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Novel Findings on the Srivastava-Luo-Raina M-Transform with Applications to Incomplete $\mathbb{W}$-Functions

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Abstract This paper introduces the Srivastava-Luo-Raina M-transform in the context of the incomplete $\mathbb{W}$-function, providing a systematic framework to explore its properties and applications. The study derives image formulas for several key cases of the incomplete $\mathbb{W}$-function, highlighting their mathematical significance and utility in broader contexts. The results can be specialized by assigning specific parameter values to recover well-known identities and generate novel insights. The presented approach has potential applications in mathematical modeling, special function theory, and integral transforms, offering a versatile tool for further research in applied mathematics and related disciplines.

1 Introduction and Preliminaries

A wide range of problems in several branches of mathematical physics, including hydrodynamics, radio physics, atomic and nuclear physics, and acoustics, are associated with special functions, which are essential in analyzing differential equation solutions [3, 9]. Incomplete special functions have found increasing applicability across various fields, including reaction and combustion processes, reaction-diffusion systems, electronics, communication, fractional differential and integral equations, theoretical mechanics, and statistical probability theory. As a result, numerous research papers [6, 7, 11] have recently been published, exploring incomplete special functions and their connections to other transcendental functions.

Integral transformations are widely applied, and a substantial body of work has already been done on their concepts and implementations, as referenced in [14, 16, 20, 21]. The computation of image formulas under these transformations for special functions, whether involving one or more variables, plays a key role in the efficient solution of differential and integral equations. These applications motivate many researchers to develop image formulas for fundamental transformations related to various special functions.

1.1 Foundational Concepts

The functions $\gamma(\kappa,\tau)$ and $\Gamma(\kappa,\tau)$, known as the lower and upper incomplete gamma functions, respectively, are defined as follows:

$$\gamma(\kappa,\tau) = \int_0^\tau t^{\kappa-1} e^{-t} dt\,, \quad (\Re(\kappa) > 0, \tau \geq 0), \tag{1.1}$$

and

$$\Gamma(\kappa,\tau) = \int_\tau^\infty t^{\kappa-1} e^{-t} dt\,, \quad (\Re(\kappa) > 0, \tau \geq 0), \tag{1.2}$$

respectively.

$$\gamma(\kappa,\tau) + \Gamma(\kappa,\tau) = \Gamma(\kappa), \qquad (\Re(\kappa) > 0).$$

Yang et al. [23] recently introduced a new category of the $\mathbb{W}$-function, which is a generalization of the $\mathbb{Y}$-function, and it is described as

$$\mathbb{W}_{r,s}^{m,n}[z;y;x] = \mathbb{W}_{r,s}^{m,n}\left[z;y;x \,\middle|\, \begin{matrix} (A_i,u_i,U_i)_1^r \\ (B_j,v_j,V_j)_1^s \end{matrix}\right] = \int_\omega \Phi_{r,s}^{m,n}(\kappa)\; z^{-\kappa}\; e^{\kappa y} \kappa^x \; d\kappa, \tag{1.3}$$

where

$$\Phi_{r,s}^{m,n}(\kappa) = \frac{\prod_{j=1}^{m} \{\Gamma(v_j - V_j\kappa)\}^{B_j} \; \prod_{i=2}^{n} \{\Gamma(1-u_i+U_i\kappa)\}^{A_i}}{\prod_{j=m+1}^{s} \{\Gamma(1-v_j+V_j\kappa)\}^{B_j} \; \prod_{i=n+1}^{r} \{\Gamma(u_i-U_i\kappa)\}^{A_i}}, \tag{1.4}$$

with complex variables x, y, z, and the contour ω in the complex plane, the orders (m,n,r,s) are non-negative integers such that $0 \le m \le s$ and $0 \le n \le r$. The parameters $U_i > 0$ and $V_j > 0$ are positive, while u_i and v_j, for $i = 1,\cdots,r$ and $j = 1,\cdots,s$, are arbitrary complex numbers satisfying the following conditions:
$U_i(v_j+\rho) \ne V_j(u_i-\rho'-1)$,
$(\rho,\rho' \in (\mathbb{N}_0(0,1,2,\cdots); i = 1,2,\cdots,r; j = 1,2,\cdots,s)$.

Kritika et al. [13] introduced the following set of incomplete $\mathbb{W}$-functions, ${}^{\gamma}\mathbb{W}_{r,s}^{m,n}[z;y;x]$ and ${}^{\Gamma}\mathbb{W}_{r,s}^{m,n}[z;y;x]$, in the study sequence mentioned above, using the incomplete gamma functions defined by [19]:

$$\begin{aligned} {}^{\gamma}\mathbb{W}_{r,s}^{m,n}[z;y;x] &= {}^{\gamma}\mathbb{W}_{r,s}^{m,n}\left[z;y;x \,\middle|\, \begin{matrix} (A_1,u_1,U_1;\tau),(A_i,u_i,U_i)_2^r \\ (B_j,v_j,V_j)_1^s \end{matrix}\right] \\ &= \frac{1}{2\pi i}\int_\omega {}^{\gamma}\Phi_{r,s}^{m,n}(\kappa,\tau) z^{-\kappa}\; e^{\kappa y}\kappa^x \; d\kappa, \end{aligned} \tag{1.5}$$

where

$${}^{\gamma}\Phi_{r,s}^{m,n}(\kappa,\tau) = \frac{\{\gamma(1-u_1+U_1\kappa,\tau)\}^{A_1} \; \prod_{j=1}^{m} \{\Gamma(v_j-V_j\kappa)\}^{B_j} \prod_{i=2}^{n} \{\Gamma(1-u_i+U_i\kappa)\}^{A_i}}{\prod_{j=m+1}^{s} \{\Gamma(1-v_j+V_j\kappa)\}^{B_j} \; \prod_{i=n+1}^{r} \{\Gamma(u_i-U_i\kappa)\}^{A_i}}, \tag{1.6}$$

and

$$\begin{aligned} {}^{\Gamma}\mathbb{W}_{r,s}^{m,n}[z;y;x] &= {}^{\Gamma}\mathbb{W}_{r,s}^{m,n}\left[z;y;x \,\middle|\, \begin{matrix} (A_1,u_1,U_1;\tau),(A_i,u_i,U_i)_2^r \\ (B_j,v_j,V_j)_1^s \end{matrix}\right] \\ &= \frac{1}{2\pi i}\int_\omega {}^{\Gamma}\Phi_{r,s}^{m,n}(\kappa,\tau) z^{-\kappa}\; e^{\kappa y}\kappa^x \; d\kappa, \end{aligned} \tag{1.7}$$

where

$${}^{\Gamma}\Phi_{r,s}^{m,n}(\kappa,\tau) = \frac{\{\Gamma(1-u_1+U_1\kappa,\tau)\}^{A_1} \; \prod_{j=1}^{m} \{\Gamma(v_j-V_j\kappa)\}^{B_j} \prod_{i=2}^{n} \{\Gamma(1-u_i+U_i\kappa)\}^{A_i}}{\prod_{j=m+1}^{s} \{\Gamma(1-v_j+V_j\kappa)\}^{B_j} \; \prod_{i=n+1}^{r} \{\Gamma(u_i-U_i\kappa)\}^{A_i}}. \tag{1.8}$$

The incomplete $\mathbb{W}$-function ${}^{\gamma}\mathbb{W}_{r,s}^{m,n}[z;y;x]$ and ${}^{\Gamma}\mathbb{W}_{r,s}^{m,n}[z;y;x]$ in (1.5) and (1.7) exist for all $\tau \ge 0$ under the same contour and the same set of conditions as above. With $A_1 = 1$, the definitions in (1.5) and (1.7) readily give the following decomposition formula:

$${}^{\gamma}\mathbb{W}_{r,s}^{m,n}[z;y;x] + {}^{\Gamma}\mathbb{W}_{r,s}^{m,n}[z;y;x] = \mathbb{W}_{r,s}^{m,n}[z;y;x], \tag{1.9}$$

for the $\mathbb{W}$-function defined by (1.3).

In 2012, Salim [17] introduced a new generalization of the Mittag-Leffler function, which is defined as:

$$E_{\alpha,\beta,p}^{\vartheta,\varsigma,q}\,(z) = \sum_{n=0}^{\infty} \frac{\vartheta_{qn}\; z^n}{\Gamma(\alpha n+\beta)\;(\varsigma)_{pn}}, \tag{1.10}$$

where $\alpha, \beta, \vartheta, \varsigma \in \mathbb{C}$, min $(\Re(\alpha), \Re(\beta), \Re(\vartheta), \Re(\varsigma)) > 0$, $p, q > 0$, and $q \leq \Re(\kappa) + p$.

