Palestine Journal of Mathematics

Vol 15(Special Issue II)(2026) , 115–125

ON INFINITE SOMBOR INDEX

Manohar S. H., Boregowda H. S. and V. Lokesha

MSC 2010 Classifications: Primary 05C09; Secondary 05C07.

Keywords and phrases: Inf-Sombor index, p-Sombor index, Sombor index.

The authors would like to thank the reviewers and editor for their constructive comments and valuable suggestions that improved the quality of our paper.

Corresponding Author: Boregowda H. S.

Abstract In this paper, we studied the inf-sombor index and computed the upper and lower bounds of various degree-based topological indices. Further, we found the graphs for the class of trees and connected unicyclic graphs that maximize and minimize the inf-sombor index. Furthermore, comparing the correlation coefficients of the p-sombor index by varying the value of p showed that the correlation coefficient of $SO_p(G)$ converges to that of $SO_\infty(G)$ as $p \to \infty$. Also showed that there does not exist any connected graph such that $SO_\infty(G) = k$ for $k = 2, 3, 5$ and 7.

1 Introduction

The Sombor index, defined as $\sum_{uv\in E(G)}[\sqrt{d(u)^2 + d(v)^2}]$, was first developed by Gutman [8] in 2021. It is based on the degree radius of an edge $e = uv \in E(G)$, i.e the distance between the origin and the vector $x = (d(u), d(v))$, this can also be viewed as the 2 norm of the vector $x = (d(u), d(v))$. Numerous problems have been solved, including determining the upper and lower bounds, finding the graphs that attain maximum and minimum $SO(G)$ in case of trees, connected unicyclic graph and connected bicyclic graph, etc., finding graphs that give integer values of sombor index[2, 3, 4, 22]. This concept was extended by Reti et al. by introducing a p-sombor index[18], which can be seen as the p norm of the vector $x = (d(u), d(v))$. As $p \to \infty, \|x\|_p \to \|x\|_\infty = max\{d(u), d(v)\}$, inspired by this inf-Sombor index is defined as $SO_\infty(G) = \sum_{uv\in E(G)} max\{d(u), d(v)\}$[14]. The $SO_(G)$ always gives an integer value.
H_1 and H_2 be two graphs with edge set $E(H_1)$ and $E(H_2)$, vertex set $V(H_1)$ and $V(H_2)$ respectively, then

(i) The graph union of $H_1 \cup H_2$ with vertex set $V(H_1) \cup V(H_2)$ and edge set is $E(H_1) \cup E(H_2)$.

(ii) The join $H_1 + H_2$ is the graph union $H_1 \cup H_2$ together with all edges joining $V(H_1)$ and $V(H_2)$.

(iii) The corona product $H_1 \circ H_2$ is obtained by taking $|V(H_1)|$ copies of H_2 and joining i^{th} vertex of H_1 to every vertex in the i^{th} copy of H_2.

For more graph theoretical terminologies and notations, we referred [23, 21].
Well-studied topological indices and their mathematical expressions are listed in the table 1 for quick reference.

2 Strength of inf-Sombor index

p-sombor index is defined as $SO_p(G) = \sum_{uv\in E(G)}[d(u)^p + d(v)^p]^{\frac{1}{p}}$ if we see this as norm p, then as $p \to \infty, [d(u)^p + d(v)^p]^{\frac{1}{p}} \to max\{d(u), d(v)\}$. Hence, $SO_p(G) \to SO_\infty(G)$. Now we show that for any class of chemical graphs and for any physicochemical property, the correlation coefficient calculated for $SO_p(G)$ with the property Y converges to the correlation coefficient for $SO_\infty(G)$ with that property Y. Let $G_1, G_2, ..., G_k$ be any class of chemical graphs. Let r_p

