mathrm{d}t, \tag{1.5}$$

with $\beta > 0$, $\Re(z) > 0$, $\Re(\nu) > \frac{1}{\beta} - 1$, $\sigma \in \mathbb{R}$. Moreover, the composition of (1.5) with left-and right-sided Riemann-Liouville fractional integrals and derivatives are investigated. A relation connecting (1.1) and (1.4) is given by [8, 1.6],

$$\left(\mathcal{L}_{\nu}^{(n)} f\right)(x) = \frac{(2\pi)^{(n-1)/2} x^{\nu n}}{n^{n\nu - \frac{1}{2}}} \left(\mathcal{L}_{\nu,0}^{(n)} t^n f\right)(x), \tag{1.6}$$

for $n \in \mathbb{N}$, $\Re(\nu) > \frac{1}{n} - 1$. In 2000, Bonilla et al. [4] studied the properties of the kernel function $\lambda_{\nu,\sigma}^{(\beta)}(z)$. A representation of $\lambda_{\nu,\sigma}^{(\beta)}(z)$ in terms of Fox's H-function is given by Gleaske et al. [7, (2.6)] as

$$\lambda_{\nu,\sigma}^{(\beta)}(z) = H_{1,2}^{2,0}\left(z \;\middle|\; \begin{matrix} (1 - (\sigma+1)/\beta, 1/\beta) \\ (0,1),\ (-\nu - \sigma/\beta, 1/\beta) \end{matrix}\right), \quad z \in \mathbb{C},\ z \neq 0, \tag{1.7}$$

where $\beta > 0$, $\Re(\nu) > \frac{1}{\beta} - 1$, and $\sigma \in \mathbb{R}$. $H_{a,b}^{m,n}(z)$ denotes the Fox's H-function, [13, 9]. The recurrence relations, representations of $\lambda_{\nu,\sigma}^{(\beta)}(z)$ by fractional integrals of elementary functions were also proved. Further, it has been proved that the function $\lambda_{\nu,\sigma}^{(\beta)}(z)$ is a solution of certain differential and integral equations. In 2019, Luo and Raina [22] established certain inequalities for $\lambda_{\nu,\sigma}^{(\beta)}(z)$ that are connected to the generalized Hurwitz-Lerch zeta function and the complementary incomplete gamma function. Many authors have explored other Bessel-type integral transforms and their kernel functions, which were introduced by Krätzel; see for example, [3, 10, 16, 25]. Motivated by these works on the Bessel-type integral transforms, it is essential to examine the further analytic properties of the transform $\mathcal{L}_{\nu,\sigma}^{(\beta)}$ and associated kernel function $\lambda_{\nu,\sigma}^{(\beta)}(z)$. In 1950, Turán [34] established an inequality which is of the form

$$[\mathscr{P}_{m+1}(x)]^2 > \mathscr{P}_m(x)\mathscr{P}_{m+2}(x) \quad \text{for all } x \in (-1,1), \ \ m \in \mathbb{N},$$

where $\mathscr{P}_m(x)$ is the Legendre polynomial. Many authors have extended similar inequalities to classical orthogonal polynomials and special functions. Notably, numerous researchers have established Turán type inequality for various special functions. For instance, refer [2]. In light of the above, we find it suitable to establish Turán type inequality for the function $\lambda_{\nu,\sigma}^{(\beta)}(z)$. It has also been demonstrated that the kernel function satisfies the Hermite-Hadamard inequality. In continuation to the above, two additional inequalities were derived for the function $\lambda_{\nu,\sigma}^{(\beta)}(z)$.

This article is divided into six sections, which are as follows: Section 2 proves the Lipschitz continuity, integrability, convexity, fixed-point properties, and various inequalities associated with the function $\lambda_{\nu,\sigma}^{(\beta)}(x)$. The behavior of the kernel function $\lambda_{\nu,\sigma}^{(\beta)}(x)$ is also studied through graphs. In Section 3, the modified Bessel-type integral transform of various generalized special functions, including the Dirac delta function and the Heaviside step function, is evaluated. A composition formula for $\mathcal{L}_{\nu,\sigma}^{(\beta)}$ with the Laplace transform is also discovered. Furthermore, a series form representation of the function $\lambda_{\nu,\sigma}^{(\beta)}(x)$ is obtained. Section 4 extends the kernel function $\lambda_{\nu,\sigma}^{(\beta)}(x)$ to the matrix-variate case, $\lambda_{\nu,\sigma}^{(\beta)}(X)$ with matrix argument X. Section 5 provides the conclusion.

2 Properties of $\lambda_{\nu,\sigma}^{(\beta)}(z)$

This section addresses the continuity, differentiability, and integrability of the kernel function $\lambda_{\nu,\sigma}^{(\beta)}(x)$. The initial finding confirms that $\lambda_{\nu,\sigma}^{(\beta)}(x)$ is Lipschitz.

Theorem 2.1. For $\beta > 0,\ \sigma,\ \nu \in \mathbb{R}$ such that $\frac{1}{\beta} - 1 < \nu < -\frac{\sigma+1}{\beta}$, $\lambda_{\nu,\sigma}^{(\beta)}(x)$ is Lipschitz on $(0,\infty)$. That is,

$$\left|\lambda_{\nu,\sigma}^{(\beta)}(x) - \lambda_{\nu,\sigma}^{(\beta)}(y)\right| \leq \frac{\Gamma\left(-\nu - \frac{\sigma+1}{\beta}\right)}{\Gamma\left(1 - \frac{\sigma+2}{\beta}\right)}|x-y|,$$

for every x, y in $(0,\infty)$.