Srivastava et al. [22] introduced the following Srivastava-Luo M-transform:

$$M_{\lambda,\mu}[f(t)](a,b) = \int_0^\infty \frac{e^{-at} f(bt)}{(t^\mu + b^\mu)^\lambda} dt, \tag{1.11}$$

provided that $\lambda \in \mathbb{C}, \Re(\lambda) \geq 0, \mu \in \mathbb{N}, a \in \mathbb{C}$, and $b \in \mathbb{R}^+$ are transform variables.

The Srivastava-Luo-Raina M-transform is closely linked to the well-known integral transforms: Laplace, Natural, and Sumudu, as stated in Equation (1.11):

The Laplace transform [11] is defined by:

$$\mathbb{L}[f(t)](p) = \int_0^\infty e^{-pt} f(t) dt, \qquad \Re(p) > 0. \tag{1.12}$$

From Equations (1.11) and (1.12), we have the following Laplace-$M_{\lambda,\mu}[f(t)](p,q)$ transform duality relation [1]:

$$\mathbb{L}[f(t)](p) = M_{0,\mu}[f(t)](p,1), \qquad \Re(p) > 0, \tag{1.13}$$

and

$$M_{\lambda,\mu}[f(t)](p,q) = \mathbb{L}\left(\frac{f(qt)}{(t^\mu + q^\mu)^\lambda}\right)(p) = \frac{1}{q}\mathbb{L}\left[\frac{f(t)}{\left((\frac{t}{q})^\mu + q^\mu\right)^\lambda}\right]\left(\frac{p}{q}\right), \qquad p, q > 0. \tag{1.14}$$

Put λ = 0 in (1.11), we get the Natural transform [5, 8] and it is defined as:

$$N[f(t)](p,q) = \int_0^\infty e^{-pt} f(qt) dt, \qquad p, q > 0. \tag{1.15}$$

From Equation (1.11) and (1.15), we have the following Natural-$M_{\lambda,\mu}[f(t)](p,q)$ transform duality relation [1]:

$$N[f(t)](p,q) = M_{0,\mu}[f(t)](p,q), \qquad p, q > 0, \tag{1.16}$$

and

$$M_{\lambda,\mu}[f(t)](p,q) = N\left[\frac{f(t)}{\left((\frac{t}{q})^\mu + q^\mu\right)^\lambda}\right](p,q), \qquad p, q > 0. \tag{1.17}$$

Sumudu transform [4] is defined by:

$$S[f(t)](q) = \int_0^\infty e^{-t} f(qt) dt, \qquad q > 0. \tag{1.18}$$

From Equations (1.11) and (1.18), we have the following Sumudu-$M_{\lambda,\mu}[f(t)](p,q)$ transform duality relation [1]:

$$S[f(t)](q) = M_{0,\mu}[f(t)](0,q), \qquad q > 0, \tag{1.19}$$

and

$$M_{\lambda,\mu}[f(t)](p,q) = \frac{1}{p}S\left[\frac{f(t)}{\left((\frac{t}{q})^\mu + q^\mu\right)^\lambda}\right]\left(\frac{q}{p}\right), \qquad p, q > 0. \tag{1.20}$$

Lemma 1.1. *For $\lambda, a \in \mathbb{C}, b \in \mathbb{R}^+, \Re(\lambda) \geq 0, \mu \in \mathbb{N}$, and $c, \nu > 0$, then we have following assertion:*

$$M_{\lambda,\mu}[t^{\nu-1}](a,b) = \frac{b^{\nu-\lambda\mu-1}a^{-\nu}}{\mu\Gamma(\lambda)}\mathbb{H}^{2,1}_{1,2}\left[ab \;\middle|\; \begin{matrix}(1,\frac{1}{\mu})\\(\nu,1),(\lambda,\frac{1}{\mu})\end{matrix}\right], \tag{1.21}$$

$$M_{\lambda,\mu}[e^{-ct}](a,b) = \frac{b^{-\lambda\mu}}{\mu(a+cb)\Gamma(\lambda)}\mathbb{H}^{2,1}_{1,2}\left[b(a+cb) \;\middle|\; \begin{matrix}(1,\frac{1}{\mu})\\(1,1),(\lambda,\frac{1}{\mu})\end{matrix}\right], \tag{1.22}$$

and

$$M_{\lambda,\mu}[t^{\nu-1}e^{-ct}](a,b) = \frac{b^{\nu-\lambda\mu-1}}{\mu(a+cb)^{\nu}\Gamma(\lambda)}\mathbb{H}^{2,1}_{1,2}\left[b(a+cb) \;\middle|\; \begin{matrix}(1,\frac{1}{\mu})\\(\nu,1),(\lambda,\frac{1}{\mu})\end{matrix}\right], \tag{1.23}$$

where $\mathbb{H}^{2,1}_{1,2}$ is the H-function [10] defined in the following manner:

$$\mathbb{H}^{2,1}_{1,2}(z;g) = \mathbb{H}^{2,1}_{1,2}\left[z;g \;\middle|\; \begin{matrix}(b,\Phi)\\(c_1,\phi_1),(c_2,\phi_2)\end{matrix}\right]. \tag{1.24}$$

2 Main Results

This section establishes the Srivastava-Luo-Raina M-transform associated with the $\mathbb{W}$-function.

Theorem 2.1. *If $\lambda, a \in \mathbb{C}, \xi_1 \geq 0, \mu \in \mathbb{N}, b \in \mathbb{R}^+$, and with the same conditions as in Equation (1.7), then the result is as follows:*

$$\begin{aligned} M_{\lambda,\mu}&\left[{}^{\Gamma}\mathbb{W}^{m,n}_{r,s}\left[zh^{\xi_1};y;x \;\middle|\; \begin{matrix}(A_1,u_1,U_1;\tau),(A_i,u_i,U_i)_2^r\\(B_j,v_j,V_j)_1^s\end{matrix}\right]\right](a,b) = \frac{b^{-\mu\lambda}}{a\mu}\frac{1}{2\pi i}\int_\Delta B(\lambda-\frac{s}{\mu},\frac{s}{\mu})\\ &\times(ab)^s\left({}^{\Gamma}\mathbb{W}^{m,n+1}_{r+1,s}\left[z\left(\frac{b}{a}\right)^{\xi_1};y;x \;\middle|\; \begin{matrix}(A_1,u_1,U_1;\tau),(s,-\xi_1;1),(A_i,u_i,U_i)_2^r\\(B_j,v_j,V_j)_1^s\end{matrix}\right]\right)ds. \end{aligned} \tag{2.1}$$

Proof. The LHS of Equation (2.1) is:

$$L_1 = M_{\lambda,\mu}\left[{}^{\Gamma}\mathbb{W}^{m,n}_{r,s}\left[zh^{\xi_1};y;x \;\middle|\; \begin{matrix}(A_1,u_1,U_1;\tau),(A_i,u_i,U_i)_2^r\\(B_j,v_j,V_j)_1^s\end{matrix}\right]\right](a,b). \tag{2.2}$$

Replace the $\mathbb{W}$-function with (1.7) and apply the definition of Srivastava-Luo-Raina M-transform defined in (1.11), we obtain:

$$L_1 = \int_0^\infty \frac{e^{-ah}}{(h^\mu+b^\mu)^\lambda}\left[\frac{1}{2\pi i}\int_\omega {}^{\Gamma}\Phi^{m,n}_{r,s}(\kappa,\tau)[z(bh)^{\xi_1}]^{-\kappa}e^{\kappa y}\kappa^x d\kappa\right]dh. \tag{2.3}$$

Change the order of integration and with the help of (1.21) of Lemma 1.1, we get:

$$\begin{aligned} L_1 &= \frac{1}{2\pi i}\int_\omega {}^{\Gamma}\Phi^{m,n}_{r,s}(\kappa,\tau)z^{-\kappa}e^{\kappa y}\kappa^x\,\frac{b^{-\xi_1\kappa+1-\mu\lambda-1}a^{\xi_1\kappa-1}}{\mu\Gamma(\lambda)}\mathbb{H}^{2,1}_{1,2}\left[ab \;\middle|\; \begin{matrix}(1,\frac{1}{\mu})\\(\xi_1\kappa+1,1),(\lambda,\frac{1}{\mu})\end{matrix}\right]d\kappa\\ &= \frac{b^{-\mu\lambda}}{a\mu}\frac{1}{2\pi i}\int_\omega {}^{\Gamma}\Phi^{m,n}_{r,s}(\kappa,\tau)\left[z\left(\frac{b}{a}\right)^{\xi_1}\right]^{-\kappa}e^{\kappa y}\kappa^x\frac{1}{2\pi i}\int_\Delta \frac{\Gamma(\xi_1\kappa+1-s)\Gamma\left(\lambda-\frac{s}{\mu}\right)\Gamma\left(\frac{s}{\mu}\right)}{\Gamma(\lambda)}(ab)^s ds d\kappa. \end{aligned} \tag{2.4}$$

