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DECOMPOSITION OF A COMPLETE BIPARTITE GRAPH PLUS A 1−FACTOR INTO PATHS AND STARS OF LENGTH FOUR

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Abstract *Let λ, α, k and m be non-negative integers. Let P_{k+1},S_{k+1} denote a path, star with k edges and $k+1$ vertices. $K_{m,m}$ denote the complete bipartite graph with m vertices in each partite set and $K_{m,m} \oplus I$ denote $K_{m,m}$ with a 1-factor added. The graph $K_{m,m} \oplus I$ has a $\{\lambda P_5, \alpha S_5\}$ −decomposition if and only if $4(\lambda+\alpha) = |E(K_{m,m} \oplus I\ )|$ and $\lambda \neq 1$. In particular, we find necessary and sufficient conditions for such decomposition in $K_{m,m} \oplus I$, when $m \equiv 3 \pmod 4$*

1 Introduction

All graphs considered here are finite and undirected. For the basic graph-theoretic terminology the reader is referred to [6]. Let $K_{m,m} \oplus I$ denote the complete bipartite graph with a 1-factor added. Let P_{k+1} is a $k-path$ on $k+1$ vertices $y_1, y_2, ..., y_{k+1}$ and S_{k+1} is a star with $k+1$ vertices and $k-edges$. S_{k+1} is isomorphic to $K_{1,k}$ with $k+1$ vertices and consist a centre vertex y_1 of degree k and k end vertices of $y_2, y_3...y_k$. It is denoted by $(y_1 : y_2y_3...y_k)$. By a definition of decomposition of G, we mean a list of edge disjoint subgraph $H_1, ..., H_m$ whose union $H_1 \oplus ... \oplus H_m$ is G. If each subgraphs in a decomposition is isomorphic to H, then we say that G has an $H-$decomposition. When G can be decomposed into λ copies of H_1 and α copies of H_2, we say that G has $\{\lambda H_1, \alpha H_2\}$-decomposition or (H_1, H_2)-multidecomposition. If such a decomposition exists for all λ and α satisfying trivial necessary conditions, then we say that G has a $\{H_1, H_2\}_{(\lambda,\alpha)}$-decomposition or fully $\{H_1, H_2\}$-decomposition.

Abueida, Daven and Roblee have established a multidecomposition of complete graphs, for additional details refer [1],[2],[3]. Jeevadoss [8] have been proved that the decomposition of complete bipartite graph into path and cycles of length k. Pauline [10] have obtained decomposition of complete graphs K_{rm}, K_{rm+1} into stars with m edges, if and only if r is even or m is odd. Nalini [9] investigated the necessary and sufficient conditions for the existence of a $\{\lambda P_5, \mu C_4\}$-decomposition of $K_{m,m} \oplus I$ when $m \equiv 3 \pmod 4$ and $\lambda \neq 1$. Focus was given to decomposing the graph into paths of length 5 and 4-cycles under these constraints. T.W.Shyu [11],[12] have settled decomposition of complete graph(complete bipartite graph) into paths and stars with k edges. Yang [14] has been proved that $k-star$ decomposition of graphs G exist, when minimum degree of G greater than or equal to $2k-1$. Jenq [7] have been showed that sufficient which is also necessary for the occurrence of the path P_{k+1} and star S_k decomposition of the complete bipartite graph with a 1-factor deleted. Ilayaraja [5] obtained necessary and sufficient condition of product graphs into paths and stars on five vertices. Recently, T.W.Shyu [13] have obtained decomposition of K_{2n+1} and $K_{2m,2n}$ into (pP_5, qS_5, rS_5).

In this paper, we investigate about the existence of a fully $\{\lambda P_5, \alpha S_5\}$-decomposition. Also, *we establish necessary and sufficient conditions for the existence of fully $\{\lambda P_5, \alpha S_5\}$-decomposition of $K_{m,m} \oplus I$.*

2 Preliminaries

Let $K_{m,m} \oplus I$ denote the complete bipartite graph with a 1-factor added and bipartition (X, Y), where $X = \{x_1, x_2, \ldots, x_m\}$, and $Y = \{y_1, y_2, \ldots, y_m\}$. Let $L_j(X, Y)$ in $K_{m,m}$ as $L_j(X, Y) = \{x_j y_j | 0 \le j \le m-1\}$, where addition in the subscript of y is considered modulo m with residues $1, 2, \ldots, m$. We say, $L_j(X, Y)$ is a $1-$factor of $K_{m,m}$ and it is called the $1-$factor of distance j. Also, $\bigcup_{j=0}^{m-1} L_j(X, Y) = K_{m,m}$.