Proof. If $x < y$ for $x, y \in (0,\infty)$, then

$$\left|\lambda_{\nu,\sigma}^{(\beta)}(x) - \lambda_{\nu,\sigma}^{(\beta)}(y)\right| \leq \frac{\beta}{\Gamma(\nu + 1 - \frac{1}{\beta})} \int_1^\infty (t^\beta - 1)^{\nu - \frac{1}{\beta}} t^\sigma \left|\exp(-xt) - \exp(-yt)\right| \mathrm{d}t. \tag{2.1}$$

Using the mean value theorem for differentiation,

$$\frac{\exp(-yt) - \exp(-xt)}{y - x} = t\exp(-ct), \tag{2.2}$$

for some $c \in (x, y)$ and $t > 1$, we have

$$\begin{aligned}\left|\lambda_{\nu,\sigma}^{(\beta)}(x) - \lambda_{\nu,\sigma}^{(\beta)}(y)\right| \ &< \ \frac{\beta}{\Gamma(\nu + 1 - \frac{1}{\beta})} \int_1^\infty (t^\beta - 1)^{\nu - \frac{1}{\beta}} t^{\sigma+1} |x-y| \ \mathrm{d}t, \\ &= \frac{\Gamma\left(-\nu - \frac{\sigma+1}{\beta}\right)}{\Gamma\left(1 - \frac{\sigma+2}{\beta}\right)}|x-y|.\end{aligned}$$

This completes the proof. □

Corollary 2.1.1. The function $\lambda_{\nu,\sigma}^{(\beta)}(x)$ is absolutely continuous function on $[a,b] \subset (0,\infty)$ for $\beta > 0,\ \frac{1}{\beta} - 1 < \nu < -\frac{\sigma+1}{\beta},\ x > 0$ and $\sigma \in \mathbb{R}$.

Corollary 2.1.2. The function $\lambda_{\nu,\sigma}^{(\beta)}(x)$ is a function of bounded variation on $[a,b] \subset (0,\infty)$ for $\beta > 0,\ x > 0,\ \frac{1}{\beta} - 1 < \nu < -\frac{\sigma+1}{\beta},\ x > 0$ and $\sigma \in \mathbb{R}$.

Corollary 2.1.3. $\lambda_{\nu,\sigma}^{(\beta)}(x)$ is Cauchy continuous function on $(0,\infty)$ for $\beta > 0,\ x > 0,\ \frac{1}{\beta} - 1 < \nu < -\frac{\sigma+1}{\beta}$, and $\sigma \in \mathbb{R}$.

Theorem 2.2. Let $\beta > 0,\ \nu > \frac{1}{\beta} - 1,\ x > 0$ and $\sigma \in \mathbb{R}$, then $\lambda_{\nu,\sigma}^{(\beta)}(x)$ is integrable function on $[a,b]$ in the Riemann sense, where $[a,b]$ is any set in $(0,\infty)$. That is

$$\int_a^b \lambda_{\nu,\sigma}^{(\beta)}(x) \ \mathrm{d}x = \lambda_{\nu,\sigma-1}^{(\beta)}(a) - \lambda_{\nu,\sigma-1}^{(\beta)}(b). \tag{2.3}$$

Proof. Since $\lambda_{\nu,\sigma}^{(\beta)}(x)$ is continuous on $[a,b] \subset (0,\infty)$, $\lambda_{\nu,\sigma}^{(\beta)}(x) \in \mathcal{R}[a,b]$. Using the formula [7, (2.13)], we obtain (2.3). □

The following results provide the convexity, fixed point properties, and various analytical inequalities of $\lambda_{\nu,\sigma}^{(\beta)}(x)$.

Theorem 2.3. The function $\lambda_{\nu,\sigma}^{(\beta)}(x)$ has a unique fixed-point in $(0,\infty)$ for $\beta > 0,\ \nu > \frac{1}{\beta} - 1,\ x > 0$ and $\sigma \in \mathbb{R}$. That is, there exists $x \in (0,\infty)$ such that

$$\lambda_{\nu,\sigma}^{(\beta)}(x) = x. \tag{2.4}$$

Proof. Using the fact that $\lambda_{\nu,\sigma}^{(\beta)}(x)$ is monotonically decreasing and continuous on $(0,\infty)$, we obtain the result. □

A real valued function f is *log-convex*, if for any α such that $0 \le \alpha \le 1$,

$$f\left(\alpha x + (1-\alpha)y\right) \le f(x)^{\alpha} f(y)^{1-\alpha}.$$

The following theorem deals the log-convexity of $\lambda_{\nu,\sigma}^{(\beta)}(x)$.

Theorem 2.4. The function $\lambda_{\nu,\sigma}^{(\beta)}(x)$ is log-convex on $(0,\infty)$, for $\beta > 0,\ \nu > \frac{1}{\beta} - 1,\ x > 0$ and $\sigma \in \mathbb{R}$.

Proof. Let $\alpha \in [0,1]$ and $x_1, x_2 \in (0,\infty)$, then

$$\begin{aligned}
&\lambda_{\nu,\sigma}^{(\beta)}\left[\alpha x_1 + (1-\alpha)x_2\right] \\
&= \frac{\beta}{\Gamma(\nu+1-\frac{1}{\beta})} \int_1^{\infty} (t^{\beta}-1)^{\nu-\frac{1}{\beta}} t^{\sigma} \exp\left[-\alpha x_1 t - (1-\alpha)x_2 t\right] \mathrm{d}t \\
&= \frac{\beta}{\Gamma(\nu+1-\frac{1}{\beta})} \int_1^{\infty} \left[(t^{\beta}-1)^{\nu-\frac{1}{\beta}} t^{\sigma} \exp(-x_1 t)\right]^{\alpha} \left[(t^{\beta}-1)^{\nu-\frac{1}{\beta}} t^{\sigma} \exp(-x_2 t)\right]^{1-\alpha} \mathrm{d}t.
\end{aligned}$$

By means of Hölder-Roger's inequality for real valued functions f and g defined on $[m,n]$, and $|f(t)|^u$, $|g(t)|^v$ integrable functions on $[m,n]$ where $u > 1$ with $\frac{1}{u} + \frac{1}{v} = 1$,

$$\int_m^n |f(t)g(t)|\mathrm{d}t \le \left[\int_m^n |f(t)|^u \mathrm{d}t\right]^{\frac{1}{u}} \left[\int_m^n |g(t)|^v \mathrm{d}t\right]^{\frac{1}{v}},$$

we have

$$\begin{aligned}
\lambda_{\nu,\sigma}^{(\beta)}\left[\alpha x_1 + (1-\alpha)x_2\right] &\le \left[\frac{\beta}{\Gamma(\nu+1-\frac{1}{\beta})} \int_1^{\infty} (t^{\beta}-1)^{\nu-1/\beta} t^{\sigma} \exp(-x_1 t)\mathrm{d}t\right]^{\alpha} \\
&\quad \times \left[\frac{\beta}{\Gamma(\nu+1-\frac{1}{\beta})} \int_1^{\infty} (t^{\beta}-1)^{\nu-1/\beta} t^{\sigma} \exp(-x_2 t)\mathrm{d}t\right]^{1-\alpha} \\
&= \left[\lambda_{\nu,\sigma}^{(\beta)}(x_1)\right]^{\alpha} \left[\lambda_{\nu,\sigma}^{(\beta)}(x_2)\right]^{1-\alpha}.
\end{aligned}$$

This completes the proof. □

Let $[a,b] \subset (0,\infty)$. From Theorem 2.4, $x \mapsto \lambda_{\nu,\sigma}^{(\beta)}(x)$ is convex on $[a,b]$. Now using the Hermite-Hadamard inequality [32, (1.1)] and Theorem 2.2, we have the following corollary.