Change the order of integration, and after some adjustment of terms, we obtain the desired outcomes. □

Theorem 2.2. *If $\lambda, a \in \mathbb{C}, \xi_1 \geq 0, \mu \in \mathbb{N}, b \in \mathbb{R}^+$, and with the same conditions as in Equation (1.5), then the result is as follows.*

$$M_{\lambda,\mu}\left[{}^{\gamma}\mathbb{W}^{m,n}_{r,s}\left[zh^{\xi_1};y;x \,\middle|\, \begin{matrix}(A_1,u_1,U_1;\tau),(A_i,u_i,U_i)_2^r\\(B_j,v_j,V_j)_1^s\end{matrix}\right]\right](a,b)=\frac{b^{-\mu\lambda}}{a\mu}\frac{1}{2\pi i}\int_{\Delta}B(\lambda-\frac{s}{\mu},\frac{s}{\mu})$$
$$\times(ab)^s\left({}^{\gamma}\mathbb{W}^{m,n+1}_{r+1,s}\left[z\left(\frac{b}{a}\right)^{\xi_1};y;x \,\middle|\, \begin{matrix}(A_1,u_1,U_1;\tau),(s,-\xi_1;1),(A_i,u_i,U_i)_2^r\\(B_j,v_j,V_j)_1^s\end{matrix}\right]\right)ds. \tag{2.5}$$

Proof. Theorem 2.2 proved to be like Theorem 2.1 but with the same conditions. □

Theorem 2.3. *If $\lambda, a, \alpha, \beta, \vartheta, \varsigma \in \mathbb{C}$, $\xi_1, \xi_2 \geq 0, \mu \in \mathbb{N}, b \in \mathbb{R}^+$, $p, q > 0$, and with the same conditions as given in Equations (1.7) and (1.10), then the result is as follows:*

$$M_{\lambda,\mu}\left[{}^{\Gamma}\mathbb{W}^{m,n}_{r,s}\left[zh^{\xi_1};y;x \,\middle|\, \begin{matrix}(A_1,u_1,U_1;\tau),(A_i,u_i,U_i)_2^r\\(B_j,v_j,V_j)_1^s\end{matrix}\right]\times E^{\vartheta,\varsigma,q}_{\alpha,\beta,p}(th^{\xi_2})\right](a,b)$$
$$=\frac{b^{-\mu\lambda}}{a\mu}\sum_{n=0}^{\infty}\frac{\vartheta_{qn}}{\Gamma(\alpha n+\beta)(\varsigma)_{pn}}\left[t\left(\frac{b}{a}\right)^{\xi_2}\right]^n\frac{1}{2\pi i}\int_{\Delta}B(\lambda-\frac{s}{\mu},\frac{s}{\mu})$$
$$\times(ab)^s\left({}^{\Gamma}\mathbb{W}^{m,n+1}_{r+1,s}\left[z\left(\frac{b}{a}\right)^{\xi_1};y;x \,\middle|\, \begin{matrix}(A_1,u_1,U_1;\tau),(s-\xi_2 n,-\xi_1;1),(A_i,u_i,U_i)_2^r\\(B_j,v_j,V_j)_1^s\end{matrix}\right]\right)ds. \tag{2.6}$$

Proof. The LHS of Equation (2.1) is

$$L_3=M_{\lambda,\mu}\left[{}^{\Gamma}\mathbb{W}^{m,n}_{r,s}\left[zh^{\xi_1};y;x \,\middle|\, \begin{matrix}(A_1,u_1,U_1;\tau),(A_i,u_i,U_i)_2^r\\(B_j,v_j,V_j)_1^s\end{matrix}\right]\times E^{\vartheta,\varsigma,q}_{\alpha,\beta,p}(th^{\xi_2})\right](a,b). \tag{2.7}$$

Replace the $\mathbb{W}$-function and Mittag-Leffler function with (1.7) and (1.10) respectively and apply the definition of Srivastava-Luo-Raina M-transform defined in (1.11), we obtain:

$$L_3=\int_0^{\infty}\frac{e^{-ah}}{(h^{\mu}+b^{\mu})^{\lambda}}\left[\frac{1}{2\pi i}\int_{\omega}{}^{\Gamma}\Phi^{m,n}_{r,s}(\kappa,\tau)[z(bh)^{\xi_1}]^{-\kappa}e^{\kappa y}\kappa^x d\kappa\sum_{n=0}^{\infty}\frac{\vartheta_{qn}[t(bh)^{\xi_2}]^n}{\Gamma(\alpha n+\beta)(\varsigma)_{pn}}\right]dh. \tag{2.8}$$

Change the order of integration and with the help of (1.21) of Lemma 1.1, we get:

$$L_3=\sum_{n=0}^{\infty}\frac{\vartheta_{qn}t^n}{\Gamma(\alpha n+\beta)(\varsigma)_{pn}}\times\frac{1}{2\pi i}\int_{\omega}{}^{\Gamma}\Phi^{m,n}_{r,s}(\kappa,\tau)z^{-\kappa}e^{\kappa y}\kappa^x\times\frac{b^{-\xi_1\kappa+\xi_2 n+1-\mu\lambda-1}a^{\xi_1\kappa-\xi_2 n-1}}{\mu\Gamma(\lambda)}$$
$$\times\mathbb{H}^{2,1}_{1,2}\left[ab \,\middle|\, \begin{matrix}(1,\frac{1}{\mu})\\(-\xi_1\kappa+\xi_2 n+1,1),(\lambda,\frac{1}{\mu})\end{matrix}\right]d\kappa$$

$$=\frac{b^{-\mu\lambda}}{a\mu}\sum_{n=0}^{\infty}\frac{\vartheta_{qn}}{\Gamma(\alpha n+\beta)(\varsigma)_{pn}}\left[t\left(\frac{b}{a}\right)^{\xi_2}\right]^n\frac{1}{2\pi i}\int_{\omega}{}^{\Gamma}\Phi^{m,n}_{r,s}(\kappa,\tau)\left[z\left(\frac{b}{a}\right)^{\xi_1}\right]^{-\kappa}e^{\kappa y}\kappa^x$$
$$\times\frac{1}{2\pi i}\int_{\Delta}\frac{\Gamma(-\xi_1\kappa+\xi_2 n+1-s)\Gamma\left(\lambda-\frac{s}{\mu}\right)\Gamma\left(\frac{s}{\mu}\right)}{\Gamma(\lambda)}(ab)^s ds d\kappa. \tag{2.9}$$

Change the order of integration, and after some adjustment of terms, we obtain the desired outcomes. □

Theorem 2.4. *If $\lambda, a, \alpha, \beta, \vartheta, \varsigma \in \mathbb{C}$, $\xi_1, \xi_2 \geq 0, \mu \in \mathbb{N}, b \in \mathbb{R}^+$, $p, q > 0$, and with the same conditions as in Equations (1.7) and (1.10), then the result is as follows:*

$$M_{\lambda,\mu}\left[{}^{\gamma}\mathbb{W}_{r,s}^{m,n}\left[zh^{\xi_1}; y; x \,\middle|\, \begin{matrix}(A_1, u_1, U_1; \tau), (A_i, u_i, U_i)_2^r \\ (B_j, v_j, V_j)_1^s\end{matrix}\right] \times E_{\alpha,\beta,p}^{\vartheta,\varsigma,q}(th^{\xi_2})\right](a, b)$$
$$= \frac{b^{-\mu\lambda}}{a\mu}\sum_{n=0}^{\infty}\frac{\vartheta_{qn}}{\Gamma(\alpha n + \beta)(\varsigma)_{pn}}\left[t\left(\frac{b}{a}\right)^{\xi_2}\right]^n \frac{1}{2\pi i}\int_{\Delta} B(\lambda - \frac{s}{\mu}, \frac{s}{\mu})$$
$$\times (ab)^s\left({}^{\gamma}\mathbb{W}_{r+1,s}^{m,n+1}\left[z\left(\frac{b}{a}\right)^{\xi_1}; y; x \,\middle|\, \begin{matrix}(A_1, u_1, U_1; \tau), (s - \xi_2 n, -\xi_1; 1), (A_i, u_i, U_i)_2^r \\ (B_j, v_j, V_j)_1^s\end{matrix}\right]\right) ds. \tag{2.10}$$

Proof. Theorem 2.4 proved to be similar to Theorem 2.3 but with the same conditions. □

