

Bridging Infra-Topology and Fuzzy Graphs: A Novel Mathematical Framework

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Abstract *This paper introduces fuzzy infra topology via fuzzy vertex infra topological spaces on simple connected fuzzy graphs. The structural interrelations between fuzzy graphs and fuzzy infra topological spaces are systematically examined. Several theorems are established and supported with illustrative examples, addressing fundamental concepts such as homeomorphism, isomorphism, weak isomorphism, co-weak isomorphism, and self-complementary fuzzy graphs. Furthermore, the interior and closure properties within this framework are analyzed in detail, thereby extending the theoretical foundation of fuzzy infra topological spaces.*

1 Introduction

Fuzzy set theory, originally proposed by Zadeh [15] in 1965, provides a framework in which elements may belong to a set with varying degrees of membership rather than in a strictly binary manner. This paradigm shift has significantly influenced many branches of mathematics and their applications. To address situations involving vagueness and uncertainty, Rosenfeld [12] extended classical graph theory in 1975 by introducing the notion of fuzzy graphs, which integrate the principles of fuzzy set theory into graph-theoretical structures.

Building on this foundation, Chang [7] introduced the concept of fuzzy topological spaces (FTS), which was later refined and studied in greater depth by Michalek and others. Over time, fuzzy topology attracted considerable attention due to its potential for modeling uncertainty in topological structures and its wide range of applications. The notion was explicitly recognized by Al-Odhari in Infra topological spaces [3]. However, Wittczak conducted significant practical research on infra-topological spaces. In 2022, Ameen and other authors introduced the concept of infra fuzzy topology, which is a subset family of fuzzy topology by omitting the condition of closedness under arbitrary union.

The interplay between fuzzy graphs and fuzzy topology has led to several significant contributions. For instance, Rosenfeld [12] studied topological structures induced by graphs, while subsequent works such as [2] investigated topological spaces associated with simple graphs. Further, [9] examined the properties of fuzzy graph complements, and Gholap and Nikumbh [6] analyzed fuzzy topological spaces in relation to fuzzy graphs. Collectively, these studies underscore the increasing importance of integrating graph-theoretical and topological perspectives within fuzzy frameworks.

This article presents new research on fuzzy graphs within the framework of fuzzy infra topo-

logical spaces. Specifically investigates fuzzy vertex infra topology graphs on simple connected fuzzy graphs, highlighting their structural properties and significance. Furthermore, we explore key theorems and provide related examples to illustrate these concepts.

2 Preliminaries

Definition 2.1. A graphs $G = (V, E)$ consists of:

- (i) V is a set of vertices (or nodes).
- (ii) E is a set of edges, which are pairs of elements from V representing connections between vertices.

Definition 2.2. A fuzzy fraph $G = (\sigma, \mu)$ is a combination of function $\sigma : V \rightarrow [0, 1]$ with $\mu : V \times V \rightarrow [0, 1]$ that exist for every $u, v \in V$, where V is the vertex set, $\mu(u, v) \leq \min(\sigma(u), \sigma(v)), \forall u, v \in V$.

Definition 2.3. For fuzzy topology, let X be the fuzzy subset $X = \{\mu : \mu \text{ in } X\}$ that fulfills the ensuring axioms.

- (i) $0, 1 \in X$
- (ii) $\alpha_1, \alpha_2 \in X, \alpha_1 \wedge \alpha_2 \in X$.
- (iii) If $\{\alpha_i : i \in J\} \subset X, j$ is known to be an index set, then $U_j \alpha_i \in X$

Definition 2.4. Let $\delta(z)$ be a subset of X , in the event that z has an fuzzy infra topology, then

- (i) $0, 1 \in X$
- (ii) $\alpha_1 \wedge \alpha_2 \in X$ whenever $\alpha_1, \alpha_2 \in X$.

Infra fuzzy topological spaces are represented by the pair (Z, τ) and $\text{IFT}(Z)$ is the set of IFT on Z . Infra fuzzy open subset of Z are defined as members of τ , and τ complements are known as infra-fuzzy closed sets. The members of τ^c are also known as IF-closed members.

Definition 2.5. A basic connected graph $G = (V, E)$ has the following properties: G is simple, which means that there are no self-loops $\forall u, v \in V$, with at most one edge connecting u and v . Because G is connected, there is a path $P = (u, v)$ in G for every pair of vertices $u, v \in V$.

Definition 2.6. In a complete fuzzy graph that is a fuzzy extension of a complete graph. When every pair of unique vertices is connected by an edge and both vertices and edges possess membership values in the interval $[0, 1]$.

Definition 2.7. A complete bipartite graph $G = (V, E)$ is a bipartite graph where: The vertex set V is divided into two disjoint sets: $V = V_1 \cup V_2, V_1 \cap V_2 = \emptyset$. The edge set E contains all possible edges between V_1 and $V_2 : E\{(u, v) | u \in V_1, v \in V_2\}$ there are no edges within V_1 or within V_2 .