Topological indices	Mathematical expressions
First Zagreb index [10]	$M_1(G) = \sum_{uv \in E(G)} [d(u) + d(v)]$
Second Zagreb index [11]	$M_2(G) = \sum_{uv \in E(G)} [d(u).d(v)]$
Forgotten index [5, 6]	$F(G) = \sum_{uv \in E(G)} [d(u)^2 + d(v)^2]$
Randic index [17]	$R(G) = \sum_{uv \in E(G)} \frac{1}{\sqrt{d(u).d(v)}}$
Reciprocal Randic index [7]	$RR(G) = \sum_{uv \in E(G)} \sqrt{d(u).d(v)}$
Inverse sum indeg index [19]	$ISI(G) = \sum_{uv \in E(G)} \frac{2d(u).d(v)}{d(u)+d(v)}$
Nirmala index [12]	$N(G) = \sum_{uv \in E(G)} \sqrt{d(u) + d(v)}$
Albertston irregularity index [1]	$Alb(G) = \sum_{uv \in E(G)} \lvert d(u) - d(v) \rvert$
Geometrical arithmetic index [20]	$GA(G) = \sum_{uv \in E(G)} \frac{2.\sqrt{d(u)d(v)}}{d(u)+d(v)}$
Sombor index [8]	$SO(G) = \sum_{uv \in E(G)} [\sqrt{d(u)^2 + d(v)^2}]$
p-sombor index [18]	$SO_p(G) = \sum_{uv \in E(G)} [d(u)^p + d(v)^p]^{1/p}$

Table 1. Topological indices and their mathematical expressions

Chemical Graphs	$SO_p(G)$	$SO_\infty(G)$	Physicochemical property (Y)
G_1	x_1^p	x_1	y_1
G_2	x_2^p	x_2	y_2
.	.	.	.
.	.	.	.
.	.	.	.
G_k	x_k^p	x_k	y_k

be the correlation coefficient relating $SO_p(G)$ values to property Y and r_∞ be the correlation coefficient of $SO_\infty(G)$ values to property Y. Then,

$$r_p = \frac{k\left(\sum_{i=1}^k x_i^p y_i\right) - \sum_{i=1}^k x_i^p \sum_{i=1}^k y_i}{\left[k\sum_{i=1}^k (x_i^p)^2 - \left(\sum_{i=1}^k x_i^p\right)^2\right]^{\frac{1}{2}} \left[k\sum_{i=1}^k (y_i)^2 - \left(\sum_{i=1}^k y_i\right)^2\right]^{\frac{1}{2}}}$$

Since $x_i^p \to x_i$ as $p \to \infty$. We have,

$$r_p \to \frac{k\left(\sum_{i=1}^k x_i y_i\right) - \sum_{i=1}^k x_i \sum_{i=1}^k y_i}{\left[k\sum_{i=1}^k (x_i)^2 - \left(\sum_{i=1}^k x_i\right)^2\right]^{\frac{1}{2}} \left[k\sum_{i=1}^k (y_i)^2 - \left(\sum_{i=1}^k y_i\right)^2\right]^{\frac{1}{2}}} = r_\infty$$

Hence $r_p \to r_\infty$ as $p \to \infty$. Figure 1 shows how the correlation coefficient of $SO_p(G)$ varies with respect to p for boiling point of alkanes. It can be observed that the correlation coefficient of $SO_p(G)$ converges to $SO_\infty(G)$ as $p \to \infty$. Figure 2 shows correlation of $SO_\infty(G)$ with boiling point of alkanes. We got a good correlation with a correlation coefficient $r = 0.9779$. The Inf-Sombor index is possibly the most significant graph invariant. First of all, in comparison to other p-sombor indices, it is simple to calculate. In addition, as previously discussed, the correlation coefficient of the p-sombor index converges to that of the inf-sombor index, resulting in a greater correlation coefficient value when compared to other p-sombor indices. Which is why the inf-sombor index is one of the efficient indices in chemical graph theory.