Theorem 2.5. *If $\lambda, a \in \mathbb{C}, \Re(\lambda), c \geq 0, \mu \in \mathbb{N}, b \in \mathbb{R}^+$, and with the same conditions given in Equation (1.7), then the result is as follows.*

$$M_{\lambda,\mu}\left[{}^{\Gamma}\mathbb{W}_{r,s}^{m,n}\left[ze^{\frac{-ch}{\kappa}}; y; x \,\middle|\, \begin{matrix}(A_1, u_1, U_1; \tau), (A_i, u_i, U_i)_2^r \\ (B_j, v_j, V_j)_1^s\end{matrix}\right]\right](a, b) = \frac{b^{-\mu\lambda}}{\mu(a + cb)}\frac{1}{2\pi i}\int_{\Delta} B(\lambda - \frac{s}{\mu}, \frac{s}{\mu})$$
$$\times [b(a + cb)]^s\left({}^{\Gamma}\mathbb{W}_{r+1,s}^{m,n+1}\left[z; y; x \,\middle|\, \begin{matrix}(A_1, u_1, U_1; \tau), (s, 0; 1), (A_i, u_i, U_i)_2^r \\ (B_j, v_j, V_j)_1^s\end{matrix}\right]\right) ds. \tag{2.11}$$

Proof. The LHS of Equation (2.16) is:

$$L_5 = M_{\lambda,\mu}\left[{}^{\Gamma}\mathbb{W}_{r,s}^{m,n}\left[ze^{\frac{-ch}{\kappa}}; y; x \,\middle|\, \begin{matrix}(A_1, u_1, U_1; \tau), (A_i, u_i, U_i)_2^r \\ (B_j, v_j, V_j)_1^s\end{matrix}\right]\right](a, b). \tag{2.12}$$

Replace the $\mathbb{W}$-function with (1.7) and apply the definition of Srivastava-Luo-Raina M-transform defined in (1.11), we obtain:

$$L_5 = \int_0^{\infty}\frac{e^{-ah}}{(h^{\mu} + b^{\mu})^{\lambda}}\left[\frac{1}{2\pi i}\int_{\omega}{}^{\Gamma}\Phi_{r,s}^{m,n}(\kappa, \tau)(ze^{\frac{-cbh}{\kappa}})^{-\kappa}e^{\kappa y}\kappa^x d\kappa\right] dh. \tag{2.13}$$

Change the integration order and with the help of (1.22) in Lemma 1.1, we get:

$$L_5 = \frac{1}{2\pi i}\int_{\omega}{}^{\Gamma}\Phi_{r,s}^{m,n}(\kappa, \tau)z^{-\kappa}e^{\kappa y}\kappa^x \times \frac{b^{-\mu\lambda}}{\mu(a + cb)\Gamma(\lambda)} \times \mathbb{H}_{1,2}^{2,1}\left[b[a + cb] \,\middle|\, \begin{matrix}(1, \frac{1}{\mu}) \\ (1, 1), (\lambda, \frac{1}{\mu})\end{matrix}\right] d\kappa$$

$$= \frac{b^{-\mu\lambda}}{\mu(a + cb)} \times \frac{1}{2\pi i}\int_{\omega}{}^{\Gamma}\Phi_{r,s}^{m,n}(\kappa, \tau)z^{-\kappa}e^{\kappa y}\kappa^x \times \frac{1}{2\pi i}\int_{\Delta}\frac{\Gamma(1 - s)\Gamma\left(\lambda - \frac{s}{\mu}\right)\Gamma\left(\frac{s}{\mu}\right)}{\Gamma(\lambda)}[b(a + cb)]^s ds d\kappa. \tag{2.14}$$

Change the order of integration and after some adjustment of terms, we get the desired outcomes. □

Theorem 2.6. *If $\lambda, a \in \mathbb{C}, \Re(\lambda), c \geq 0, \mu \in \mathbb{N}, b \in \mathbb{R}^+$, and with same conditions given in Equation (1.5), then the result is as follows.*

$$M_{\lambda,\mu}\left[{}^{\gamma}\mathbb{W}_{r,s}^{m,n}\left[ze^{\frac{-ch}{\kappa}}; y; x \,\middle|\, \begin{matrix}(A_1, u_1, U_1; \tau), (A_i, u_i, U_i)_2^r \\ (B_j, v_j, V_j)_1^s\end{matrix}\right]\right](a, b) = \frac{b^{-\mu\lambda}}{\mu(a + cb)}\frac{1}{2\pi i}\int_{\Delta} B(\lambda - \frac{s}{\mu}, \frac{s}{\mu})$$
$$\times [b(a + cb)]^s\left({}^{\gamma}\mathbb{W}_{r+1,s}^{m,n+1}\left[z; y; x \,\middle|\, \begin{matrix}(A_1, u_1, U_1; \tau), (s, 0; 1), (A_i, u_i, U_i)_2^r \\ (B_j, v_j, V_j)_1^s\end{matrix}\right]\right) ds. \tag{2.15}$$

Proof. Theorem 2.6 is proved to be like Theorem 2.5, but with the same conditions. □

Theorem 2.7. *If* $\lambda, a, \alpha, \beta, \vartheta, \varsigma \in \mathbb{C}, \Re(\lambda), c, d \geq 0, \mu \in \mathbb{N}, b \in \mathbb{R}^+$, $p, q > 0$, *and with the same conditions as given in Equations* (1.7) *and* (1.10)*, then the result is as follows.*

$$M_{\lambda,\mu}\left[{}^{\Gamma}\mathbb{W}^{m,n}_{r,s}\left[ze^{\frac{-ch}{\kappa}};y;x \,\middle|\, \begin{matrix}(A_1,u_1,U_1;\tau),(A_i,u_i,U_i)_2^r\\(B_j,v_j,V_j)_1^s\end{matrix}\right] \times E^{\vartheta,\varsigma,q}_{\alpha,\beta,p}(te^{\frac{-dh}{n}})\right](a,b)$$
$$= \frac{b^{-\mu\lambda}}{\mu(a+(c+d)b)}E^{\vartheta,\varsigma,q}_{\alpha,\beta,p}(t)\frac{1}{2\pi i}\int_\Delta B(\lambda-\frac{s}{\mu},\frac{s}{\mu}) \times [b(a+(c+d)b)]^s$$
$$\times\left({}^{\Gamma}\mathbb{W}^{m,n+1}_{r+1,s}\left[z;y;x \,\middle|\, \begin{matrix}(A_1,u_1,U_1;\tau),(s,0;1),(A_i,u_i,U_i)_2^r\\(B_j,v_j,V_j)_1^s\end{matrix}\right]\right)ds. \tag{2.16}$$

Proof. The LHS of Equation (2.16) is:

$$L_7 = M_{\lambda,\mu}\left[{}^{\Gamma}\mathbb{W}^{m,n}_{r,s}\left[ze^{\frac{-ch}{\kappa}};y;x \,\middle|\, \begin{matrix}(A_1,u_1,U_1;\tau),(A_i,u_i,U_i)_2^r\\(B_j,v_j,V_j)_1^s\end{matrix}\right] E^{\vartheta,\varsigma,q}_{\alpha,\beta,p}(te^{\frac{-dh}{n}})\right](a,b). \tag{2.17}$$

Replace the $\mathbb{W}$-function and Mittag-Leffler function with (1.7) and (1.10) respectively and apply the definition of Srivastava-Luo-Raina M-transform defined in (1.11), we obtain:

$$L_7 = \int_0^\infty \frac{e^{-ah}}{(h^\mu+b^\mu)^\lambda}\left[\frac{1}{2\pi i}\int_\omega {}^{\Gamma}\Phi^{m,n}_{r,s}(\kappa,\tau)(ze^{\frac{-ch}{\kappa}})^{-\kappa}e^{\kappa y}\kappa^x d\kappa \sum_{n=0}^\infty \frac{\vartheta_{qn}(te^{\frac{-dh}{n}})^n}{\Gamma(\alpha n+\beta)(\varsigma)_{pn}}\right]dh. \tag{2.18}$$