Corollary 2.4.1. Let $\beta > 0,\ \nu > \frac{1}{\beta} - 1$, $\sigma \in \mathbb{R}$, and $b > a > 0$, then

$$\lambda_{\nu,\sigma}^{(\beta)}\left(\frac{a+b}{2}\right) \le \frac{\lambda_{\nu,\sigma-1}^{(\beta)}(a) - \lambda_{\nu,\sigma-1}^{(\beta)}(b)}{b-a} \le \frac{\lambda_{\nu,\sigma}^{(\beta)}(a) + \lambda_{\nu,\sigma}^{(\beta)}(b)}{2}. \tag{2.5}$$

Similar to Theorem 2.4, one can show that the function $\sigma \mapsto \lambda_{\nu,\sigma}^{(\beta)}(x)$ is log-convex on $\mathbb{R}$ with $\nu > \frac{1}{\beta} - 1,\ x > 0,\ \beta > 0$. This leads to the following corollary.

Corollary 2.4.2. Let $\beta > 0,\ \nu > \frac{1}{\beta} - 1,\ x > 0$ and $\sigma \in \mathbb{R}$, then the function $\sigma \mapsto \lambda_{\nu,\sigma}^{(\beta)}(x)$ satisfies the Turán type inequality,

$$\left[\lambda_{\nu,\sigma}^{(\beta)}(x)\right]^2 \le \left[\lambda_{\nu,\sigma+\beta}^{(\beta)}(x)\right] \left[\lambda_{\nu,\sigma-\beta}^{(\beta)}(x)\right]. \tag{2.6}$$

Theorem 2.5. Let $\beta > 0,\ \nu > \frac{1}{\beta} - 1,\ \sigma \in \mathbb{R},\ x,\ y > 1$ such that $\frac{1}{x} + \frac{1}{y} \leq 1$, and $p,\ q > 0$ such that $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\lambda_{\nu,\sigma}^{(\beta)}\left(\frac{x^p}{p} + \frac{y^q}{q}\right) \leq \left[\lambda_{\nu,\sigma}^{(\beta)}(px)\right]^{\frac{1}{p}} \left[\lambda_{\nu,\sigma}^{(\beta)}(qy)\right]^{\frac{1}{q}}. \tag{2.7}$$

Proof. Using the Young's inequality,

$$uv \leq \frac{u^p}{p} + \frac{v^q}{q},$$

where $u,\ v > 0$ and $\frac{1}{p} + \frac{1}{q} = 1,\ p,\ q > 0$, and using (1.5), we have

$$\lambda_{\nu,\sigma}^{(\beta)}\left(\frac{x^p}{p} + \frac{y^q}{q}\right) \leq \frac{\beta}{\Gamma(\nu + 1 - \frac{1}{\beta})} \int_1^\infty (t^\beta - 1)^{\nu - 1/\beta} t^\sigma \exp(-xyt)\mathrm{d}t. \tag{2.8}$$

Using Hölder-Roger's inequality, we obtain

$$\begin{aligned}\lambda_{\nu,\sigma}^{(\beta)}\left(\frac{x^p}{p} + \frac{y^q}{q}\right) \leq & \left[\frac{\beta}{\Gamma(\nu + 1 - \frac{1}{\beta})} \int_1^\infty (t^\beta - 1)^{\nu - 1/\beta} t^\sigma \exp(-pxt)\mathrm{d}t\right]^{\frac{1}{p}} \\ & \times \left[\frac{\beta}{\Gamma(\nu + 1 - \frac{1}{\beta})} \int_1^\infty (t^\beta - 1)^{\nu - 1/\beta} t^\sigma \exp(-qyt)\mathrm{d}t\right]^{\frac{1}{q}}.\end{aligned}$$

This completes the proof. □

Theorem 2.6. Let $\beta > 0,\ \frac{1}{\beta} - 1 < \nu < -\frac{\sigma}{\beta},\ x > 0$ and $\sigma < -1$, then $\lambda_{\nu,\sigma}^{(\beta)}(x)$ satisfies the following inequality,

$$\Gamma(\sigma + 1, x)\frac{\Gamma\left(-\frac{\sigma}{\beta} - \nu\right)}{\Gamma\left(1 - \frac{\sigma+1}{\beta}\right)} \leq \frac{x^{\sigma+1}}{\sigma + 1}\lambda_{\nu,\sigma}^{(\beta)}(x), \tag{2.9}$$

where $\Gamma(\cdot\ ,\cdot)$ is the incomplete gamma function [5, (2)].

Proof. Let $f(t) = \exp(-xt),\ g(t) = \frac{\beta}{\Gamma(\nu+1-\frac{1}{\beta})}(t^\beta - 1)^{\nu - \frac{1}{\beta}}$, and $h(t) = t^\sigma$, then $f,\ g$ are decreasing and integrable on $(1, \infty)$. Moreover,

$$\int_1^\infty h(t)\mathrm{d}t = -\frac{1}{\sigma + 1}. \tag{2.10}$$

Using the incomplete gamma function, we obtain

$$\int_1^\infty f(t)h(t)\mathrm{d}t = \int_1^\infty \exp(-xt)t^\sigma \mathrm{d}t = \frac{\Gamma(\sigma + 1, x)}{x^{\sigma+1}} \tag{2.11}$$

and using the beta function, we have

$$\int_1^\infty g(t)h(t)\mathrm{d}t = \int_1^\infty \frac{\beta}{\Gamma(\nu + 1 - \frac{1}{\beta})}(t^\beta - 1)^{\nu - \frac{1}{\beta}} t^\sigma \mathrm{d}t = \frac{\Gamma\left(-\frac{\sigma}{\beta} - \nu\right)}{\Gamma\left(1 - \frac{\sigma+1}{\beta}\right)}. \tag{2.12}$$

Using, the Chebyshev integral inequality [31, p. 40] the result (2.9) follows. □

The following result gives a computable series representation of $\lambda_{\nu,\sigma}^{(\beta)}(z)$.