3 inf-Sombor index of standard class of graphs

This section discusses $SO_\infty(G)$ for some standard classes of graphs.

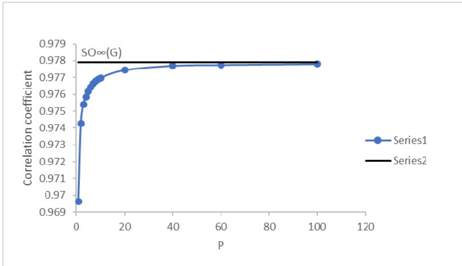


Figure 1.

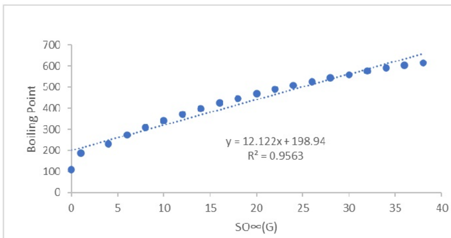


Figure 2.

Theorem 3.1. *Let G be an order n regular graph with regularity r then*

$$SO_\infty(G) = \frac{nr^2}{2}$$

Proof. The number of edges in a graph G of order n and regularity r is $\frac{nr}{2}$ therefore

$$\begin{aligned} SO_\infty(G) &= \frac{nr}{2}.r \\ &= \frac{nr^2}{2} \end{aligned}$$

□

Corollary 3.2. *(i)* $SO_\infty(K_n) = \frac{n.(n-1)^2}{2}$

(ii) $SO_\infty(C_n) = 2n$

Theorem 3.3. *Let K_{m_1,m_2} denotes the complete bipartite graph, then*

$$SO_\infty(K_{m_1,m_2}) = m_1 m_2 \cdot max\{m_1, m_2\}$$

Proof. Number of edges in K_{m_1,m_2} is $m_1 m_2$, hence

$$SO_\infty(K_{m_1,m_2}) = \sum_{i=1}^{m_1 m_2} max\{d(u_i), d(v_i)\}$$

$$SO_\infty(K_{m_1,m_2}) = m_1 m_2 \cdot max\{m_1, m_2\}$$

□

Corollary 3.4. $SO_\infty(S_n) = (n-1)^2$, *where S_n represents star graph of order $n \geq 3$.*

Theorem 3.5. $SO_\infty(P_n) = 2(n-1)$, *where P_n represents path graph of order $n \geq 3$.*

4 Graphs with given k inf-Sombor index

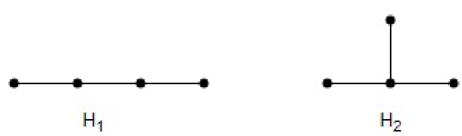
One question that comes to mind is: Is there a graph G such that $SO_\infty(G) = k$, given any positive integer k? The answer is yes if we consider the graph $G = k.K_2$, then $SO_\infty(G) = k$. Is there a connected graph G such that $SO_\infty(G) = k$, given any positive integer k? The following theorem answers this question.

Lemma 4.1. *If H is a subgraph of G, then $SO_\infty(H) \leq SO_\infty(G)$.*

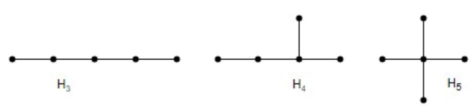
Theorem 4.2. *For $k = 2, 3, 5$ and 7, there doesn't exist any connected graph G such that $SO_\infty(G) = k$.*

Proof. For $k = 2$ or 3, let G be any connected graph with more than 2 vertices, then P_2 is always a subgraph of G hence by lemma 4.1 $SO_\infty(P_2) \leq SO_\infty(G) \implies SO_\infty(G) \geq 4$. Any connected graph with 1 or 2 vertices is isomorphic to K_1 or K_2, hence $SO_\infty(G)$ is 0 or 1. Which proves that the inf sombor index of a connected graph cannot be 2 or 3.