Remark 2.1. If H_1 and H_2 have a $\{\lambda P_5, \alpha S_5\}$-decomposition, then $H_1 \oplus H_2$ has a such decomposition.

Theorem 2.2. *[4] A non trivial connected graph G has a P_3-decomposition if and only if G has even size.*

construction:1 Let S_5^β and S_5^δ be two stars with 5 vertices, where $S_5^\beta = x_1 : uy_3y_4v$ and $S_5^\delta = z_1 : uvw_3w_4$. If u and v are common vertices of S_5^β and, S_5^δ then we have two edge-disjoint paths of length 4, say P_5^β and P_5^δ from S_5^β and S_5^δ as follows: $P_5^\beta = y_3x_1uz_1w_4$ and $P_5^\delta = y_4x_1vz_1w_3$.

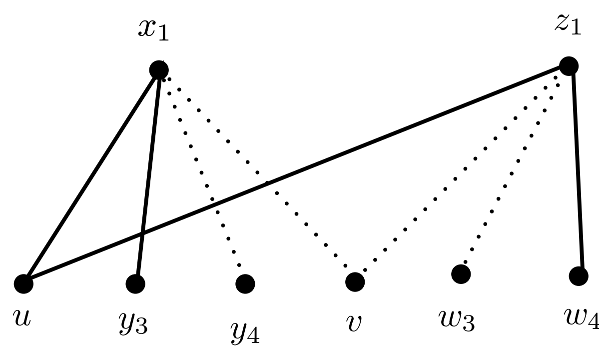

Figure 1. $P_5^\beta = (S_5^\beta - x_1y_4 - x_1v) \cup z_1u \cup z_1w_4, P_5^\delta = (S_5^\delta - z_1u - z_1w_4) \cup x_1y_4 \cup x_1v$

construction:2 Let S_5^β,S_5^δ and S_5^γ be three stars with $S_5^\beta \cup S_5^\delta = P_5^\beta \cup P_5^\delta = 4$ edges, where $S_5^\beta = x_1 : y_2y_3y_4y_5$, $S_5^\delta = z_1 : w_2w_3w_4w_5$ and $S_5^\gamma = q_1 : r_2r_3r_4r_5$. If z is a common vertex of S_β,S_5^δ and S_5^γ then we have three edge-disjoint paths of length 4, say P_5^β, P_5^δ and P_5^γ from S_5^β,S_5^δ and S_5^γ, as follows: $P_5^\beta = y_2x_1uz_1v$, $P_5^\delta = y_3x_1zq_1v$, $P_5^\gamma = zz_1wq_1r_5$.

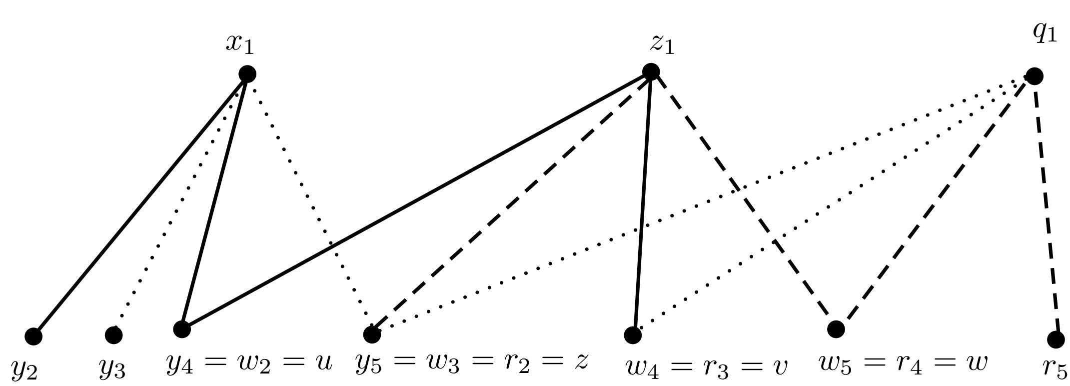

Figure 2. $P_5^\beta = (S_5^\beta - x_1y_3 - x_1z) \cup uz_1 \cup z_1v, P_5^\delta = (S_5^\delta - z_1y_4 - z_1w_4) \cup wq_1 \cup q_1r_5, P_5^\gamma = (S_5^\gamma - q_1w - q_1r_5) \cup zx_1 \cup x_1y_3$

3 Necessary Condition

Lemma 3.1. *Let λ, α be non-negative integers, and let $m \in \mathbb{N}$ such that $m \equiv 3 \pmod 4$. If $K_{m,m} \oplus I$ admits a $(\lambda P_5, \alpha S_5)$-decomposition, then the following hold:* $4(\lambda+\alpha) = |E(K_{m,m} \oplus I)|$, *and* $\lambda \neq 1$.