Change the integration order and with the help of (1.22) in Lemma 1.1, we get:

$$L_7 = \sum_{n=0}^\infty \frac{\vartheta_{qn}t^n}{\Gamma(\alpha n+\beta)(\varsigma)_{pn}} \times \frac{1}{2\pi i}\int_\omega {}^{\Gamma}\Phi^{m,n}_{r,s}(\kappa,\tau)z^{-\kappa}e^{\kappa y}\kappa^x \times \frac{b^{-\mu\lambda}}{\mu(a+(c+d)b)\Gamma(\lambda)}$$
$$\times \mathbb{H}^{2,1}_{1,2}\left[b[a+(c+d)b] \,\middle|\, \begin{matrix}(1,\frac{1}{\mu})\\(1,1),(\lambda,\frac{1}{\mu})\end{matrix}\right]d\kappa$$
$$= \frac{b^{-\mu\lambda}}{\mu(a+(c+d)b)}\sum_{n=0}^\infty \frac{\vartheta_{qn}t^n}{\Gamma(\alpha n+\beta)(\varsigma)_{pn}} \times \frac{1}{2\pi i}\int_\omega {}^{\Gamma}\Phi^{m,n}_{r,s}(\kappa,\tau)z^{-\kappa}e^{\kappa y}\kappa^x$$
$$\times \frac{1}{2\pi i}\int_\Delta \frac{\Gamma(1-s)\Gamma\left(\lambda-\frac{s}{\mu}\right)\Gamma\left(\frac{s}{\mu}\right)}{\Gamma(\lambda)}[b(a+(c+d)b)]^s ds d\kappa. \tag{2.19}$$

Change the order of integration and after some adjustment of terms, we get the desired outcomes. □

Theorem 2.8. *If* $\lambda, a, \alpha, \beta, \vartheta, \varsigma \in \mathbb{C}, \Re(\lambda), c, d \geq 0, \mu \in \mathbb{N}, b \in \mathbb{R}^+$, $p, q > 0$, *and with the same conditions as given in Equations* (1.7) *and* (1.10)*, then the result is as follows.*

$$M_{\lambda,\mu}\left[{}^{\gamma}\mathbb{W}^{m,n}_{r,s}\left[ze^{\frac{-ch}{\kappa}};y;x \,\middle|\, \begin{matrix}(A_1,u_1,U_1;\tau),(A_i,u_i,U_i)_2^r\\(B_j,v_j,V_j)_1^s\end{matrix}\right] \times E^{\vartheta,\varsigma,q}_{\alpha,\beta,p}(te^{\frac{-dh}{n}})\right](a,b)$$
$$= \frac{b^{-\mu\lambda}}{\mu(a+(c+d)b)}E^{\vartheta,\varsigma,q}_{\alpha,\beta,p}(t)\frac{1}{2\pi i}\int_\Delta B(\lambda-\frac{s}{\mu},\frac{s}{\mu}) \times [b(a+(c+d)b)]^s$$
$$\times\left({}^{\gamma}\mathbb{W}^{m,n+1}_{r+1,s}\left[z;y;x \,\middle|\, \begin{matrix}(A_1,u_1,U_1;\tau),(s,0;1),(A_i,u_i,U_i)_2^r\\(B_j,v_j,V_j)_1^s\end{matrix}\right]\right)ds. \tag{2.20}$$

Proof. Theorem 2.8 is proved to be like Theorem 2.7, but with the same conditions. □

Theorem 2.9. *If $\lambda, a \in \mathbb{C}, \Re(\lambda), \xi_1 \geq 0, \mu \in \mathbb{N}, b \in \mathbb{R}^+$, and with same conditions given in Equation (1.7), then the result is as follows.*

$$M_{\lambda,\mu}\left[{}^{\Gamma}\mathbb{W}_{r,s}^{m,n}\left[zh^{\xi_1}e^{\frac{-ch}{\kappa}};y;x \,\middle|\, \begin{matrix}(A_1,u_1,U_1;\tau),(A_i,u_i,U_i)_2^r\\(B_j,v_j,V_j)_1^s\end{matrix}\right]\right](a,b) = \frac{b^{-\mu\lambda}}{\mu(a+cb)}\frac{1}{2\pi i}\int_{\Delta}B(\lambda-\frac{s}{\mu},\frac{s}{\mu})$$
$$\times [b(a+cb)]^s\left({}^{\Gamma}\mathbb{W}_{r+1,s}^{m,n+1}\left[z\left(\frac{b}{a+cb}\right)^{\xi_1};y;x \,\middle|\, \begin{matrix}(A_1,u_1,U_1;\tau),(s,-\xi_1;1),(A_i,u_i,U_i)_2^r\\(B_j,v_j,V_j)_1^s\end{matrix}\right]\right)ds. \tag{2.21}$$

Proof. The LHS of Equation (2.21) is:

$$L_9 = M_{\lambda,\mu}\left[{}^{\Gamma}\mathbb{W}_{r,s}^{m,n}\left[zh^{\xi_1}e^{\frac{-ch}{\kappa}};y;x \,\middle|\, \begin{matrix}(A_1,u_1,U_1;\tau),(A_i,u_i,U_i)_2^r\\(B_j,v_j,V_j)_1^s\end{matrix}\right]\right](a,b). \tag{2.22}$$

Replace the $\mathbb{W}$-function with (1.7) and apply the definition of Srivastava-Luo-Raina M-transform defined in (1.11), we obtain:

$$L_9 = \int_0^{\infty}\frac{e^{-ah}}{(h^{\mu}+b^{\mu})^{\lambda}}\left[\frac{1}{2\pi i}\int_{\omega}{}^{\Gamma}\Phi_{r,s}^{m,n}(\kappa,\tau)[z(bh)^{\xi_1}e^{\frac{-cbh}{\kappa}}]^{-\kappa}e^{\kappa y}\kappa^x d\kappa\right]dh. \tag{2.23}$$

Change the integration order and with the help of (1.23) in Lemma 1.1, we get:

$$\frac{1}{2\pi i}\int_{\omega}{}^{\Gamma}\Phi_{r,s}^{m,n}(\kappa,\tau)z^{-\kappa}e^{\kappa y}\kappa^x\frac{b^{-\xi_1\kappa+1-\mu\lambda-1}}{\mu(a+cb)^{-\xi_1\kappa+1}\Gamma(\lambda)}\times\mathbb{H}_{1,2}^{2,1}\left[b[a+cb]\,\middle|\,\begin{matrix}(1,\frac{1}{\mu})\\(-\xi_1\kappa+1,1),(\lambda,\frac{1}{\mu})\end{matrix}\right]d\kappa$$
$$= \frac{b^{-\mu\lambda}}{\mu(a+cb)}\times\frac{1}{2\pi i}\int_{\omega}{}^{\Gamma}\Phi_{r,s}^{m,n}(\kappa,\tau)\left[z\left(\frac{b}{a+cb}\right)^{\xi_1}\right]^{-\kappa}e^{\kappa y}\kappa^x$$
$$\times\frac{1}{2\pi i}\int_{\Delta}\frac{\Gamma(-\xi_1\kappa+1-s)\Gamma\left(\lambda-\frac{s}{\mu}\right)\Gamma\left(\frac{s}{\mu}\right)}{\Gamma(\lambda)}[b(a+cb)]^s ds d\kappa. \tag{2.24}$$

Change the order of integration and after some adjustment of terms, we get the desired outcomes. □

Theorem 2.10. *If $\lambda, a \in \mathbb{C}, \Re(\lambda), c \geq 0, \mu \in \mathbb{N}, b \in \mathbb{R}^+$, and with same conditions given in Equation (1.5), then the result is as follows.*

$$M_{\lambda,\mu}\left[{}^{\gamma}\mathbb{W}_{r,s}^{m,n}\left[zh^{\xi_1}e^{\frac{-ah}{\kappa}};y;x \,\middle|\, \begin{matrix}(A_1,u_1,U_1;\tau),(A_i,u_i,U_i)_2^r\\(B_j,v_j,V_j)_1^s\end{matrix}\right]\right](a,b) = \frac{b^{-\mu\lambda}}{\mu(a+cb)}\frac{1}{2\pi i}\int_{\Delta}B(\lambda-\frac{s}{\mu},\frac{s}{\mu})$$
$$\times [b(a+cb)]^s\left({}^{\gamma}\mathbb{W}_{r+1,s}^{m,n+1}\left[z\left(\frac{b}{a+cb}\right)^{\xi_1};y;x \,\middle|\, \begin{matrix}(A_1,u_1,U_1;\tau),(s,-\xi_1;1),(A_i,u_i,U_i)_2^r\\(B_j,v_j,V_j)_1^s\end{matrix}\right]\right)ds. \tag{2.25}$$

Proof. Theorem 2.10 is proved to be like Theorem 2.9, but with the same conditions. □