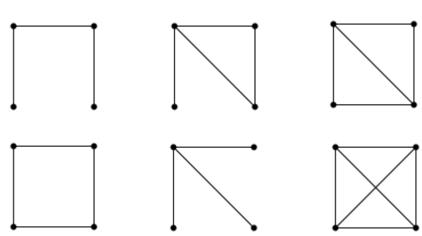
For $k = 5$, let G be a connected graph with more than 3 vertices, then either H_1 or H_2 is a subgraph of G. Hence by lemma 4.1 $SO_\infty(G) \geq SO_\infty(H_1) \implies SO_\infty(G) \geq 6$. Any connected



graph with 3 vertices is isomorphic to P_3 or K_3. None of which has an index 5. Hence, it proves the fact that there doesn't exist any connected graph G with inf sombor index 5. For $k = 7$, let



G be any connected graph with more than 4 vertices then either H_3 or H_4, H_5 is a subgraph of G hence by lemma 4.1 $SO_\infty(G) \geq SO_\infty(H_5) \implies SO_\infty(G) \geq 8$. If G is a graph with 1 or 2 vertices than G is isomorphic to K_1 or K_2. If G is a connected graph with 3 vertices, then G is isomorphic to P_3 or K_3, neither of which has inf sombor index 7. If G is a connected graph with 4 vertices, then G must be isomorphic to any of 6 possible graphs given below. None of which



has an index 7. Hence the result. □

Here we show that for any $k \geq 8$ there exists a connected graph such that $SO_\infty(G) = k$. For any even positive integer $k = 2n \geq 4$, consider the graph $G = P_{n+1}$ then $SO_\infty(G) = k$.
For any odd positive integer $k = 2n + 1 \geq 8$, consider the graph H shown in figure 3 $n - 1$ vertices whose inf sombor index is $SO_\infty(H) = k$. Now consider this question for any positive integer k other than 2, 3, 5 and 7, does there exist a tree T such that $SO_\infty(T) = k$?

For any even integer $k \geq 4$, consider the path graph with $\frac{k}{2}+1$ vertices then $SO_\infty(P_{\frac{k}{2}+1}) = k$. For any odd positive integer $k = 2n+1 \geq 9$, consider the tree T shown in figure 4 with n vertices, which has infinite sombor index $SO_\infty(T) = k$.

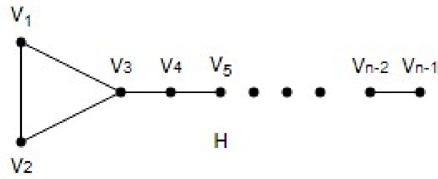


Figure 3.

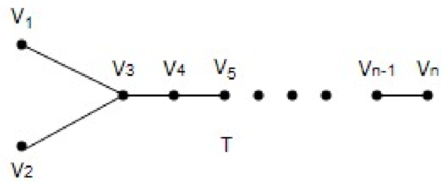


Figure 4.

5 Bounds on other degree-based topological indices

In this section, we discuss bounds of $SO_\infty(G)$ with respect to other degree-based topological indices.

Theorem 5.1. *For any graph G of size m,*

$$M_1(G) - \Delta(G)m \leq SO_\infty(G) \leq M_1(G) - \delta(G)m.$$

Furthermore equality holds if the graph is regular.