Proof. First, note that $K_{m,m}$ has m^2 edges and the 1-factor I on $2m$ vertices has m edges. Therefore, $|E(K_{m,m} \oplus I)| = m^2 + m$. Each copy of P_5 and S_5 contains exactly 4 edges, so a $(\lambda P_5, \alpha S_5)$-decomposition covers $4(\lambda + \alpha)$ edges. Equating, we get $4(\lambda + \alpha) = m^2 + m = m(m+1)$. Hence, the first statement is proven. Now we prove $\lambda \neq 1$ using degree arguments. In the graph $K_{m,m} \oplus I$, every vertex in $K_{m,m}$ has degree m, and the 1-factor in I increase the degree of m vertices by 1. Thus, in $K_{m,m} \oplus I$, there are m vertices of degree $m + 1$, and m vertices of degree m. Each P_5 has degree sequence $\{1, 2, 2, 2, 1\}$, contributing two vertices of degree 1. Each S_5 has degree sequence $\{4, 1, 1, 1, 1\}$, contributing four vertices of degree 1. Therefore, the total number of degree-1 vertices in the decomposition is $2\lambda + 4\alpha$. Assume $\lambda = 1$. Then using the edge count condition: $\lambda + \alpha = \frac{m(m+1)}{4} \Rightarrow \alpha = \frac{m(m+1)}{4} - 1$. Substituting into the degree1 count: $2\lambda+4\alpha = 2+4\left(\frac{m(m+1)}{4} - 1\right) = m(m+1)-2$. This implies the decomposition requires $m(m + 1) - 2$ degree-1 vertices. However, $K_{m,m} \oplus I$ has only $2m$ vertices. For $m \geq 3$, and since $m \equiv 3 \pmod 4$, we have $m(m + 1) - 2 > 2m$, which is a contradiction. Therefore, the graph does not have enough vertices to accommodate that many degree 1 vertices when $\lambda = 1$. Hence, $\lambda \neq 1$. □

Lemma 3.2. *There exists a $\{\lambda P_5, \alpha S_5\}$-decomposition of $K_{7,7} \oplus I$, where $\lambda \neq 1$.*

Proof. Let $V(K_{7,7} \oplus I$)=$\{x_1, x_2, x_3, x_4, x_5, x_6, x_7\} \cup \{y_1, y_2, y_3, y_4, y_5, y_6, y_7\}$.The $(\lambda P_5, \alpha S_5)-$ *decomposition* of $K_{7,7} \oplus I$ is obtained using Constructions and , as follows.
(1)λ=0 and α=14.The required stars are
$(x_1 : y_1y_2y_3y_4)$, $(x_2 : y_1y_2y_3y_4)$, $(x_1 : y_1y_5y_6y_7)$, $(x_2 : y_2y_5y_6y_7)$, $(x_3 : y_1y_2y_3y_7)$, $(x_4 : y_1y_2y_3y_4)$, $(x_4 : y_4y_5y_6y_7)$, $(x_3 : y_3y_4y_5y_6)$, $(x_5 : y_1y_2y_3y_5)$, $(x_5 : y_4y_5y_6y_7)$, $(x_6 : y_1y_2y_3y_6)$,$(x_6 : y_4y_5y_6y_7)$, $(x_7 : y_1y_2y_3y_7)$, $(x_7 : y_4y_5y_6y_7)$

(2)λ=2 and α=12. The required paths and stars are
$x_1y_1x_2y_2x_3$, $x_1y_2x_4y_1x_3$, $(y_1 : x_1x_5x_6x_7)$, $(y_2 : x_2x_5x_6x_7)$, $(y_3 : x_1x_2x_3x_4)$, $(y_3 : x_3x_5x_6x_7)$, $(y_4 : x_1x_2x_3x_4)$, $(y_4 : x_4x_5x_6x_7)$, $(y_5 : x_1x_2x_3x_5)$, $(y_5 : x_4x_5x_6x_7)$, $(y_6 : x_1x_2x_3x_6)$, $(y_6 : x_4x_5x_6x_7)$, $(y_7 : x_1x_2x_3x_7)$, $(y_7 : x_4x_5x_6x_7)$