Definition 2.8. The numbers m and n represent the number of vertices in V_1 and V_2

- (i) Consider a pair fuzzy graphs, $M = (P, \delta, \gamma)$ and $M' = (P', \delta', \gamma')$. If $\delta(g) \leq \delta'(f(g)) \forall g \in P$ and $\gamma(g, h) \leq \gamma'(f(g), f(h)) \forall g, h \in P$, then a map $f : M \rightarrow M'$ is a homomorphism from M to M' .
- (ii) If $\delta(g) = \delta'(f(g)) \forall g \in P$ along with $\gamma(g, h) = \gamma'(f(g), f(h)) \forall g, h \in P$, then a bijective map $f : M \rightarrow M'$ is an isomorphism from M to M' .
- (iii) If $\delta(g) = \delta'(f(g))$ every an in P , then a bijective homomorphism $f : M \rightarrow M'$ is a weak isomorphism from M to M' .
- (iv) If $\gamma(g, h) = \gamma'(f(g), f(h))$ for all $g, h \in P$, then a bijective homomorphism $f : M \rightarrow M'$ is a co-weak isomorphism from M to M' .

Definition 2.9. A graph that is isomorphic to its complement is said to be self-complementary. The 4-vertex path graph and the 5-vertex cycle graph are the most basic non-trivial self-complementary graphs.

Definition 2.10. If (X, τ_{fix}) be a FITS then σ be fuzzy set if $Ijcl\sigma = \wedge\{\psi : \psi \geq \sigma, \psi \in FIjC(X)\}$ is called a fuzzy infra j closure.

If (X, τ_{fix}) be FITS and σ be fuzzy set if $Ijint\sigma = \vee\{\psi : \psi \leq \sigma, \psi \in FIjO(X)\}$ called a fuzzy infra j interior.

3 Fuzzy Vertex Infra Topological Graphs

Definition 3.1. A fuzzy infra topological space is defined as a pair (X, τ_{fix}) , where X denotes the set of vertices and edges in fuzzy graph, and τ_{fix} is a collection of fuzzy subsets of X that satisfies the axioms of an infra topology.

Example 3.2. Let $K = \{E_1, E_2, E_3, E_4\}$ represent the cities.

Membership values of vertices:

$$\delta(E_1) = 0.9, \quad \delta(E_2) = 0.7, \quad \delta(E_3) = 0.5, \quad \delta(E_4) = 0.4$$

Membership values of edges:

$$\begin{aligned} \gamma(E_1, E_2) = 0.8, \quad \gamma(E_2, E_3) &= 0.6, \quad \gamma(E_1, E_3) = 0.5, \quad \gamma(E_3, E_4) &= 0.4, \\ \gamma(E_1, E_4) = 0.2 \end{aligned}$$

Define infra set U (membership value 0.5):

$$U = \{E_1, E_2, E_3, (E_1, E_2), (E_1, E_3), (E_2, E_3)\}$$

U represents the *core region* of well-connected cities.

Define infra set V (membership value 0.2 to 0.5):

$$V = \{E_2, E_4, (E_2, E_3), (E_3, E_4), (E_1, E_4)\}$$

V represents the *peripheral region* of less-connected cities.

Here, E_3 acts as a boundary city between the well-connected cities (E_1, E_2) and the less-connected city E_4 . Analyzing E_3 's position could help in infrastructure planning to improve connectivity between the core region and the peripheral region.

Definition 3.3. Consider a simple connected fuzzy graph $M = (P, \delta, \gamma)$ with its fuzzy vertex infra set $P(M)$. We define an adjacency relation R on $P(M)$ by declaring that $(k, \delta(k)), (l, \delta(l)) \in R$, Whenever $\gamma(k, l) > 0$, for all $(k, \delta(k)), (l, \delta(l)) \in P(M)$. For each vertex $(k, \delta(k)) \in P(M)$, define

$$R[k] = \{(l, \delta(l)) \in P(M) : \gamma(k, l) > 0\}.$$

The collection $S = \{R[k] : (k, \delta(k)) \in P(M)\}$ provides a subbasis for a topology on $P(M)$. Let η denote the family of finite intersections of members of S . Then η forms a basis. By considering all possible unions of elements of η , we obtained a infra topology on $P(M)$. The infra topology is known as the fuzzy vertex infra topology generated by the fuzzy graph M . the structure $(P(M), \tau_{fix})$ is recognized as the fuzzy vertex topological spaces determined by M .

Example 3.4. Consider a fuzzy graph M_1 whose vertex infra set is,

$$P(M_1) = \{(I, 0.3), (J, 0.6), (K, 0.5), (L, 0.8)\}$$

Then,

$$\begin{aligned} R[I] &= \{(J, 0.6)\}, \quad R[J] = \{(I, 0.3), (K, 0.5), (L, 0.8)\}, \\ R[K] &= \{(J, 0.6)\}, \quad R[L] = \{(J, 0.6)\} \end{aligned}$$

$$S_1 = \{R[I], R[J], R[K], R[L]\} = \{\{(J, 0.6)\}, \{(I, 0.3), (K, 0.5), (L, 0.8)\}\}$$

And:

$$\eta_1 = \{\emptyset, \{(J, 0.6)\}, \{(I, 0.3), (K, 0.5), (L, 0.8)\}\}$$

The fuzzy infra topology τ_{fix} is the collection of all unions of sets in η_1 :

$$\tau_{fix} = \{\emptyset, \{(J, 0.6)\}, \{(I, 0.3), (K, 0.5), (L, 0.8)\}, P(M_1)\}$$

is generated by the fuzzy vertex infra topology of M_1 .