Proof. Let $e_1 = u_1v_1, e_2 = u_2v_2, \dots e_n = u_nv_n$ be the edges of G and say $max\{d(u_i), d(v_i)\} = d(u_i)$ for every $i = 1, 2, \dots m$,

$$\text{Consider,} \qquad M_1(G) = \sum_{i=1}^{m}[d(u_i) + d(v_i)]$$

$$M_1(G) \geq \sum_{i=1}^{m}[d(u_i) + \delta(G)]$$

$$M_1(G) \geq SO_\infty(G) + \delta(G)m \tag{5.1}$$

$$\text{similarly} \quad M_1(G) \leq \sum_{i=1}^{m}[d(u_i) + \Delta(G)]$$

$$M_1(G) \leq SO_\infty(G) + \Delta(G)m \tag{5.2}$$

from equations 5.1 and 5.2, we get $M_1(G) - \Delta(G)m \leq SO_\infty(G) \leq M_1(G) - \delta(G)m$. Equality holds when the graph is regular. □

Theorem 5.2. *G be any graph with $\delta(G) > 0$, then*

$$\frac{M_2(G)}{\Delta(G)} \leq SO_\infty(G) \leq \frac{M_2(G)}{\delta(G)}$$

Proof. Let $e_1 = u_1v_1, e_2 = u_2v_2, \dots e_n = u_nv_n$ be the edges of G and say $max\{d(u_i), d(v_i)\} = d(u_i)$ for every $i = 1, 2, \dots m$.

$$\text{Consider} \quad M_2(G) = \sum_{uv \in E(G)} d(u)d(v)$$
$$\geq \delta(G)d(u_1) + \delta(G)d(u_2) + \cdots + \delta(G)d(u_m))$$
$$M_2(G) \geq \delta(G)SO_\infty(G) \tag{5.3}$$
$$\text{similarly} \quad M_2(G) \leq \Delta(G)\left[d(u_1) + d(u_2) + \cdots + d(u_m)\right]$$
$$M_2(G) \leq \Delta(G)SO_\infty(G) \tag{5.4}$$

from equations 5.3 and 5.4, we get $\frac{M_2(G)}{\Delta(G)} \leq SO_\infty(G) \leq \frac{M_2(G)}{\delta(G)}$.

□

Theorem 5.3. *G be any graph, then*

$$m\delta(G) \leq SO_\infty(G) \leq m\Delta(G).$$

Equality holds when G is regular.

Proof. Let G be any $(m, n)-$graph. Then,

$$SO_\infty(G) = \sum_{uv \in E(G)} max\{d(u), d(v)\}$$
$$\leq m\Delta(G)$$
$$\text{Similarly} \quad SO_\infty(G) \geq m\delta(G)$$
$$m\delta(G) \leq SO_\infty(G) \leq m\Delta(G).$$

The equality holds when the graph is regular. □

Theorem 5.4. *G be any graph with $\delta(G) \geq 1$, then*

$$\frac{\delta(G)}{\Delta(G)} ISI(G) \leq SO_\infty(G) \leq \frac{\Delta(G)}{\delta(G)} ISI(G)$$

Proof. Let $e_1 = u_1v_1, e_2 = u_2v_2, \dots e_n = u_nv_n$ be the edges of G and say $max\{d(u_i), d(v_i)\}$ $= d(u_i)$for every $i = 1, 2, \dots m$.

$$\text{Consider} \quad ISI(G) = \sum_{e_i = u_iv_i \in E(G)} \frac{2d(u_i)d(v_i)}{d(u_i) + d(v_i)}$$
$$\leq \sum_{e_i = u_iv_i \in E(G)} \frac{2\Delta(G)d(u_i)}{2\delta(G)}$$
$$\leq \frac{\Delta(G)}{\delta(G)} SO_\infty(G) \tag{5.5}$$
$$\text{Similarly} \quad ISI(G) \geq \sum_{e_i = u_iv_i \in E(G)} \frac{2\delta(G)d(u_i)}{2\Delta(G)}$$
$$\geq \frac{\delta(G)}{\Delta(G)} SO_\infty(G) \tag{5.6}$$

from equations 5.5 and 5.6, we get $\frac{\delta(G)}{\Delta(G)} SO_\infty(G) \leq ISI(G) \leq \frac{\Delta(G)}{\delta(G)} SO_\infty(G)$

$$\frac{\delta(G)}{\Delta(G)} ISI(G) \leq SO_\infty(G) \leq \frac{\Delta(G)}{\delta(G)} ISI(G)$$