Theorem 2.7. For $\beta > 0,\ \sigma,\ \nu \in \mathbb{R}, z \neq 0$, we have

$$\lambda_{\nu,\sigma}^{(\beta)}(z) = \sum_{n=0}^{\infty} \frac{\Gamma\left(-\nu - \frac{\sigma}{\beta} - \frac{n}{\beta}\right)}{n!\Gamma\left(1 - \frac{(\sigma+1)}{\beta} - \frac{n}{\beta}\right)} (-z)^n + \beta z^{-\sigma-\beta\nu} \sum_{m=0}^{\infty} \frac{\Gamma\left(\beta\nu + \sigma - \beta m\right)}{m!\Gamma\left(1 - \frac{1}{\beta} + \nu - m\right)} \left(-z^{\beta}\right)^m. \tag{2.13}$$

Proof. Using the Mellin-Barnes integral representation of (1.7), we have

$$\lambda_{\nu,\sigma}^{(\beta)}(z) = \frac{1}{2\pi i} \int_{\mathcal{L}} \frac{\Gamma(s)\Gamma(-\nu - \frac{\sigma}{\beta} + \frac{s}{\beta})}{\Gamma(1 - \frac{\sigma+1}{\beta} + \frac{s}{\beta})} z^{-s} \mathrm{d}s, \tag{2.14}$$

$\beta > 0,\ \Re(\nu) > \frac{1}{\beta} - 1,\ x > 0, \sigma \in \mathbb{R}$ and $\Re(s) > \max[0,\ \beta\nu + \sigma]$.
Consider the function

$$g(s) = \frac{\Gamma(s)\Gamma(-\nu - \frac{\sigma}{\beta} + \frac{s}{\beta})}{\Gamma(1 - \frac{\sigma+1}{\beta} + \frac{s}{\beta})} z^{-s}, \tag{2.15}$$

where $\mathcal{L} = \mathscr{L}_{-\infty}$, is the contour which encloses the poles $s = \beta\nu + \sigma - \beta m$ $(m = 0, 1, \dots)$ of $\Gamma(-\nu - \sigma/\beta + s/\beta)$ and the poles $s = -n$ $(n = 0, 1, \dots)$ of $\Gamma(s)$. Suppose $\beta\nu + \sigma - \beta m \neq -n$ for any $m, n \in \mathbb{Z}_+$, then using the Cauchy residue theorem, we have

$$\lambda_{\nu,\sigma}^{(\beta)}(z) = \sum_{n=0}^{\infty} \mathscr{R}es\left\{g(s) : s = -n\right\} + \sum_{m=0}^{\infty} \mathscr{R}es\left\{g(s) : s = \beta\nu + \sigma - \beta m\right\}. \tag{2.16}$$

Since,

$$\lim_{s \mapsto -n} (s+n)\Gamma(s) = \lim_{s \mapsto -n} \frac{(s+n)(s+n-1)\cdots s\Gamma(s)}{(s+n-1)\cdots s} = \frac{(-1)^n}{n!}, \tag{2.17}$$

we have

$$\mathscr{R}es\{g(s) : s = -n\} = \lim_{s \mapsto -n} (s+n)g(s) = \frac{\Gamma\left(-\nu - \frac{\sigma}{\beta} - \frac{n}{\beta}\right)}{n!\Gamma\left(1 - \frac{(\sigma+1)}{\beta} - \frac{n}{\beta}\right)} (-z)^n, \tag{2.18}$$

and the residues of g at $s = \beta\nu + \sigma - \beta m$ is

$$\mathscr{R}es\left\{g(s) : s = \beta\nu + \sigma - \beta m\right\} = \lim_{s \to \beta\nu + \sigma - \beta m} \left(s - \beta\nu - \sigma + \beta m\right) g(s) = \beta z^{-\sigma-\beta\nu} \frac{\Gamma\left(\beta\nu + \sigma - \beta m\right)}{m!\Gamma\left(1 - \frac{1}{\beta} + \nu - m\right)} \left(-z^{\beta}\right)^m. \tag{2.19}$$

Substituting equations (2.18) and (2.19) in equation (2.16), we have the series form of $\lambda_{\nu,\sigma}^{(\beta)}(z)$ in (2.13). □

Consider the kernel function in (1.5) with $\beta = 1$, the behaviour of the function for various values of $\sigma \in \mathbb{R}$ is plotted below.

Figure 1 represents the graph of $\lambda_{\nu,\sigma}^{(\beta)}(x)$ for $x > 0$ and various values of σ by fixing $\nu = 1$ and $\beta = 1$. Figure 2 represents the graph of $\lambda_{\nu,\sigma}^{(\beta)}(x)$ for $x > 0$ and various values of σ by fixing $\nu = 2$ and $\beta = 1$. In both cases, it is clear that, as σ increases, the curves are heavy peaked. Further, as the value of ν increases, the curves translate towards right from axes. The function $\lambda_{\nu,\sigma}^{(\beta)}(x)$ shows two key behaviors based on the parameters σ and ν. As σ increases, the curves become more sharply peaked, indicating that σ controls the steepness and localization of the peak around a specific point on the x-axis. In contrast, as ν increases, the peak shifts to the right, showing that ν influences the horizontal position of the peak, with higher ν values pushing the peak to larger x-values. The combined effect of increasing σ and ν produces curves that are both more sharply peaked and translated to the right.