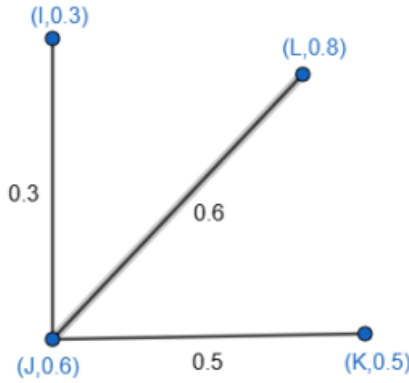


Figure 1. A Simple connected fuzzy graph M_1

Description Figure 1: A simple connected fuzzy graph M_1 with vertex set $P(M_1) = \{(I, 0.3), (J, 0.6), (K, 0.5), (L, 0.8)\}$. Each vertex is associated with a membership value in $[0, 1]$, and edges are assigned fuzzy weights $\mu(I, J) = 0.3, \mu(J, K) = 0.5,$ and $\mu(J, L) = 0.6$. Vertex J acts as the central connector, ensuring connectivity of the graph. The corresponding fuzzy infra topology $\tau_{fix} = \{0, \{(J, 0.6)\}, \{(I, 0.3), (K, 0.5), (L, 0.8)\}, P(M_1)\}$ is generated from its neighborhood structure.

Theorem 3.5. If N is a simple connected spanning fuzzy subgraph of fuzzy graph M . Then the fuzzy vertex infra topology associated with N is finer than the fuzzy vertex infra topology associated with M .

Proof:

Now, consider the simple connected spanning fuzzy subgraph $N = (k, \tau, l)$ of the fuzzy graph $M = (P, \delta, \gamma)$, where $k = l$. Consider the sub-basis of the vertex infra topology w_1 on N , along with fuzzy vertex infra topology w_2 on M , these are given by, $S_1 = \{R[k] : (k : \gamma(k)) \in k\}$ and $S_2 = \{R[K] : (k, \delta(k)) \in l\}$.

Since N is a spanning subgraph of M , every element of S_1 is also an element of S_2 . Hence $S_1 \subseteq S_2$, which implies that $\tau_{fix2} \subseteq \tau_{fix1}$.

Therefore, the fuzzy vertex infra topology associated with N is finer than that associated with M .

Theorem 3.6. Let $M_1 = (P_1, \delta_1, \gamma_1)$ and $M_2 = (P_2, \delta_2, \gamma_2)$ be fuzzy graphs. If M_1 and M_2 are isomorphic, then the fuzzy vertex infra topologies τ_{fix1} and τ_{fix2} generated by M_1 and M_2 , respectively are homeomorphic.

Proof:

Suppose $M_1 = (P_1, \delta_1, \gamma_1)$ along with $M_2 = (P_2, \delta_2, \gamma_2)$ are isomorphic fuzzy graph. Then there exists a bijective mapping $f : P_1 \rightarrow P_2$ such that $\delta_1(g) = \delta_2(f(g)) \forall g \in P_1$, and $\gamma_1(g, h) = \gamma_2(f(g), f(h)) \forall g, h \in P_1$. By definition, the subbasis of the fuzzy vertex infra topology τ_{fix1} on M_1 is $S_1 = \{R[g] : (g, \delta_1(g)) \in P_1\}$, while the subbasis of fuzzy vertex infra topology τ_{fix2} on M_2 is $S_2 = \{R[f(g)] : (f(g), \delta_2(f(g))) \in P_2\}$. Since the isomorphism f preserves vertex membership and adjacency, the mapping between S_1 and S_2 is bijective. Hence, the subbases S_1 and S_2 generate topologies τ_{fix1} and τ_{fix2} that are homeomorphic.

Example 3.7. Both M_1 and M_2 are fuzzy graphs that are isomorphic. Let τ_{fix1} represent the fuzzy vertex infra topology produced by M_1 , and τ_{fix2} represent the fuzzy vertex infra topology produced by M_2 . After that, we have:

$$P(M_1) = \{(I, 0.7), (J, 0.6), (K, 0.8), (L, 0.3)\}$$

$$P(M_2) = \{(a, 0.7), (b, 0.6), (c, 0.8), (d, 0.3)\}$$

$$\begin{aligned} \tau_{fix1} &= \{\emptyset, \{(J, 0.6)\}, \{(L, 0.3)\}, \{(J, 0.6), (L, 0.3)\}, \\ &\quad \{(I, 0.7), (K, 0.8)\}, \{(I, 0.7), (K, 0.8), (L, 0.3)\}, \\ &\quad \{(I, 0.7), (J, 0.6), (K, 0.8)\}P(M_1)\} \\ \tau_{fix2} &= \{\emptyset, \{(b, 0.6)\}, \{(d, 0.3)\}, \{(b, 0.6), (d, 0.3)\}, \\ &\quad \{(a, 0.7), (c, 0.8)\}, \{(a, 0.7), (c, 0.8), (d, 0.3)\}, \\ &\quad \{(a, 0.7), (b, 0.6), (c, 0.8)\}P(M_2)\} \end{aligned}$$