□

Theorem 5.5. *For any graph G, $SO_\infty(G) \leq SO_p(G) \leq 2^{\frac{1}{p}} SO_\infty(G)$.*

Proof. Let $e_1 = u_1v_1, e_2 = u_2v_2, \dots e_n = u_nv_n$ be the edges of G and say $max\{d(u_i), d(v_i)\} = d(u_i)$ for every $i = 1, 2, \dots m$.

$$\text{Consider } SO_p(G) = \sum_{e_i=u_iv_i \in E(G)} [d(u_i)^p + d(v_i)^p]^{\frac{1}{p}}$$
$$\leq \sum_{e_i=u_iv_i \in E(G)} [d(u_i)^p 2]^{\frac{1}{p}}$$
$$\leq 2^{\frac{1}{p}} \sum_{e_i=u_iv_i \in E(G)} d(u_i)$$
$$SO_p(G) \leq 2^{\frac{1}{p}} SO_\infty(G) \tag{5.7}$$
$$\text{Similarly } SO_p(G) \geq \sum_{e_i=u_iv_i \in E(G)} [d(u_i)^p]^{\frac{1}{p}}$$
$$\geq SO_\infty(G) \tag{5.8}$$

from equations 5.7 and 5.8, we get $SO_\infty(G) \leq SO_p(G) \leq 2^{\frac{1}{p}} SO_\infty(G)$. □

6 On extremal graphs of inf-sombor index

Here we discuss the graphs that maximize and minimize the $SO_\infty(G)$ for connected unicyclic graphs and trees.

Theorem 6.1. *The star graph maximizes the inf-sombor index in the case of trees.*

Proof. The number of edges in any tree with n vertices is $n - 1$; therefore,

$$SO_\infty(T_n) = \sum_{i=1}^{n-1} max\{d(u), d(v)\}$$

we know that $d(v) \leq n - 1$ for any $v \in V(G)$, therefore

$$\leq (n-1)^2$$

$$SO_\infty(T_n) \leq SO_\infty(S_n) \qquad \text{by corollary 3.4}$$

. □

Theorem 6.2. *In the case of trees, the path graph P_n minimizes the inf-Sombor index.*

Proof. For $n = 2$ the result is obvious, let T_n be the tree with $n > 2$, for any $uv \in E(T_n)$, $max\{d(v), d(u)\} \geq 2$ Thus,

$$SO_\infty(T_n) = \sum_{uv \in E(G)} max\{d(v), d(u)\}$$
$$SO_\infty(T_n) \geq \sum_{uv \in E(G)} 2 = 2m$$
$$SO_\infty(T_n) \geq SO_\infty(P_n) \qquad \text{by theorem 3.5.}$$

□

Theorem 6.3. *In the class of connected unicyclic graphs, the cyclic graph minimizes the inf-sombor index.*

Proof. Let U_n be any connected unicyclic graph of order n; then the number of edges in U_n is exactly n. Therefore

$$SO_\infty(G) = \sum_{e_i=u_iv_i\in E(G)} max\{d(u_i), d(v_i)\}$$

where $e_i = u_iv_i,$ for $i = 1, 2, ..., n.$
Since $n \geq 3,$ $max\{d(u_i), d(v_i)\} \geq 2$

$$SO_\infty(U_n) \geq \sum_{e_i\in E(G)} 2 = 2.n$$

$$SO_\infty(U_n) \geq SO_\infty(C_n) \qquad \text{by corollary 3.2.}$$

□

Definition 6.4. H_n is the graph obtained by joining any two pendant vertices of the star graph, as shown in the Figure 5.