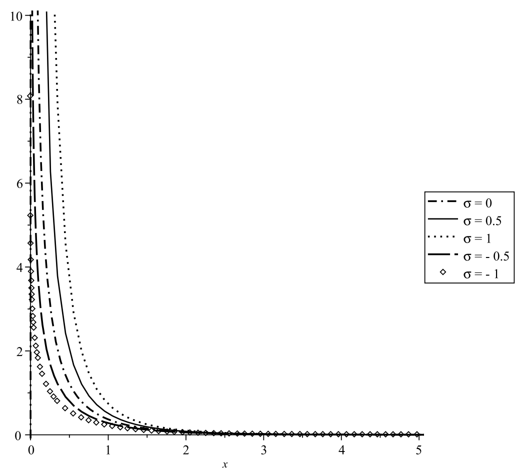

Figure 1. $\lambda_{\nu,\sigma}^{(\beta)}(x)$ for $\nu = \beta = 1$ and $\sigma = -1,\ -0.5,\ 0,\ 0.5,\ 1$

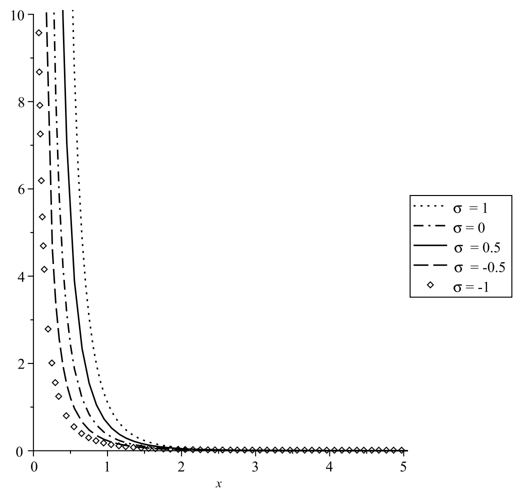

Figure 2. $\lambda_{\nu,\sigma}^{(\beta)}(x)$ for $\beta = 1$, $\nu = 2$ and $\sigma = -1,\ -0.5,\ 0,\ 0.5,\ 1$

3 $\mathcal{L}_{\nu,\sigma}^{(\beta)}$ and Generalized Special functions

Here, we establish $\mathcal{L}_{\nu,\sigma}^{(\beta)}$ transform of some generalized special functions. First result is for the power function $x^{\lambda-1}$, $(\Re(\lambda) > 0)$.

Theorem 3.1. For $\beta > 0,\ \Re(\nu) > \frac{1}{\beta} - 1,\ \ x > 0,\ \Re(\lambda) > 0,\ \sigma \in \mathbb{R}$ such that $\Re(\lambda - \sigma) > \beta\Re(\nu - 1)$, then

$$\left(\mathcal{L}_{\nu,\sigma}^{(\beta)} x^{\lambda-1}\right) = x^{-\lambda}\Gamma(\lambda)\frac{\Gamma\left(1-\nu+\frac{\lambda-\sigma}{\beta}\right)}{\Gamma\left(\frac{\lambda-\sigma-1}{\beta}+1\right)}. \tag{3.1}$$

Proof. Using (1.4) and (1.5), we obtain

$$\begin{aligned}\left(\mathcal{L}_{\nu,\sigma}^{(\beta)} x^{\lambda-1}\right)(x) &= \int_0^{+\infty} \lambda_{\nu,\sigma}^{(\beta)}(xt)t^{\lambda-1}\mathrm{d}t.\\ &= \frac{\beta}{\Gamma(\nu+1-\frac{1}{\beta})}\int_0^{+\infty}\int_1^{\infty}(y^\beta-1)^{\nu-1/\beta}y^\sigma\exp(-xty)\mathrm{d}yt^{\lambda-1}\mathrm{d}t.\end{aligned}$$

By interchanging the order of integrals and simplifying, we have

$$\begin{aligned}\left(\mathcal{L}_{\nu,\sigma}^{(\beta)} x^{\lambda-1}\right)(x) &= \frac{\beta}{\Gamma(\nu+1-\frac{1}{\beta})}\int_0^{+\infty} t^{\lambda-1}\exp(-xty)\mathrm{d}t\int_1^{\infty}(y^\beta-1)^{\nu-1/\beta}y^\sigma\mathrm{d}y.\\ &= \frac{x^{-\lambda}\beta}{\Gamma(\nu+1-\frac{1}{\beta})}\Gamma(\lambda)\int_1^{\infty}(y^\beta-1)^{\nu-1/\beta}y^{\sigma-\lambda}\mathrm{d}y.\end{aligned}$$

This completes the proof. □

The Dirac delta function or the unit impulse function is defined as [1]

$$\delta(x) := \begin{cases} +\infty & \text{if } x = 0\\ 0 & \text{if } x \neq 0\end{cases} \quad \text{and} \quad \int_{-\infty}^{\infty}\delta(x)\mathrm{d}x = 1.$$

For $\epsilon > 0$, consider the function

$$\delta_\epsilon(x) := \begin{cases} \frac{1}{\epsilon} & \text{if } x \in\ [0,\epsilon]\\ 0 & \text{otherwise}\end{cases},$$

then

$$\int_{-\infty}^{\infty} \delta_\epsilon(x)\mathrm{d}x = 1 \quad \text{and} \quad \lim_{\epsilon\to 0} \delta_\epsilon(x) = \delta(x).$$

The following theorem gives the composition of $\mathcal{L}_{\nu,\sigma}^{(\beta)}$ with the Dirac delta function.

Theorem 3.2. Let $\beta > 0,\ \Re(\nu) > \frac{1}{\beta} - 1,\ x > 0, \sigma \in \mathbb{R}$, and $a > 0$, then

$$\delta(t-a) \xrightarrow{\mathcal{L}_{\nu,\sigma}^{(\beta)}} \lambda_{\nu,\sigma}^{(\beta)}(ax).$$

Proof. Using (1.4) and (1.5), we have

$$\begin{aligned}\left(\mathcal{L}_{\nu,\sigma}^{(\beta)}\delta_\epsilon(t-a)\right)(x) &= \int_0^{+\infty} \lambda_{\nu,\sigma}^{(\beta)}(xt)\delta_\epsilon(t-a)\mathrm{d}t \\ &= \frac{\beta}{\Gamma(\nu+1-\frac{1}{\beta})}\int_0^{+\infty}\int_1^{\infty}(y^\beta-1)^{\nu-\frac{1}{\beta}}y^\sigma e^{-xty}\mathrm{d}y\delta_\epsilon(t-a)\mathrm{d}t.\end{aligned}$$