Complements of τ_{fix1} and τ_{fix2}

$$\begin{aligned} \tau_{fix1} &= \{\emptyset, \{(J, 0.6)\}, \{(K, 0.3)\}, \{(J, 0.6), (K, 0.3)\}, \\ &\quad \{(I, 0.7), (K, 0.8)\}, \{(I, 0.7), (K, 0.8), (L, 0.3)\}, \\ &\quad \{(I, 0.7), (J, 0.6), (K, 0.8)\}, \mathcal{P}(M_1)\} \\ \tau_{fix2} &= \{\emptyset, \{(b, 0.6)\}, \{(d, 0.3)\}, \{(b, 0.6), (d, 0.3)\}, \\ &\quad \{(a, 0.7), (c, 0.8)\}, \{(a, 0.7), (c, 0.8), (d, 0.3)\}, \\ &\quad \{(a, 0.7), (b, 0.6), (c, 0.8)\}, \mathcal{P}(M_2)\} \end{aligned}$$

Therefore, $\tau_{fix1} = \tau_{fix2}$.

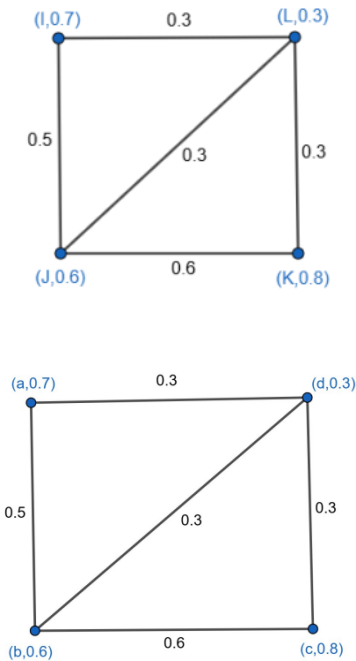


Figure 2. Isomorphic fuzzy graph M_1 and M_2

Description Figure 2: Figure 1 and 2 show two fuzzy graphs M_1 and M_2 with the same structure. Each vertex is assigned a membership value in $[0, 1]$, and the edges carry fuzzy membership values that do not exceed the minimum of their end vertices. The mapping $a \leftrightarrow I, b \leftrightarrow J, c \leftrightarrow K$ and $d \leftrightarrow L$ preserves these values, proving that M_1 and M_2 are isomorphic fuzzy graphs inducing the same fuzzy vertex infra-topology.

Corollary 3.8. Fuzzy vertex infra topologies τ_{fix1} and τ_{fix2} , produced by fuzzy graph $M_1 = (P_1, \delta_1, \gamma_1)$ and $M_2 = (P_2, \delta_2, \gamma_2)$ respectively, are homomorphic, if they are weak isomorphic.

Remark 3.9. The fuzzy vertex infra topologies τ_{fix1} and τ_{fix2} produced by two fuzzy graph, $M_1 = (P_1, \delta_1, \gamma_1)$ and $M_2 = (P_2, \delta_2, \gamma_2)$ might not be homomorphic if they are co-weak isomorphic.

Example 3.10. M_1 and M_2 are fuzzy graph that are co-weak isomorphic. Assume that M_1 produced the fuzzy vertex infra topologies τ_{fix1} and M_2 produced the fuzzy vertex infra topologies τ_{fix2} . After that we have,

$$\begin{aligned}
 P(M_1) &= \{(I, 0.7), (J, 0.6), (K, 0.8), (L, 0.3)\}, \\
 P(M_2) &= \{(a, 0.7), (b, 0.7), (c, 0.9), (d, 1)\}, \\
 \tau_{fix1} &= \{\emptyset, \{(J, 0.6)\}, \{(L, 0.3)\}, \{(J, 0.6), (L, 0.3)\}, \{(I, 0.7), (K, 0.8)\}, \\
 &\quad \{(I, 0.7), (K, 0.8), (L, 0.3)\}, \{(I, 0.7), (J, 0.6), (K, 0.8)\}, P(M_1)\}, \\
 \tau_{fix2} &= \{\emptyset, \{(b, 0.7)\}, \{(d, 1)\}, \{(b, 0.7), (d, 1)\}, \{(a, 0.7), (c, 0.9)\}, \\
 &\quad \{(a, 0.7), (c, 0.9), (d, 1)\}, \{(a, 0.7), (b, 0.7), (c, 0.9)\}, P(M_2)\}.
 \end{aligned}$$

Therefore, τ_{fix1} and τ_{fix2} are distinct; hence τ_{fix1} is not homeomorphic to τ_{fix2} .