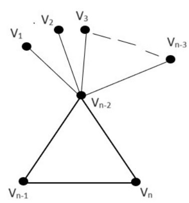


Figure 5. H_n

Lemma 6.5. *Let G be an unicyclic connected graph of order* $n \geq 6$*. If G maximizes the* $SO_\infty(G)$*, then G cannot have more than* 1 *edges of edge degree* 2*.*

Proof. Let $e_i = u_iv_i,$ for $i = 1, 2, ..., n$ be the edges of G. Assume that G has more than 2 edges of edge degree 2, say e_1 and e_2. Consider

$$SO_\infty(G) = \sum_{e_i\in E(G)} max\{d(u_i), d(v_i)\}$$

$$= max\{d(v_1), d(u_1)\} + max\{d(v_2), d(u_2)\} + \sum_{i=3}^{n} max\{d(u_i), d(v_i)\}$$

since $max\{d(u_i), d(v_i)\} \leq n-1$ for $i = 3, 4, ..., n.$ and $max\{d(v_j), d(u_j)\} \leq 3,$ for $j = 1, 2.$

$$SO_\infty(G) \leq 3 + 3 + (n-1)(n-2) = 6 + (n-1)(n-2) \tag{6.1}$$

$$\text{If } (n-1)^2 + 2 - [(n-1)(n-2) + 6] \leq 0$$

$$n^2 - 2n + 1 + 2 - n^2 + 3n - 2 - 6 \leq 0$$

$$n \leq 5$$

which leads to a contradiction, since $n \geq 6$
hence $(n-2)(n-1) + 6 < (n-1)^2 + 2$
From equation 6.1 we have

$$SO_\infty(G) \leq (n-1)(n-2) + 6 < (n-1)^2 + 2 = SO_\infty(H_n)$$

which leads to contradiction, since G maximizes inf-sombor index. □

Lemma 6.6. *Let G be an unicyclic connected graph of order* $n \geq 6$*. If G maximizes* $SO_\infty(G)$*, then G cannot have an edge of edge degree* 1*.*

Proof. Let $e_i = u_iv_i$ for $i = 1, 2, ..., n$ be the edges of G. Assume that G has an edge, say e_1, of edge degree 1. Hence $max\{d(u_1), d(v_1)\} = 2$ and also G has no vertex of degree $n-1$ hence $max\{d(u_i), d(v_i)\} \leq n-2 \ \ for \quad i = 2, 3, ..., n.$

$$\begin{aligned} SO_\infty(G) &= \sum_{e_i \in E(G)} max\{d(u_i), d(v_i)\} \\ &= max\{d(v_1), d(u_1)\} + \sum_{i=2}^{n} max\{d(u_i), d(v_i)\} \\ &\leq 2 + (n-1)(n-2) < 2 + (n-1)^2 \\ SO_\infty(G) &< SO_\infty(H_n) \end{aligned}$$

Which is a contradiction to the fact that G maximizes the inf-sombor index. □

Lemma 6.7. *Let G^* be the graph of order $n \geq 6$ depicted in figure 6, which will not maximize the inf-sombor index in case of unicyclic graphs.*

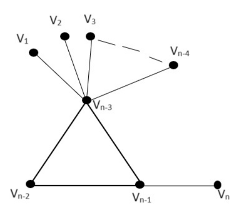


Figure 6. G^*

Proof. The inf-sombor index of the graph G^* is given by

$$SO_\infty(G^*) = (n-2)^2 + 6 \tag{6.2}$$

$$\begin{aligned} \text{If } \ (n-1)^2 + 2 - [(n-2)^2 + 6] &\leq 0 \\ 2n - 7 &\leq 0 \\ n &\leq \frac{7}{2} \qquad \text{which is a contradiction.} \\ \text{Hence, } \ [(n-2)^2 + 6] &< 2 + (n-1)^2 \end{aligned}$$

from equation 6.2, we have

$$\begin{aligned} SO_\infty(G^*) &= 6 + (n-2)^2 < (n-1)^2 + 2 \\ SO_\infty(G^*) &< SO_\infty(H_n) \end{aligned}$$