Interchanging the order of integrals, we have

$$\begin{aligned}\left(\mathcal{L}_{\nu,\sigma}^{(\beta)}\delta_\epsilon(t-a)\right)(x) &= \frac{\beta}{\Gamma(\nu+1-\frac{1}{\beta})}\int_1^{\infty}(y^\beta-1)^{\nu-\frac{1}{\beta}}y^\sigma\int_0^{+\infty} e^{-xty}\delta_\epsilon(t-a)\,\mathrm{d}t\,\mathrm{d}y \\ &= \frac{\beta}{\Gamma(\nu+1-\frac{1}{\beta})}\int_1^{\infty}(y^\beta-1)^{\nu-\frac{1}{\beta}}y^\sigma e^{-cxy}\mathrm{d}y = \lambda_{\nu,\sigma}^{(\beta)}(cx), \end{aligned} \tag{3.2}$$

for some $c \in [a, a+\epsilon]$. Taking the limit as $\epsilon \to 0$, we obtain

$$\left(\mathcal{L}_{\nu,\sigma}^{(\beta)}\delta(t-a)\right)(x) = \frac{\beta}{\Gamma(\nu+1-\frac{1}{\beta})}\int_1^{\infty}(y^\beta-1)^{\nu-\frac{1}{\beta}}y^\sigma e^{-axy}\mathrm{d}y = \lambda_{\nu,\sigma}^{(\beta)}(ax).$$

This completes the proof. □

The next result gives the modified Bessel-type transform $\mathcal{L}_{\nu,\sigma}^{(\beta)}$ of the Heaviside step function, which is given by [33],

$$\mathscr{H}(x) := \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0. \end{cases}$$

Then $\frac{\mathrm{d}}{\mathrm{d}x}\mathscr{H}(x) = \delta(x)$.

Theorem 3.3. Let $\beta > 0,\ \Re(\nu) > \frac{1}{\beta} - 1,\ x > 0, \sigma \in \mathbb{R}$, and $b > 0$ then

$$\mathscr{H}(t-b) \xrightarrow{\mathcal{L}_{\nu,\sigma}^{(\beta)}} \frac{1}{x}\lambda_{\nu,\sigma-1}^{(\beta)}(bx), \tag{3.3}$$

Proof. Using (1.4) and (1.5), we have

$$\begin{aligned}\left(\mathcal{L}_{\nu,\sigma}^{(\beta)}\mathscr{H}(t-b)\right)(x) &= \int_0^{+\infty}\lambda_{\nu,\sigma}^{(\beta)}(xt)\mathscr{H}(t-b)\mathrm{d}t \\ &= \frac{\beta}{\Gamma(\nu+1-\frac{1}{\beta})}\int_0^{+\infty}\int_1^{\infty}(y^\beta-1)^{\nu-1/\beta}y^\sigma e^{-xty}\mathrm{d}y\mathscr{H}(t-b)\mathrm{d}t.\end{aligned}$$

changing the order of integrals, we obtain

$$\begin{aligned}\left(\mathcal{L}_{\nu,\sigma}^{(\beta)}\mathscr{H}(t-b)\right)(x) &= \frac{\beta}{\Gamma(\nu+1-\frac{1}{\beta})}\int_1^{\infty}(y^\beta-1)^{\nu-1/\beta}y^\sigma\mathrm{d}y\int_0^{+\infty}e^{-xty}\mathscr{H}(t-b)\mathrm{d}t \\ &= \frac{\beta}{x\Gamma(\nu+1-\frac{1}{\beta})}\int_1^{\infty}(y^\beta-1)^{\nu-1/\beta}y^{\sigma-1}e^{-bxy}\mathrm{d}y \\ &= \frac{1}{x}\lambda_{\nu,\sigma-1}^{(\beta)}(bx).\end{aligned}$$

By means of equation (1.5), we obtain the result in (3.3). □

Next, consider the function $g(x) = x^{\eta-1}H^{m,n}_{p,q}(bx^{\rho})$, $\eta \in \mathbb{C}$ and $\rho > 0$. Then the following theorem gives the formula for $\left(\mathcal{L}^{(\beta)}_{\nu,\sigma} g\right)(x)$.

Theorem 3.4. Let $\eta \in \mathbb{C}$, $\beta, \rho > 0$, $\Re(\nu) > \frac{1}{\beta} - 1$, $x > 0$ and $\sigma \in \mathbb{R}$, then

$$\begin{aligned}&\left(\mathcal{L}^{(\beta)}_{\nu,\sigma} x^{\eta-1}H^{m,n}_{p,q}(bx^{\rho})\right)(x)\\&= \frac{\mathcal{C}}{x^{\eta}} H^{m,n+2}_{p+2,q+1}\left(\frac{b}{x^{\rho}}\middle| \begin{matrix}(u_1,\mathcal{A}_1),\dots,(u_p,\mathcal{A}_p),(1-\eta,\rho),(1+\nu-\frac{\eta-\sigma}{\beta},\frac{\rho}{\beta})\\ (v_1,\mathcal{B}_1),\dots,(v_q,\mathcal{B}q),(-\frac{\eta-\sigma-1}{\beta},\frac{\rho}{\beta})\end{matrix}\right),\end{aligned} \tag{3.4}$$

where $\mathcal{C} = \dfrac{\beta}{\Gamma(\nu+1-\frac{1}{\beta})}$, for $i = 1, 2, \dots p$, $j = 1, 2, \dots q$, $u_i,\ v_j \in \mathbb{C}$ and $\mathcal{A}_i, \mathcal{B}_j \in \mathbb{R}_+$.