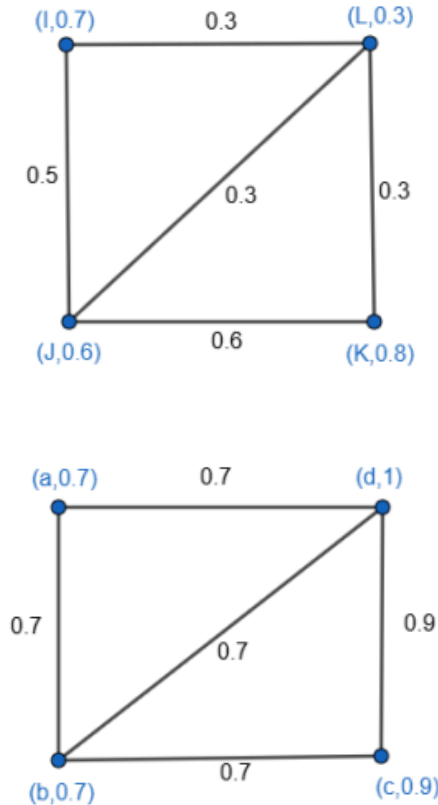


Figure 3. Co-weak isomorphic fuzzy graph M_1 and M_2

Description Figure 3: Figures 1 and 2 shows the fuzzy graphs M_1 and M_2 respectively. Each node is labeled with its membership value, and edge represent the degree of connection between vertices. Although M_1 and M_2 are co-weak isomorphic fuzzy graphs, meaning there exists a bijective mapping between their vertex sets preserving membership degrees, the corresponding fuzzy vertex infra topologies τ_{fix1} and τ_{fix2} are not homeomorphic. This is because the infra topologies derived from each graph involve distinct fuzzy infra sets, confirming that topological equivalence does not hold despite the graph-level isomorphism.

Theorem 3.11. Assume that all simple connected fuzzy graph are in the set X , with $\tau_{f_{ix1}}$ and $\tau_{f_{ix2}}$ being fuzzy vertex infra topologies produced by M_1 and M_2 , then the relation \sim established on X , $M_1 \sim M_2$ if and only if $\tau_{f_{ix1}} \simeq \tau_{f_{ix2}}$ is an equivalence relation on X .

Proof:

- (i) Reflexive: $\tau_{f_{ix1}} \simeq \tau_{f_{ix2}}, M_1 \sim M_2$
- (ii) Symmetry: It is evident that $\tau_{f_{ix1}} \simeq \tau_{f_{ix2}}$, since $\tau_{f_{ix2}} \simeq \tau_{f_{ix1}}$ thus, $M_1 \sim M_2$ suggests that $M_2 \sim M_1$.
- (iii) Transitive: $\tau_{f_{ix1}} \simeq \tau_{f_{ix2}}$ and $\tau_{f_{ix2}} \simeq \tau_{f_{ix3}}$, which indicates $\tau_{f_{ix1}} \simeq \tau_{f_{ix3}}$, since $M_1 \sim M_2$ and $M_2 \sim M_3$, $M_1 \sim M_3$. Therefore the equivalence of the relation \sim defined on X respectively.

Theorem 3.12. Assuming the fuzzy graph $M = (P, \delta, \gamma)$ is complete, then w_1 is the fuzzy vertex infra topology $\tau_{f_{ix}}$ that produced M .

Proof:

The complete fuzzy graph $M = (P, \delta, \gamma)$ implies that $\gamma(c, d) = \delta(c) \wedge \delta(d) \forall c, d \in P$.

Let two of the different element be a and $d \in P$ by definition, then $c \in R[d]$ but $d \notin R[d]$ and $d \in R[c]$ but $d \notin R[d]$. The elements $R[c]$ and $R[d]$ are open set since they are parts of $\tau_{f_{ix}}$. Accordingly, for each pair of unique fuzzy point $c, d \in P$, there are fuzzy open sets $R[c]$ as well as $R[d]$ in $(P(M), \tau_{f_{ix}})$ where $d \in R[c]$ but $d \notin R[d]$ and $c \in R[d]$ but $c \notin R[c]$. w_1 is therefore $\tau_{f_{ix}}$.

Remark 3.13. The previous theorem does not have to be true in reverse. In other words, the fuzzy graph $M = (P, \delta, \gamma)$ might not be complete even if the fuzzy vertex infra topology $\tau_{f_{ix}}$ produced by M is w_1 . For instance, the fuzzy graph M has a fuzzy vertex infra topology but is not complete:

$$\tau_{f_{ix}} = \{\emptyset, \{(S, 0.5), (T, 0.7), (U, 0.9)\}, \{(S, 0.5), (T, 0.7)\}, \\ \{(S, 0.7), (U, 0.9)\}, \{(U, 0.9), (S, 0.5)\}, P(M)\}$$

which is w_1 .

Theorem 3.14. Assuming the fuzzy graph $M = (P, \delta, \gamma)$ is complete, then w_2 is the fuzzy vertex infra topology $\tau_{f_{ix}}$ that produces M .