(3)λ=3 and α=11. The required paths and stars are
$x_1y_1x_2y_6x_3$, $x_5y_3x_3y_1x_4$, $x_7y_3x_6y_6x_1$, $(y_1 : x_1x_5x_6x_7)$, $(y_2 : x_1x_2x_3x_4)$, $(y_2 : x_2x_5x_6x_7)$, $(y_3 : x_1x_2x_3x_4)$, $(y_4 : x_1x_2x_3x_4)$, $(y_4 : x_4x_5x_6x_7)$, $(y_5 : x_1x_2x_3x_5)$, $(y_5 : x_4x_5x_6x_7)$, $(y_6 : x_4x_5x_6x_7)$, $(y_7 : x_1x_2x_3x_7)$, $(y_7 : x_4x_5x_6x_7)$

(4)λ=4 and α=10. The required paths and stars are
$x_1y_1x_2y_2x_3$, $x_1y_2x_4y_1x_3$, $x_1y_1x_7y_2x_6$, $x_2y_2x_5y_1x_6$, $(y_3 : x_1x_2x_3x_4)$, $(y_3 : x_3x_5x_6x_7)$, $(y_4 : x_1x_2x_3x_4)$, $(y_4 : x_4x_5x_6x_7)$, $(y_5 : x_1x_2x_3x_5)$, $(y_5 : x_4x_5x_6x_7)$, $(y_6 : x_1x_2x_3x_6)$, $(y_6 : x_4x_5x_6x_7)$, $(y_7 : x_1x_2x_3x_7)$, $(y_7 : x_4x_5x_6x_7)$

(5)λ=5 and α=9. The required paths and stars are
$x_1y_1x_2y_6x_3$, $x_5y_3x_3y_1x_4$, $x_7y_3x_6y_6x_1$, $x_4y_6x_5y_7x_6$, $x_4y_7x_7y_6x_6$, $(y_1 : x_1x_5x_6x_7)$, $(y_2 : x_1x_2x_3x_4)$, $(y_2 : x_2x_5x_6x_7)$, $(y_3 : x_1x_2x_3x_4)$, $(y_4 : x_1x_2x_3x_4)$, $(y_4 : x_4x_5x_6x_7)$, $(y_5 : x_1x_2x_3x_5)$, $(y_5 : x_4x_5x_6x_7)$, $(y_7 : x_1x_2x_3x_7)$

(6)λ=6 and α=8. The required paths and stars are
$x_1y_1x_2y_2x_3$, $x_1y_2x_4y_1x_3$, $x_1y_1x_7y_2x_6$, $x_2y_2x_5y_1x_6$, $x_4y_5x_5y_7x_6$, $x_4y_7x_7y_5x_6$, $(y_3 : x_1x_2x_3x_4)$, $(y_3 : x_3x_5x_6x_7)$, $(y_4 : x_1x_2x_3x_4)$, $(y_4 : x_4x_5x_6x_7)$, $(y_5 : x_1x_2x_3x_5)$, $(y_6 : x_1x_2x_3x_6)$, $(y_6 : x_4x_5x_6x_7)$, $(y_7 : x_1x_2x_3x_7)$

(7)λ=7 and α=7. The required paths and stars are
$x_1y_1x_2y_6x_3$, $x_5y_3x_3y_1x_4$, $x_4y_6x_5y_7x_6$, $x_4y_7x_7y_6x_6$, $x_1y_6x_6y_5x_4$,

$x_1y_2x_2y_3x_3$, $x_1y_3x_4y_2x_3$, $(y_1 : x_1x_5x_6x_7)$, $(y_2 : x_2x_5x_6x_7)$, $(y_4 : x_1x_2x_3x_4)$, $(y_4 : x_4x_5x_6x_7)$, $(y_5 : x_1x_2x_3x_5)$, $(y_5 : x_4x_5x_6x_7)$, $(y_7 : x_1x_2x_3x_7)$

(8)λ=8 and α=6. The required paths and stars are
$x_1y_1x_2y_2x_3$, $x_1y_2x_4y_1x_3$, $x_1y_1x_7y_2x_6$, $x_1y_4x_4y_3x_3$, $x_1y_3x_2y_4x_3$, $x_2y_2x_5y_1x_6$, $x_4y_5x_5y_7x_6$, $x_4y_7x_7y_5x_6$, $(y_3 : x_3x_5x_6x_7)$, $(y_4 : x_4x_5x_6x_7)$, $(y_5 : x_1x_2x_3x_5)$, $(y_6 : x_1x_2x_3x_6)$, $(y_6 : x_4x_5x_6x_7)$, $(y_7 : x_1x_2x_3x_7)$