Proof. Using (1.4), let us consider,

$$\begin{aligned}\left(\mathcal{L}^{(\beta)}_{\nu,\sigma} x^{\eta-1}H^{m,n}_{p,q}(bx^{\rho})\right)(x) &= \frac{\beta}{\Gamma(\nu+1-\frac{1}{\beta})}\int_0^{+\infty} \lambda^{(\beta)}_{\nu,\sigma}(xt)t^{\eta-1}H^{m,n}_{p,q}(bt^{\rho})\mathrm{d}t\\ &= \frac{\beta}{\Gamma(\nu+1-\frac{1}{\beta})}\int_0^{+\infty} \lambda^{(\beta)}_{\nu,\sigma}(xt)t^{\eta-1}\frac{1}{2\pi i}\int_{\mathcal{L}}\phi(s)(bt^{\rho})^{-s}\mathrm{d}s\mathrm{d}t.\end{aligned} \tag{3.5}$$

Changing the order of integration, we have

$$\left(\mathcal{L}^{(\beta)}_{\nu,\sigma} g(x)\right)(x) = \frac{\beta}{\Gamma(\nu+1-\frac{1}{\beta})}\frac{1}{2\pi i}\int_{\mathcal{L}}\phi(s)b^{-s}\int_0^{+\infty} t^{\eta-\rho s-1}\lambda^{(\beta)}_{\nu,\sigma}(xt)\mathrm{d}t\mathrm{d}s.$$

Now using (3.1), we obtain the result in (3.4). □

Corollary 3.4.1. Let $\eta \in \mathbb{C}$, $\beta,\ \rho > 0$, $\Re(\nu) > \frac{1}{\beta} - 1$, $x > 0$ and $\sigma \in \mathbb{R}$, then

$$\begin{aligned}\left(\mathcal{L}^{(\beta)}_{\nu,\sigma} x^{\eta-1}{}_pF_q(bx^{\rho})\right)(x) &= \frac{\beta}{\Gamma(\nu+1-\frac{1}{\beta})}\frac{1}{x^{\eta}}\frac{\Gamma(b_1)\Gamma(b_2)\cdots\Gamma(b_q)}{\Gamma(a_1)\Gamma(a_2)\cdots\Gamma(a_p)}\\ &\times H^{1,p+2}_{p+2,q+2}\left(\frac{-b}{x^{\rho}}\middle| \begin{matrix}(1-a_1,1),\dots,(1-a_p,1),(1-\eta,\rho),(1+\nu-\frac{\eta-\sigma}{\beta},\frac{\rho}{\beta})\\ (0,1),(1-b_1,1),\dots,(1-b_q,1),(-\frac{\eta-\sigma-1}{\beta},\frac{\rho}{\beta})\end{matrix}\right),\end{aligned} \tag{3.6}$$

where $a_i,\ b_j \in \mathbb{C}$ and $\alpha_i, \beta_j \in \mathbb{R}_+$ for $i = 1, 2, \dots p$, $j = 1, 2, \dots q$, and ${}_pF_q(z)$ is the generalized hypergeometric function, which is defined by [6, 4.1(1)]

$${}_pF_q(u_1,u_2,\dots,u_p;v_1,v_2,\dots,v_q;z) := \sum_{n=0}^{\infty}\frac{(u_1)_n(u_2)_n\cdots(u_p)_n}{(v_1)_n(v_2)_n\cdots(v_q)_n}\frac{z^n}{n!}, \tag{3.7}$$

where $u_i, v_j \in \mathbb{C}$; $u_i, v_j \neq -n$, $n = 0, 1, 2, \cdots$ for $i = 1, \dots, p$ and $j = 1, \dots, q$. Here $(u)_n$ denotes the pochhammer symbol defined for $u \in \mathbb{C}$ by [6, 4.1(2)]

$$(u)_0 = 1,\ (a)_k = u(u+1)\cdots(u+k-1),\ \ k = 1, 2, \dots. \tag{3.8}$$

The following result gives a composition formula for $\mathcal{L}^{(\beta)}_{\nu,\sigma}$ with the Laplace transform. The Laplace transform of a function $f(t)$ $(t > 0)$ at $\zeta \in \mathbb{C}$ is defined as

$$(\mathbb{L}f)(\zeta) = \int_0^{+\infty} e^{-\zeta t}f(t)\mathrm{d}t, \tag{3.9}$$

whenever the integral in (3.9) exists.

Theorem 3.5. Let $\zeta \in \mathbb{C}$, $\zeta \neq 0$, $\beta > 0$, $\Re(\nu) > \frac{1}{\beta} - 1$, $\sigma \in \mathbb{R}$ and $f \in \mathscr{L}_{\gamma,\tau}(0,\infty)$, then

$$\left(\mathbb{L}\mathcal{L}^{(\beta)}_{\nu,\sigma} f\right)(\zeta) = \int_0^{+\infty} H^{2,1}_{2,2}\left(\frac{x}{\zeta}\middle| \begin{matrix}(0,1),\ (1-(\sigma+1)/\beta, 1/\beta)\\ (0,1),\ (-\nu-\sigma/\beta, 1/\beta)\end{matrix}\right) f(x)\mathrm{d}x.$$

Proof. Suppose $f \in \mathscr{L}_{\gamma,\tau}(0,\infty)$. Let $\beta > 0,\ \Re(\nu) > \frac{1}{\beta} - 1,\ x > 0$ and $\sigma \in \mathbb{R}$, then

$$\begin{aligned}\left(\mathbb{L}\mathcal{L}_{\nu,\sigma}^{(\beta)} f\right)(\zeta) &= \int_0^{+\infty} e^{-\zeta t} \left(\mathcal{L}_{\nu,\sigma}^{(\beta)} f\right)(t)\mathrm{d}t \\ &= \int_0^{+\infty} e^{-\zeta t} \int_0^{+\infty} \lambda_{\nu,\sigma}^{(\beta)}(xt) f(x)\mathrm{d}x\mathrm{d}t.\end{aligned}$$

Using (1.7), we obtain

$$\left(\mathbb{L}\mathcal{L}_{\nu,\sigma}^{(\beta)} f\right)(\zeta) = \int_0^{+\infty} e^{-\zeta t} \int_0^{+\infty} \frac{1}{2\pi i} \int_{\mathcal{L}} \frac{\Gamma(s)\Gamma(-\gamma - \sigma/\beta + s/\beta)}{\Gamma(1 - (\sigma+1)/\beta + s/\beta)} (tx)^{-s}\mathrm{d}s f(x)\mathrm{d}x\mathrm{d}t.$$