Proof:

Since $M = (P, \delta, \gamma)$ is a complete fuzzy graph $R[c] = P - \{(c, \delta(c))\}$ for every $c \in P(M)$. A sub base for $\tau_{f_{ix}}$ then $c = \{P - \{(c, \delta(c))\} : (c, \delta(c)) \in P(M)\}$. Assume that η is the finite intersection of S 's members. Consequently $(c, \delta(c)) \in P(M)$. As all $(c, \delta(c)), (d, \delta(d)) \in P(M)$ which means $(c, \delta(c)) \neq (d, \delta(d)), \{(c, \delta(c))\} \in \eta$ since $(c, \delta(c)) \notin R(c)$ but $(c, \delta(c)) \in R(d)$ to ensure, $\cap R[d] = \{(c, \delta(c))\}$,

When all $(d, \delta(d)) \in P(M)$ are taken to intersect so that $(d, \delta(d)) \neq (c, \delta(c))$ for every $(c, \delta(c)) \in P(M)$, the fuzzy vertex infra topology $\tau_{f_{ix}}$ produced by M must include a singleton set of the form $\{(c, \delta(c))\}$, if $(c, \delta(c)), (d, \delta(d)) \in P(M)$ then there exists open sets, says $k = (c, \delta(c))$ and $l = (d, \delta(d))$ in $\tau_{f_{ix}}$ such that $(c, \delta(c)) \in k$ $(d, \delta(d)) \in l$ and $k \cap l = \emptyset$, with $(c, \delta(c)) \neq (d, \delta(d))$ therefore $\tau_{f_{ix}}$ is w_2 .

Remark 3.15. The converse of the preceding theorem need not be necessarily true. In other words, the fuzzy graphs $M = (P, \delta, \gamma)$ might not be complete if the fuzzy vertex infra topology $\tau_{f_{ix}}$ produced M is w_2 . In figure 3.4 the fuzzy graph M , for instance is fuzzy vertex infra topology but incomplete

$$\tau_{f_{ix}} = \{\emptyset, \{(S, 0.5), (T, 0.7), (U, 0.9)\}, \{(S, 0.5), (T, 0.7)\}, \\ \{(S, 0.7), (U, 0.9)\}, \{(U, 0.9), (S, 0.5)\}, P(M)\}$$

which is w_2 .

Corollary 3.16. If complete bipartite fuzzy graph is $M = (P, \delta, \gamma)$, then w_1 and w_2 are created by M 's fuzzy vertex infra topology $\tau_{f_{ix}}$.

Theorem 3.17. For a simple connected self-complementary fuzzy graph $M = (P, \delta, \gamma)$ the fuzzy vertex infra topology produced by M its complement \overline{M} are identical.

Proof:

Since M is self-complementary, M and \overline{M} are isomorphic. By proposition implies that the fuzzy vertex infra topology induced by M and \overline{M} is clearly equivalent. Therefore the FV (Fuzzy Vertex) infra topologies generated by M and \overline{M} are identical.

Example 3.18. From Figure 4, M is identified as a self-complementary fuzzy graph. Let τ_{fix} denote the fuzzy vertex infra topology induced by M , and $\overline{\tau_{fix}}$ represent the fuzzy vertex infra topology induced by \overline{M} . Consequently, we get:

$$\begin{aligned} \tau_{fix} &= \{\emptyset, \{(S, 0.5)\}, \{(T, 0.7)\}, \{(U, 0.9)\}, \\ &\quad \{(S, 0.5), (T, 0.7)\}, \{(S, 0.5), (U, 0.9)\}, \{(T, 0.7), (U, 0.9)\}, P(M)\}, \\ \overline{\tau_{fix}} &= \{\emptyset, \{(S, 0.5)\}, \{(T, 0.7)\}, \{(U, 0.9)\}, \\ &\quad \{(S, 0.5), (T, 0.7)\}, \{(S, 0.5), (U, 0.9)\}, \{(T, 0.7), (U, 0.9)\}, P(\overline{M})\}. \end{aligned}$$

Here, clearly

$$\tau_{fix} = \overline{\tau_{fix}}.$$

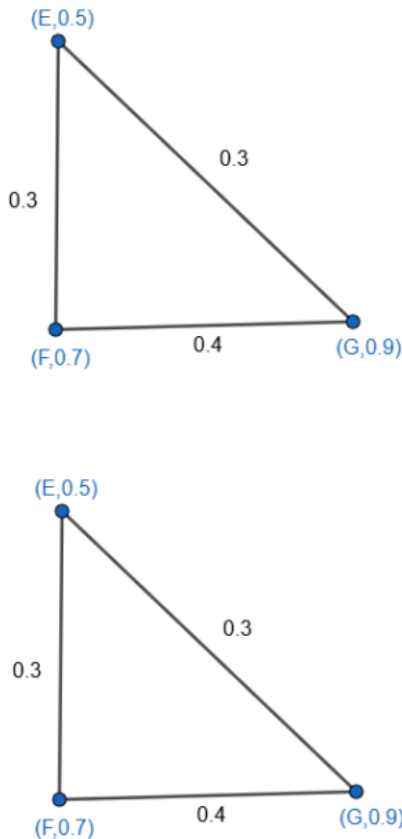


Figure 4. Figure 4

Description From the figure 4: the fuzzy graph M is observed to be self-complementary, meaning that M and its complement \overline{M} share the same structural properties in terms of fuzzy vertex connections. Let τ_{fix} denotes the fuzzy vertex infra topology induced by M , and let $(\overline{\tau_{fix}})$ denotes the fuzzy vertex infra topology induced by \overline{M} . This highlights the self-complementary nature of M at the level of fuzzy vertex infra topological structure.