Theorem 2.11. *If $\lambda, c, d, \alpha, \beta, \vartheta, \varsigma \in \mathbb{C}, \Re(\lambda), \xi_1, \xi_2 \geq 0, \mu \in \mathbb{N}, b \in \mathbb{R}^+$, $p, q > 0$, and with the same conditions as given in Equations (1.7) and (1.10), then the result is as follows:*

$$M_{\lambda,\mu}\left[{}^{\Gamma}\mathbb{W}_{r,s}^{m,n}\left[zh^{\xi_1}e^{\frac{-ch}{\kappa}};y;x \,\middle|\, \begin{matrix}(A_1,u_1,U_1;\tau),(A_i,u_i,U_i)_2^r\\(B_j,v_j,V_j)_1^s\end{matrix}\right]\times E_{\alpha,\beta,p}^{\vartheta,\varsigma,q}(th^{\xi_2}e^{\frac{-dbh}{n}})\right](a,b)$$
$$= \frac{b^{-\mu\lambda}}{\mu(a+(c+d)b)}\sum_{n=0}^{\infty}\frac{\vartheta_{qn}}{\Gamma(\alpha n+\beta)(\varsigma)_{pn}}\left(t\left(\frac{b}{a+(c+d)b}\right)^{\xi_2}\right)^n$$
$$\times\frac{1}{2\pi i}\int_{\Delta}B\left(\lambda-\frac{s}{\mu},\frac{s}{\mu}\right)[b(a+(c+d)b)]^s$$
$$\times\left({}^{\Gamma}\mathbb{W}_{r+1,s}^{m,n+1}\left[z\left(\frac{b}{a+(c+d)b}\right)^{\xi_1};y;x \,\middle|\, \begin{matrix}(A_1,u_1,U_1;\tau),(s-\xi_2 n,-\xi_1;1),(A_i,u_i,U_i)_2^r\\(B_j,v_j,V_j)_1^s\end{matrix}\right]\right)ds. \tag{2.26}$$

Proof. The LHS of Equation (3.11) is:

$$L_{11} = M_{\lambda,\mu}\left[{}^{\Gamma}\mathbb{W}_{r,s}^{m,n}\left[zh^{\xi_1}e^{\frac{-ah}{\kappa}};y;x \,\middle|\, \begin{matrix}(A_1,u_1,U_1;\tau),(A_i,u_i,U_i)_2^r\\(B_j,v_j,V_j)_1^s\end{matrix}\right]\times E_{\alpha,\beta,p}^{\vartheta,\varsigma,q}(th^{\xi_2}e^{\frac{-bh}{n}})\right](a,b) \tag{2.27}$$

Replace the $\mathbb{W}$-function and Mittag-Leffler function with (1.7) and (1.10) respectively and apply the definition of Srivastava-Luo-Raina M-transform defined in (1.11), we obtain:

$$L_{11} = \int_0^\infty \frac{e^{-ah}}{(h^\mu+b^\mu)^\lambda}\left[\frac{1}{2\pi i}\int_\omega {}^{\Gamma}\Phi_{r,s}^{m,n}(\kappa,\tau)[z(bh)^{\xi_1}e^{\frac{-cbh}{\kappa}}]^{-\kappa}e^{\kappa y}\kappa^x d\kappa\times E_{\alpha,\beta,p}^{\vartheta,\varsigma,q}(th^{\xi_2}e^{\frac{-dh}{n}})^n\right]dh. \tag{2.28}$$

Change the integration order and with the help of (1.23) in Lemma 1.1, we get:

$$\begin{aligned} L_{11} &= \sum_{n=0}^\infty \frac{\vartheta_{qn}t^n}{\Gamma(\alpha n+\beta)(\varsigma)_{pn}}\times\frac{1}{2\pi i}\int_\omega {}^{\Gamma}\Phi_{r,s}^{m,n}(\kappa,\tau)z^{-\kappa}e^{\kappa y}\kappa^x\times\frac{b^{-\xi_1\kappa+\xi_2 n+1-\mu\lambda-1}}{\mu(a+(c+d)b)^{-\xi_1\kappa+\xi_2 n+1}\Gamma(\lambda)}\\ &\times\mathbb{H}_{1,2}^{2,1}\left[b[a+(c+d)b]\,\middle|\,\begin{matrix}(1,\frac{1}{\mu})\\(-\xi_1\kappa+\xi_2 n+1,1),(\lambda,\frac{1}{\mu})\end{matrix}\right]d\kappa\\ &= \frac{b^{-\mu\lambda}}{\mu(a+(c+d)b)}\sum_{n=0}^\infty\frac{\vartheta_{qn}}{\Gamma(\alpha n+\beta)(\varsigma)_{pn}}\left(t\left(\frac{b}{a+(c+d)b}\right)^{\xi_2}\right)^n\\ &\times\frac{1}{2\pi i}\int_\omega {}^{\Gamma}\Phi_{r,s}^{m,n}(\kappa,\tau)\left[z\left(\frac{b}{a+(c+d)b}\right)^{\xi_1}\right]^{-\kappa}e^{\kappa y}\kappa^x\\ &\times\frac{1}{2\pi i}\int_\Delta\frac{\Gamma(-\xi_1\kappa+\xi_2 n+1-s)\Gamma\left(\lambda-\frac{s}{\mu}\right)\Gamma\left(\frac{s}{\mu}\right)}{\Gamma(\lambda)}[b(a+(c+d)b)]^s ds d\kappa. \end{aligned} \tag{2.29}$$

Change the order of integration, and after some adjustment of terms, we get the desired outcomes. □

Theorem 2.12. *If* $\lambda,c,d,\alpha,\beta,\vartheta,\varsigma\in\mathbb{C},\Re(\lambda),\xi_1,\xi_2\geq 0,\mu\in\mathbb{N},b\in\mathbb{R}^+$, $p,q>0$, *and with the same conditions as given in Equations* (1.7) *and* (1.10)*, then the result is as follows:*

$$\begin{aligned} &M_{\lambda,\mu}\left[{}^{\gamma}\mathbb{W}_{r,s}^{m,n}\left[zh^{\xi_1}e^{\frac{-ch}{\kappa}};y;x\,\middle|\,\begin{matrix}(A_1,u_1,U_1;\tau),(A_i,u_i,U_i)_2^r\\(B_j,v_j,V_j)_1^s\end{matrix}\right]\times E_{\alpha,\beta,p}^{\vartheta,\varsigma,q}(th^{\xi_2}e^{\frac{-dbh}{n}})\right](a,b)\\ &=\frac{b^{-\mu\lambda}}{\mu(a+(c+d)b)}\sum_{n=0}^\infty\frac{\vartheta_{qn}}{\Gamma(\alpha n+\beta)(\varsigma)_{pn}}\left(t\left(\frac{b}{a+(c+d)b}\right)^{\xi_2}\right)^n\\ &\times\frac{1}{2\pi i}\int_\Delta B\left(\lambda-\frac{s}{\mu},\frac{s}{\mu}\right)[b(a+(c+d)b)]^s\\ &\times\left({}^{\gamma}\mathbb{W}_{r+1,s}^{m,n+1}\left[z\left(\frac{b}{a+(c+d)b}\right)^{\xi_1};y;x\,\middle|\,\begin{matrix}(A_1,u_1,U_1;\tau),(s-\xi_2 n,-\xi_1;1),(A_i,u_i,U_i)_2^r\\(B_j,v_j,V_j)_1^s\end{matrix}\right]\right)ds. \end{aligned} \tag{2.30}$$

Proof. Theorem 2.12 is proved to be like Theorem 2.11, but with the same conditions. □

3 Special Cases

In this section, as a specific application of Theorem 2.11 and Theorem 2.12, we derive the Srivastava-Luo-Raina M-transform for the product of Mittag-Leffler function with the incomplete $\mathbb{Y}$-function, the incomplete I-function and the incomplete H-function. By specifying the parameters of certain features, we obtain the following special cases which illustrate the application of the fundamental results.

(i) Incomplete I-Function: If we substitute $x = y = 0$ and $\frac{1}{z}$ in place of z in Equation (1.5), they get reduced to the incomplete $\mathbb{I}$-functions [12]:

$$
{}^{\gamma}\mathbb{W}_{r,s}^{m,n}\left[\frac{1}{z};0;0 \;\middle|\; \begin{matrix} (A_1,u_1,U_1;\tau),(A_i,u_i,U_i)_2^r \\ (B_j,v_j,V_j)_1^s \end{matrix}\right] = {}^{\gamma}\mathbb{I}_{r,s}^{m,n}\left[z \;\middle|\; \begin{matrix} (A_1,u_1,U_1;\tau),(A_i,u_i,U_i)_2^r \\ (B_j,v_j,V_j)_1^s \end{matrix}\right]
$$
$$
= \frac{1}{2\pi i}\int_{\omega} {}^{\gamma}\Phi_{r,s}^{m,n}(\kappa,\tau) z^{\kappa}\, d\kappa, \tag{3.1}
$$

where

$$
{}^{\gamma}\Phi_{r,s}^{m,n}(\kappa,\tau) = \frac{\{\gamma(1-u_1+U_1\kappa,\tau)\}^{A_1}\prod_{j=1}^{m}\{\Gamma(v_j-V_j\kappa)\}^{B_j}\prod_{i=2}^{n}\{\Gamma(1-u_i+U_i\kappa)\}^{A_i}}{\prod_{j=m+1}^{s}\{\Gamma(1-v_j+V_j\kappa)\}^{B_j}\prod_{i=n+1}^{r}\{\Gamma(u_i-U_i\kappa)\}^{A_i}}. \tag{3.2}
$$

Applying the relationships outlined in Theorems (2.11) and (2.12), we obtain the following corollaries.