Changing the order of integration and simplifying, we get

$$\left(\mathbb{L}\mathcal{L}_{\nu,\sigma}^{(\beta)} f\right)(\zeta) = \frac{1}{2\pi i} \int_0^{+\infty} \int_{\mathcal{L}} \frac{\Gamma(s)\Gamma(1-s)\Gamma(-\nu - \frac{\sigma}{\beta} + \frac{s}{\beta})}{\Gamma(1 - \frac{\sigma+1}{\beta} + \frac{s}{\beta})} \left(\frac{x}{\zeta}\right)^{-s} \mathrm{d}s f(x)\mathrm{d}x.$$

This completes the proof. □

4 Matrix analog of $\lambda_{\nu,\sigma}^{(n)}(x)$

In this section, we present the matrix variate case of the function $\lambda_{\nu,\sigma}^{(\beta)}$. In the following discussion, we consider two types of matrices of order $q \times q$, one with elements in real domain and other with complex domain. There is no possiblity of confusion since we use X for real scalar matrix and $\tilde{X}$ for complex scalar matrix. The determinant of a real matrix X is denoted by $|X|$ or $\det(X)$, and in the complex domain, the absolute value of the determinant is denoted by $|\det(\tilde{X})|$. The identity matrix of order $q \times q$ is denoted by I_q, $\mathrm{tr}(X)$ represents the trace of the matrix X. $X > O$ denotes a real positive definite matrix of order $q \times q$, and we write $O < A < X < B$ in the sense that $A,\ B,\ X,\ X - A,$ and $B - X$ are positive definite matrices, where A and B are constant matrices. $X^{\frac{1}{2}}$ will stand for the symmetric positive definite square root of a symmetric positive definite matrix X. The integral

$$\int_A^B f(X)\,\mathrm{d}X = \int_{O<A<X<B} f(X)\,\mathrm{d}X = \int_{A<X<B} f(X)\,\mathrm{d}X \tag{4.1}$$

means that a real valued scalar function $f(X)$ of $q \times q$ real positive definite matrix X is integrated over all X such that $X > O,\ X - A > O,\ B - X > O$. Here $\mathrm{d}X$ stands for the wedge product of differentials [24]. The corresponding integral in complex domain is denoted by

$$\int_A^B f(\tilde{X})\,\mathrm{d}\tilde{X} = \int_{A<\tilde{X}<B} f(\tilde{X})\,\mathrm{d}\tilde{X}, \tag{4.2}$$

integrated over all positive definite Hermitian matrix $\tilde{X}$ such that $\tilde{X} > O,\ \tilde{X} - \tilde{A} > O,\ \tilde{B} - \tilde{X} > O$.

The real matrix variate gamma function is defined as [24, (5.1.2)]

$$\begin{aligned}\Gamma_q(\alpha) =& \int_{X>O} |X|^{\alpha - \frac{q+1}{2}} \mathrm{e}^{-\mathrm{tr}(X)}\,\mathrm{d}X \\ =& \pi^{\frac{q(q-1)}{4}} \Gamma(\alpha)\Gamma\left(\alpha - \frac{1}{2}\right) \cdots \Gamma\left(\alpha - \frac{q-1}{2}\right),\end{aligned} \tag{4.3}$$

where $\Re(\alpha) > \frac{q-1}{2}$. Similiarly, $\tilde{\Gamma}_q(\cdot)$ denotes the complex matrix variate gamma function, given by

$$\tilde{\Gamma}_q(\alpha) = \int_{\tilde{Y}>O} |\det(\tilde{Y})|^{\alpha - q} \mathrm{e}^{-\mathrm{tr}(\tilde{Y})} \mathrm{d}\tilde{Y} = \pi^{\frac{q(q-1)}{2}} \Gamma(\alpha)\Gamma(\alpha - 1) \cdots \Gamma(\alpha - q + 1), \tag{4.4}$$

where $\Re(\alpha) > q-1$.

For $n = 1, 2, \ldots$, let the real matrix variate case of $\lambda^{(n)}_{\nu,\sigma}(x)$ be denoted by $\Lambda^{(n)}_{\nu,\sigma}(X)$ and defined as

$$\Lambda^{(n)}_{\nu,\sigma}(X) := \frac{n}{\Gamma_q\left(\nu + \frac{q+1}{2} - \frac{1}{n}\right)} \int_{Y > I_q} |Y^n - I_q|^{\nu - \frac{1}{n}} |Y|^{\sigma} \exp\left(-\mathrm{tr}\left[XY\right]\right) \mathrm{d}Y, \tag{4.5}$$

where $\Re\left(\nu\right) > \frac{1}{n} - 1$. Immediately, one can note that when $q = 1$ and $\beta = n$, $\Lambda^{(n)}_{\nu,\sigma}(X)$ reduces to the univariate case in (1.5). Analogously in complex domain, $\Lambda^{(n)}_{\nu,\sigma}(\tilde{X})$ is defined as

$$\Lambda^{(n)}_{\nu,\sigma}(\tilde{X}) := \frac{n}{\tilde{\Gamma}_q\left(\nu + \frac{q+1}{2} - \frac{1}{\beta}\right)} \int_{\tilde{Y} > I_q} |\tilde{Y}^n - I_q|^{\nu - \frac{1}{n}} |\tilde{Y}|^{\sigma} \exp\left(-\mathrm{tr}\left[\tilde{X}\tilde{Y}\right]\right) \mathrm{d}\tilde{Y}, \tag{4.6}$$

where $\Re\left(\nu\right) > \frac{1}{n} - 1$, $\tilde{X} > O$.

Remark 4.1. Note that the newly defined matrix variate extensions of Bessel type functions in (4.5) and (4.6) are not explicitly solvable at present. However for $n = 1$, by applying the M-transform, one can observe that they belong to the class of Meijer's G-functions of matrix arguments.

5 Concluding Remarks

The kernel function satisfies the essential analytical properties of classical calculus and obey to several intriguing inequalities. In order to undrestand the behaviour of the kernel function, a computable series representation has been obtained along with a graphical representations. Driven by mathematical curiosity, we extended the kernel function to matrix arguments and observed that its key properties remain intact in this generalized context.

Acknowledgement
The authors express their gratitude to the University of Kerala for providing the necessary facilities for this research work.
Data Availability Statement Data sharing is not applicable to this article because no datasets were generated/analyzed during the preparation of the paper.
Declarations
Competing Interests Authors have no relevant financial or non-financial interests to disclose.

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