Theorem 3.19. Consider $M = (P, \delta, \gamma)$ as a fuzzy infra topology, where $\gamma : P \times P \rightarrow [0, 1]$ is defined as $\gamma(k, l) = 1/2(\delta(k) \wedge \delta(l)) \forall k, l \in P$.

Proof:

If $M = (P, \delta, \gamma)$ be a fuzzy infra topology, where $\gamma(k, l) = 1/2(\delta(k) \wedge \delta(l))$, for all $k, l \in P$

Now consider the complement fuzzy infra topology, \overline{M}

$\overline{\gamma}(k, l) = 1 - \gamma(k, l)$ for all $k, l \in P$.

$\overline{\gamma}(k, l) = 1 - 1/2(\delta(k) \wedge \delta(l))$

. $f; p \rightarrow p$ such that M and \overline{M} are identical

$\gamma(f(k), f(l)) = 1/2(\delta(k) \wedge \delta(l)) = \overline{\gamma}(k, l)$.

Thus, $M \approx \overline{M}$ order the identity map on P .

4 Interior and closure of a fuzzy infra graph

Definition 4.1. Consider $M = (P, \delta, \gamma)$ a simple connected fuzzy graph, with $(P(M), \tau_{fix})$ as its corresponding fuzzy vertex infra topological spaces. For any subset H of $P(M)$, the interior and closure of H denoted by $int(H)$ and $cl(H)$ are defined as

$int(H) = \vee \{I \in \tau_{fix} : I \leq H\}$ and $cl(H) = \wedge \{I \in \tau'_{fix} : I \geq H\}$ Where τ'_{fix} is the complement of τ_{fix}

Theorem 4.2. Consider $M = (P, \delta, \gamma)$ a simple connected fuzzy graph, with $P(M, \tau_{fix})$ as its corresponding fuzzy vertex infra topological spaces. If H is a subset of $P(M)$ then

- (i) $Iint(H) \leq H$
- (ii) $Iint(int(H)) = Iint(H)$
- (iii) $H \leq J \Rightarrow Iint(H) \leq Iint(J)$
- (iv) $Iint(H \wedge J) = Iint(H) \wedge Iint(J)$
- (v) $H \leq Icl(H)$
- (vi) $Icl(cl(H)) = Icl(H)$
- (vii) $H \leq J \Rightarrow Icl(H) \leq Icl(J)$
- (viii) $Icl(H \vee J) = Icl(H) \vee Icl(J)$.

Proof:

- (i) $Iint(H) \leq H$
By definition, the infra interior is the largest infra open set contained in H . So $Iint(H) \subseteq H$ (in fuzzy terms: membership values of $Iint(H)$ do not exceed those of H). Hence $Iint(H) \leq H$.
- (ii) $Iint(Iint(H)) = Iint(H)$
Let $U = Iint(H)$. Then U is infra open. Taking the infra interior of U gives the largest infra open subset of U . But U is already infra open.
- (iii) $H \leq J \Rightarrow Iint(H) \leq Iint(J)$
If $H \subseteq J$, then every infra open set $U \subseteq H$ is also contained in J . Taking the supremum of all infra open subset of H is contained in the largest infra open subset of J .
- (iv) $Iint(H \wedge J) = Iint(H) \wedge Iint(J)$
Let $U = Iint(H)$ and $V = Iint(J)$ both infra open. $U \wedge V \subseteq H \wedge J$ (intersection of infra open subsets is infra open).
- (v) $H \leq Icl(H)$
By definition, the infra closure is the smallest infra-closed set containing H . So H is contained in $Icl(H)$.
- (vi) $Icl(cl(H)) = Icl(H)$
Let $C = Icl(H)$, which is infra closed by definition. The infra closure of C is the smallest infra closed set containing C . Since C is already infra closed.

- (vii) $H \leq J \Rightarrow Icl(H) \leq Icl(J)$
 If $H \subseteq J$, then every infra closed set containing J also contains H . so the smallest infra closed set containing H is contained in the smallest infra closed set containing J .
- (viii) $Icl(H \vee J) = Icl(H) \vee Icl(J)$.
 Let $C_H = Icl(H)$ and $C_J = Icl(J)$ both infra closed. $C_H \vee C_J$ is infra closed and contains $H \vee J$.