Corollary 3.1. *If $\lambda, c, d, \alpha, \beta, \vartheta, \varsigma \in \mathbb{C}, \Re(\lambda), \xi_1, \xi_2 \geq 0, \mu \in \mathbb{N}, b \in \mathbb{R}^+$, $p, q > 0$, and with the same conditions as given in Equations (1.10) and (3.1) , then the result is as follows:*

$$
M_{\lambda,\mu}\left[{}^{\Gamma}\mathbb{I}_{r,s}^{m,n}\left[zh^{\xi_1}e^{\frac{-ch}{\kappa}} \;\middle|\; \begin{matrix} (A_1,u_1,U_1;\tau),(A_i,u_i,U_i)_2^r \\ (B_j,v_j,V_j)_1^s \end{matrix}\right] \times E_{\alpha,\beta,p}^{\vartheta,\varsigma,q}(th^{\xi_2}e^{\frac{-dbh}{n}})\right](a,b)
$$
$$
= \frac{b^{-\mu\lambda}}{\mu(a+(c+d)b)}\sum_{n=0}^{\infty}\frac{\vartheta_{qn}}{\Gamma(\alpha n+\beta)(\varsigma)_{pn}}\left(t\left(\frac{b}{a+(c+d)b}\right)^{\xi_2}\right)^n
$$
$$
\times \frac{1}{2\pi i}\int_{\Delta} B\left(\lambda-\frac{s}{\mu},\frac{s}{\mu}\right)[b(a+(c+d)b)]^s
$$
$$
\times\left({}^{\Gamma}\mathbb{I}_{r+1,s}^{m,n+1}\left[z\left(\frac{b}{a+(c+d)b}\right)^{\xi_1} \;\middle|\; \begin{matrix} (A_1,u_1,U_1;\tau),(s-\xi_2 n,\xi_1;1),(A_i,u_i,U_i)_2^r \\ (B_j,v_j,V_j)_1^s \end{matrix}\right]\right) ds. \tag{3.3}
$$

Corollary 3.2. *If $\lambda, c, d, \alpha, \beta, \vartheta, \varsigma \in \mathbb{C}, \Re(\lambda), \xi_1, \xi_2 \geq 0, \mu \in \mathbb{N}, b \in \mathbb{R}^+$, $p, q > 0$, and with the same conditions as given in Equations (1.10) and (3.1), then the result is as follows:*

$$
M_{\lambda,\mu}\left[{}^{\gamma}\mathbb{I}_{r,s}^{m,n}\left[zh^{\xi_1}e^{\frac{-ch}{\kappa}} \;\middle|\; \begin{matrix} (A_1,u_1,U_1;\tau),(A_i,u_i,U_i)_2^r \\ (B_j,v_j,V_j)_1^s \end{matrix}\right] \times E_{\alpha,\beta,p}^{\vartheta,\varsigma,q}(th^{\xi_2}e^{\frac{-dbh}{n}})\right](a,b)
$$
$$
= \frac{b^{-\mu\lambda}}{\mu(a+(c+d)b)}\sum_{n=0}^{\infty}\frac{\vartheta_{qn}}{\Gamma(\alpha n+\beta)(\varsigma)_{pn}}\left(t\left(\frac{b}{a+(c+d)b}\right)^{\xi_2}\right)^n
$$
$$
\times \frac{1}{2\pi i}\int_{\Delta} B\left(\lambda-\frac{s}{\mu},\frac{s}{\mu}\right)[b(a+(c+d)b)]^s
$$
$$
\times\left({}^{\gamma}\mathbb{I}_{r+1,s}^{m,n+1}\left[z\left(\frac{b}{a+(c+d)b}\right)^{\xi_1} \;\middle|\; \begin{matrix} (A_1,u_1,U_1;\tau),(s-\xi_2 n,\xi_1;1),(A_i,u_i,U_i)_2^r \\ (B_j,v_j,V_j)_1^s \end{matrix}\right]\right) ds. \tag{3.4}
$$

(ii) Incomplete $\mathbb{Y}$-Function: On setting $A_1 = A_2 = \cdots = A_r = 1, B_1 = B_2 = \cdots = B_s = 1$ in Equation (1.5), $\mathbb{W}$-function reduce to the $\mathbb{Y}$-function proposed by Yang et al. [23] and we reach the following conclusion:

$$
{}^{\gamma}\mathbb{W}_{r,s}^{m,n}\left[z;y;x \;\middle|\; \begin{matrix} (1,u_1,U_1;\tau),(1,u_i,U_i)_2^r \\ (1,v_j,V_j)_1^s \end{matrix}\right] = {}^{\gamma}\mathbb{Y}_{r,s}^{m,n}\left[z;y;x; \;\middle|\; \begin{matrix} (u_1,U_1;\tau),(U_i,U_i)_2^r \\ (v_j,V_j)_1^s \end{matrix}\right]
$$
$$
= \frac{1}{2\pi i}\int_{\omega} {}^{\gamma}\Phi_{r,s}^{m,n}(\kappa,\tau) z^{-\kappa}\, e^{\kappa y}\kappa^x\, d\kappa, \tag{3.5}
$$

where

$$ {}^{\gamma}\Phi_{r,s}^{m,n}(\kappa,\tau) = \frac{\{\gamma(1-u_1+U_1\kappa,\tau)\} \prod_{j=1}^{m}\{\Gamma(v_j-V_j\kappa)\}\prod_{i=2}^{n}\{\Gamma(1-u_i+U_i\kappa)\}}{\prod_{j=m+1}^{s}\{\Gamma(1-v_j+V_j\kappa)\} \prod_{i=n+1}^{r}\{\Gamma(u_i-U_i\kappa)\}}. \tag{3.6} $$

Applying the relationships outlined in Theorems (2.11) and (2.12), we obtain the following corollaries.

Corollary 3.3. *If* $\lambda, c, d, \alpha, \beta, \vartheta, \varsigma \in \mathbb{C}, \Re(\lambda), \xi_1, \xi_2 \geq 0, \mu \in \mathbb{N}, b \in \mathbb{R}^+$, $p, q > 0$, *and with the same conditions as given in Equations* (1.10) *and* (3.5)*, then the result is as follows:*

$$ \begin{aligned} & M_{\lambda,\mu}\left[{}^{\Gamma}\mathbb{Y}_{r,s}^{m,n}\left[zh^{\xi_1}e^{\frac{-ch}{\kappa}};y;x \,\middle|\, \begin{matrix}(u_1,U_1;\tau),(u_i,U_i)_2^r\\(v_j,V_j)_1^s\end{matrix}\right] \times E_{\alpha,\beta,p}^{\vartheta,\varsigma,q}(th^{\xi_2}e^{\frac{-dbh}{n}})\right](a,b) \\ & = \frac{b^{-\mu\lambda}}{\mu(a+(c+d)b)}\sum_{n=0}^{\infty}\frac{\vartheta_{qn}}{\Gamma(\alpha n+\beta)(\varsigma)_{pn}}\left(t\left(\frac{b}{a+(c+d)b}\right)^{\xi_2}\right)^n \\ & \times \frac{1}{2\pi i}\int_{\Delta} B\left(\lambda-\frac{s}{\mu},\frac{s}{\mu}\right)[b(a+(c+d)b)]^s \\ & \times \left({}^{\Gamma}\mathbb{Y}_{r+1,s}^{m,n+1}\left[z\left(\frac{b}{a+(c+d)b}\right)^{\xi_1};y;x \,\middle|\, \begin{matrix}(u_1,U_1;\tau),(s-\xi_2 n,-\xi_1;1),(u_i,U_i)_2^r\\(v_j,V_j)_1^s\end{matrix}\right]\right)ds. \end{aligned} \tag{3.7} $$