Hence G^* will not maximize the inf-sombor index. □

Theorem 6.8. *H_n maximizes the inf-sombor index for unicyclic connected graphs.*

Proof. Let G be any unicyclic connected graph that maximizes the inf-sombor index, and let $u_1, u_2, ..., u_k$ be the k vertices on the cycle. Remove any edge $e = u_iu_{i+1}$, the resulting graph is a tree T_n with n vertices.
Case (i): If T_n is a star, then the graph is isomorphic to H_n.
Case (ii): If T_n is not a star graph then $\Delta(G) \leq n-2$.
Subcase (i): If $\Delta(G) < n-1$ or $\Delta(G) \leq n-2$ then

$$\begin{aligned} SO_\infty(G) &\leq (n-2)n < 2 + (n-1)^2 \\ SO_\infty(G) &< SO_\infty(H_n) \end{aligned}$$

which leads to a contradiction since G maximizes the inf-sombor index.
Subcase (ii): If $\Delta(T_n) = n-2$ then we obtain the graph T_n which is shown in figure 7.
Furthermore, an edge is added between the vertices of T_n to obtain the original graph G. If

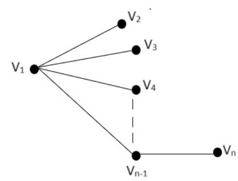


Figure 7. T_n

any of the vertices $v_2, v_3, ..., v_{n-2}$ is adjacent, then the edge $e = v_{n-1}v_n$ will be an edge of edge degree 1 and hence by Lemma 6.6, G is not a maximal graph, leading to a contradiction. If any vertex $v_i \quad i = 2, 3, ..., n-2$ is adjacent to the vertex v_n, then the edges $e_1 = v_iv_n$ and $e_2 = v_{n-1}v_n$ are two edges of edge degree 2 and hence by Lemma 6.5, G is not maximal, leading to a contradiction. If any vertex $v_i \quad i = 2, 3, ..., n-2$ is adjacent to the vertex v_{n-1}, then by Lemma 6.7, G does not maximize the inf-sombor index, leading to a contradiction. If v_1 is adjacent to v_n, then G is isomorphic to H_n, which completes the proof. □

7 Conclusion and further research

Firstly, we computed $SO_\infty(G)$ for some standard class of graphs and determined bounds relating to other degree-based topological indices. Further, we obtained the graphs that maximize and minimize the inf-sombor index in the case of trees and unicyclic graphs and found for what values of $k \in \mathbb{Z}^+$ there exists a graph G such that $SO_\infty(G) = k$. Furthermore, we compared the correlation coefficient of $SO_p(G)$ varying p and came to the conclusion that the correlation coefficient of $SO_p(G)$ converges to the correlation coefficient of that of $SO_\infty(G)$, and lastly, we correlated $SO_\infty(G)$ and boiling point and got a good correlation. To conduct an additional study, one can consider:

(i) Finding maximal and minimal graphs for higher cyclomatic number graphs.

(ii) Bounds in terms of other graph invariants can be considered.

(iii) One can consider optimizing problems like given any positive integer k, what is the minimum order graph G such that $SO_\infty(G) = k$

(iv) Studying the spectral properties of this index will be an interesting task.

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Author information

Manohar S. H., Department of Studies and Research in Mathematics, Tumkur University, Tumkur-572103, Karnataka.
Department of Mathematics, GFGC Madhugiri-572132., India.
E-mail: `manoharsh146@gmail.com`

Boregowda H. S., Department of Studies and Research in Mathematics, Tumkur University, Tumkur-572103, Karnataka, India., India.
E-mail: `bgsamarasa@gmail.com`

V. Lokesha, Department of Studies in Mathematics, Vijaya Nagara Sri Krishnadevaraya University, Bellary-583105, Karnataka., India.
E-mail: `v.lokesha@gmail.com`