Theorem 4.3. $M = (P, \delta, \gamma)$ be a fuzzy vertex infra topology, with τ_{fix} for a fuzzy vertex infra set $H \subseteq P(M)$, the following condition are satisfied,

- (i) $int(H)$ is an open set in τ_{fix} .
 (ii) If K is open in τ_{fix} , such that $K \subseteq H, K \subseteq int(H)$
 (iii) $cl(H)$ is closed in τ_{fix} .
 (iv) If L is closed in τ_{fix} . Such that $L \subseteq H, L \subseteq cl(H)$.

Proof:

- (i) $int(H) = \vee\{K \in \tau_{fix} | K \subseteq H\}$
 Since $int(H)$ is a union (supremum) of open set in τ_{fix} and union of open sets are open in a fuzzy infra topology
 $int(H) \in \tau_{fix}$
 Hence, the interior of H is open.
- (ii) Let $K \subseteq H$ be open in τ_{fix} . by definition of $int(H)$ as the largest open set inside H :
 $K \subseteq int(H)$
 Hence, any open set $K \subseteq H$ is contained in $int(H)$.
- (iii) $cl(H) = \wedge\{L \subseteq P(M) | L \text{ is closed in } \tau_{fix}, H \subseteq L\}$
 Since $cl(H)$ is an intersection (infimum) of closed sets, and intersections of closed sets are closed in a fuzzy infra topology, $cl(H)$ is closed.
- (iv) Let $L \subseteq P(M)$ be a closed such that $L \subseteq H$. By definition, $cl(H)$ is the smallest closed set containing H . Therefore, if L is closed and $L \subseteq H$, then $L \subseteq cl(H)$
 Hence, any closed set inside H is contained in the closure of H .

5 Conclusion remarks

We defined the notion of fuzzy infra topological spaces within the graph framework of simple connected fuzzy infra topological vertex graphs. Analyzing the relationships between fuzzy graphs and infra topological spaces allows us to comment on certain aspects of infra topological properties. Moreover, the discovery of homeomorphism, isomorphism, weak isomorphism, co-weak isomorphism, and self-complementary fuzzy graphs, along with the study of interior and closure properties, helps in understanding fuzzy infra topological structures more effectively.

References

- [1] A.Kilicman and K. Abdulkalek, Topological spaces associated with simple graphs, *Journal of Mathematical Analysis* 9 (2018) 44-52.
- [2] Ali DM, Srivastava AK. On fuzzy connectedness. *Fuzzy Sets and Systems*. 1988 Nov 1; 28(2):203-8.
- [3] Al-Odhari A.M. On infra topological spaces. *International Journal of Mathematical Archive*. 2015; 6(11):179-84.
- [4] Alzubaidi AM, Dammak M. Graphic topology on fuzzy graphs. *Adv. Math.: Sci. J.* 2022;11:853-68.
- [5] D. Sasikala R.Mahima Generalized fuzzy infra j topological sets and structure. *Indian journal of natural science*. 2024 Dec; Issue 87; Vol.15.
- [6] D. Sasikala R. Mahima Exploring fuzzy separation axioms within fuzzy infra topologies, *Panamerican Mathematical journal*, 2025, Vol.35, No.4s.

- [7] Gholap PS, Nikumbh VE. Fuzzy topological spaces on fuzzy graphs. *Annals of Fuzzy Mathematics and Informatics*. 2023 Jun;25(3):279-91.
- [8] L. Chang, Fuzzy topological spaces, *J. Math. Anal. Appl.* 24 (1968) 182 - 190.
- [9] H. Sari and A. Kopuzlu, On topological spaces generated by simple undirected graphs, *AIMS Mathematics* 5 (2020) 5541 - 5550.
- [10] M. S. Sunitha and A. Vijaya Kumar, Complement of fuzzy graph, *Indian Journal of Pure and Applied Mathematics* 33 (9) (2002) 1451 - 1464.
- [11] Michalek J. Fuzzy topologies. *Kybernetika*. 1975;11(5):345 - 54.
- [12] Mufti ZS, Tabraiz A, Xin Q, Almutairi B, Anjum R. Fuzzy topological analysis of pizza graph. *AIMS Mathematics*. 2023 Jan 1;8(6):12841 - 56.
- [13] P. S. Gholap and V. E. Nikumbh, Topological spaces generated by graph, *Jnanabha* 52 (2022) 1 - 7.
- [14] P.S Gholap, Nikumbh VE. Fuzzy topological spaces on fuzzy graphs. *Annals of Fuzzy Mathematics and Informatics*. 2023 Jun;25(3):279 - 91.
- [15] Rosenfield, *Fuzzy Graphs*, Academic Press, Newyork 1975 77 - 95.
- [16] Sunitha MS, Vijayakumar A. *Studies on fuzzy graphs (Doctoral dissertation, Department of Mathematics)*.
- [17] T. Ibraheem and A. Nagim, On topological spaces generated by graphs and vice versa, *Journal of Al-Quadisiyah for Computer Science and Mathematics* 13 (2021) 13 - 24.
- [18] Zadeh L.A information and control of fuzzy set. 1965, 338 - 353.

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