Corollary 3.4. *If* $\lambda, c, d, \alpha, \beta, \vartheta, \varsigma \in \mathbb{C}, \Re(\lambda), \xi_1, \xi_2 \geq 0, \mu \in \mathbb{N}, b \in \mathbb{R}^+$, $p, q > 0$, *and with the same conditions as given in Equations* (1.10) *and* (3.5)*, then the result is as follows:*

$$ \begin{aligned} & M_{\lambda,\mu}\left[{}^{\gamma}\mathbb{Y}_{r,s}^{m,n}\left[zh^{\xi_1}e^{\frac{-ch}{\kappa}};y;x \,\middle|\, \begin{matrix}(u_1,U_1;\tau),(u_i,U_i)_2^r\\(v_j,V_j)_1^s\end{matrix}\right] \times E_{\alpha,\beta,p}^{\vartheta,\varsigma,q}(th^{\xi_2}e^{\frac{-dbh}{n}})\right](a,b) \\ & = \frac{b^{-\mu\lambda}}{\mu(a+(c+d)b)}\sum_{n=0}^{\infty}\frac{\vartheta_{qn}}{\Gamma(\alpha n+\beta)(\varsigma)_{pn}}\left(t\left(\frac{b}{a+(c+d)b}\right)^{\xi_2}\right)^n \\ & \times \frac{1}{2\pi i}\int_{\Delta} B\left(\lambda-\frac{s}{\mu},\frac{s}{\mu}\right)[b(a+(c+d)b)]^s \\ & \times \left({}^{\gamma}\mathbb{Y}_{r+1,s}^{m,n+1}\left[z\left(\frac{b}{a+(c+d)b}\right)^{\xi_1};y;x \,\middle|\, \begin{matrix}(u_1,U_1;\tau),(s-\xi_2 n,-\xi_1;1),(u_i,U_i)_2^r\\(v_j,V_j)_1^s\end{matrix}\right]\right)ds. \end{aligned} \tag{3.8} $$

(iii) Incomplete H-Function: If we replace $x = y = 0$ and $\frac{1}{z}$ in place of z in (3.5), it reduces to the incomplete H-function [2], as follows:

$$ \begin{aligned} {}^{\gamma}\mathbb{Y}_{r,s}^{m.n}\left[\frac{1}{z};0;0 \,\middle|\, \begin{matrix}(u_1,U_1;\tau),(u_i,U_i)_2^r\\(v_j,V_j)_1^s\end{matrix}\right] &= \gamma_{r,s}^{m,n}\left[z \,\middle|\, \begin{matrix}(u_1,U_1;\tau),(u_i,U_i)_2^r\\(v_j,V_j)_1^s\end{matrix}\right] \\ &= \frac{1}{2\pi i}\int_{\omega} {}^{\gamma}\Phi_{r,s}^{m,n}(\kappa,\tau)z^{\kappa}\,d\kappa, \end{aligned} \tag{3.9} $$

where

$$ {}^{\gamma}\Phi_{r,s}^{m,n}(\kappa,\tau) = \frac{\{\gamma(1-u_1+U_1\kappa,\tau)\} \prod_{j=1}^{m}\{\Gamma(v_j-V_j\kappa)\}\prod_{i=2}^{n}\{\Gamma(1-u_i+U_i\kappa)\}}{\prod_{j=m+1}^{s}\{\Gamma(1-v_j+V_j\kappa)\} \prod_{i=n+1}^{r}\{\Gamma(u_i-U_i\kappa)\}}. \tag{3.10} $$

Applying the relationships outlined in Theorems (2.11) and (2.12), we obtain the following corollaries.

Corollary 3.5. *If* $\lambda, c, d, \alpha, \beta, \vartheta, \varsigma \in \mathbb{C}, \Re(\lambda), \xi_1, \xi_2 \geq 0, \mu \in \mathbb{N}, b \in \mathbb{R}^+$, $p, q > 0$, *and with the* *same conditions as given in Equations* (1.10) *and* (3.9)*, then the result is as follows:*

$$M_{\lambda,\mu}\left[\Gamma^{m,n}_{r,s}\left[zh^{\xi_1}e^{\frac{-ch}{\kappa}}\;\middle|\;\begin{matrix}(u_1,U_1;\tau),(u_i,U_i)_2^r\\(v_j,V_j)_1^s\end{matrix}\right]\times E^{\vartheta,\varsigma,q}_{\alpha,\beta,p}(th^{\xi_2}e^{\frac{-dbh}{n}})\right](a,b)$$
$$=\frac{b^{-\mu\lambda}}{\mu(a+(c+d)b)}\sum_{n=0}^{\infty}\frac{\vartheta_{qn}}{\Gamma(\alpha n+\beta)(\varsigma)_{pn}}\left(t\left(\frac{b}{a+(c+d)b}\right)^{\xi_2}\right)^n$$
$$\times\frac{1}{2\pi i}\int_{\Delta}B\left(\lambda-\frac{s}{\mu},\frac{s}{\mu}\right)[b(a+(c+d)b)]^s$$
$$\times\left(\Gamma^{m,n+1}_{r+1,s}\left[z\left(\frac{b}{a+(c+d)b}\right)^{\xi_1}\;\middle|\;\begin{matrix}(u_1,U_1;\tau),(s-\xi_2 n,\xi_1;1),(u_i,U_i)_2^r\\(v_j,V_j)_1^s\end{matrix}\right]\right)ds. \tag{3.11}$$

Corollary 3.6. *If* $\lambda,c,d,\alpha,\beta,\vartheta,\varsigma\in\mathbb{C},\Re(\lambda),\xi_1,\xi_2\geq 0,\mu\in\mathbb{N},b\in\mathbb{R}^+$*,* $p,q>0$*, and with the same conditions as given in Equations* (1.10) *and* (3.9)*, then the result is as follows:*

$$M_{\lambda,\mu}\left[\gamma^{m,n}_{r,s}\left[zh^{\xi_1}e^{\frac{-ch}{\kappa}}\;\middle|\;\begin{matrix}(u_1,U_1;\tau),(u_i,U_i)_2^r\\(v_j,V_j)_1^s\end{matrix}\right]\times E^{\vartheta,\varsigma,q}_{\alpha,\beta,p}(th^{\xi_2}e^{\frac{-dbh}{n}})\right](a,b)$$
$$=\frac{b^{-\mu\lambda}}{\mu(a+(c+d)b)}\sum_{n=0}^{\infty}\frac{\vartheta_{qn}}{\Gamma(\alpha n+\beta)(\varsigma)_{pn}}\left(t\left(\frac{b}{a+(c+d)b}\right)^{\xi_2}\right)^n$$
$$\times\frac{1}{2\pi i}\int_{\Delta}B\left(\lambda-\frac{s}{\mu},\frac{s}{\mu}\right)[b(a+(c+d)b)]^s$$
$$\times\left(\gamma^{m,n+1}_{r+1,s}\left[z\left(\frac{b}{a+(c+d)b}\right)^{\xi_1}\;\middle|\;\begin{matrix}(u_1,U_1;\tau),(s-\xi_2 n,\xi_1;1),(u_i,U_i)_2^r\\(v_j,V_j)_1^s\end{matrix}\right]\right)ds. \tag{3.12}$$

3.1 Remark

The following observations illustrate specific cases of the stated theorems by applying appropriate parameter settings, yielding well-known Mittag-Leffler functions and related results.

- Setting $\varsigma=p=1$ in Theorems 2.3, 2.4, 2.7, 2.8, 2.11, and 2.12, then we get a Mittag-Leffler function of 4 parameters defined by Shukla and Prajapati [18].
- Setting $\varsigma=p=q=1$ in Theorems 2.3, 2.4, 2.7, 2.8, 2.11, and 2.12 yields the Mittag-Leffler function of three parameters defined by Prabhakar [15].
- Setting $\varsigma=\vartheta=p=q=1$ in Theorems 2.3, 2.4, 2.7, 2.8, 2.11, and 2.12 yields the Mittag-Leffler function of two parameters defined by Wiman [24]. Moreover if $\beta=1$, we get Mittag-Leffler $E_\alpha(t)$ in theorems.
- By setting $\xi_1,\xi_2=0$ in Theorems 2.11 and 2.12 then the result is same as that of Theorems 2.7 and 2.8 respectively.
- By setting $c=d=0$ in Theorems 2.11 and 2.12 then the result is same as that of Theorems 2.3 and 2.4 respectively.

4 Conclusion

In this article, we successfully derived the Srivastava-Luo-Raina M-transform for the incomplete $\mathbb{W}$-function, representing a significant extension of the Y-function previously investigated by Yang et al. [23]. Moreover, we explored and derived related transforms for other notable functions, including the $\mathbb{Y}$-function, I-function, and Fox H-function. These results broaden the scope of the Srivastava-Luo-Raina M-transform and highlight its versatility and applicability across various special functions.

The framework established in this study paves the way for further exploration of connections between integral transforms and incomplete special functions. Such advancements can inspire new applications in diverse fields, including mathematical physics, engineering, and applied sciences. By providing a unified approach to these transformations, this work contributes significantly to the ongoing development of mathematical tools for solving complex problems in both theoretical and practical domains.

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