(9)λ=9 and α=5. The required paths and stars are
$x_1y_1x_2y_6x_3$, $x_1y_2x_2y_3x_3$, $x_1y_3x_4y_2x_3$, $x_1y_6x_6y_3x_7$, $x_4y_1x_3y_3x_5$, $x_4y_6x_5y_7x_6$, $x_4y_7x_7y_6x_6$, $x_1y_1x_7y_2x_5$, $x_2y_2x_6y_1x_3$, $(y_4 : x_1x_2x_3x_4)$, $(y_4 : x_4x_5x_6x_7)$, $(y_5 : x_1x_2x_3x_5)$,$(y_5 : x_4x_5x_6x_7)$, $(y_7 : x_1x_2x_3x_7)$

(10)λ=10 and α=4. The required paths and stars are
$x_1y_1x_2y_2x_3$, $x_1y_2x_4y_1x_3$, $x_1y_1x_7y_2x_6$, $x_1y_4x_4y_3x_3$, $x_1y_3x_2y_4x_3$, $x_2y_2x_5y_1x_6$, $x_4y_5x_5y_7x_6$, $x_4y_7x_7y_5x_6$, $x_4y_6x_7y_4x_6$, $x_4y_4x_5y_6x_6$, $(y_3 : x_3x_5x_6x_7)$, $(y_5 : x_1x_2x_3x_5)$, $(y_6 : x_1x_2x_3x_6)$, $(y_7 : x_1x_2x_3x_7)$

(11)λ=11 and α=3. The required paths and stars are
$x_1y_1x_2y_6x_3$, $x_1y_2x_2y_3x_3$, $x_1y_3x_4y_2x_3$, $x_1y_6x_6y_3x_7$, $x_4y_1x_3y_3x_5$, $x_4y_6x_5y_7x_6$, $x_4y_7x_7y_6x_6$, $x_1y_1x_7y_2x_5$, $x_2y_2x_6y_1x_3$, $x_1y_4x_3y_7x_7$, $x_4y_4x_2y_7x_1$, $(y_4 : x_4x_5x_6x_7)$, $(y_5 : x_1x_2x_3x_5)$, $(y_5 : x_4x_5x_6x_7)$

(12)λ=12 and α=2. The required paths and stars are
$x_1y_1x_2y_2x_3$, $x_1y_2x_4y_1x_3$, $x_1y_1x_7y_2x_6$, $x_1y_4x_4y_3x_3$, $x_1y_3x_2y_4x_3$, $x_2y_2x_5y_1x_6$, $x_4y_5x_5y_7x_6$, $x_4y_7x_7y_5x_6$, $x_4y_6x_7y_4x_6$, $x_4y_4x_5y_6x_6$, $x_1y_5x_3y_6x_6$, $x_5y_5x_2y_6x_1$, $(y_3 : x_3x_5x_6x_7)$, $(y_7 : x_1x_2x_3x_7)$

(13)λ=13 and α=1. The required paths and stars are
$x_1y_1x_2y_6x_3$, $x_1y_2x_2y_3x_3$, $x_1y_3x_4y_2x_3$, $x_1y_6x_6y_3x_7$, $x_4y_1x_3y_3x_5$, $x_4y_6x_5y_7x_6$, $x_4y_7x_7y_6x_6$, $x_1y_1x_7y_2x_5$, $x_2y_2x_6y_1x_3$, $x_1y_4x_3y_7x_7$, $x_4y_4x_2y_7x_1$, $x_4y_4x_5y_5x_6$, $x_4y_5x_7y_4x_6$, $(y_5 : x_1x_2x_3x_5)$

(14)λ=14 and α=0. The required paths and stars are
$x_1y_1x_2y_2x_3$, $x_1y_2x_4y_1x_3$, $x_1y_1x_7y_2x_6$, $x_1y_4x_4y_3x_3$, $x_1y_3x_2y_4x_3$, $x_2y_2x_5y_1x_6$, $x_4y_5x_5y_7x_6$, $x_4y_7x_7y_5x_6$, $x_4y_6x_7y_4x_6$, $x_4y_4x_5y_6x_6$, $x_1y_5x_3y_6x_6$, $x_5y_5x_2y_6x_1$, $x_2y_7x_3y_3x_5$, $x_1y_7x_7y_3x_6$

□

Lemma 3.3. *There exists a* $\{\lambda P_5, \alpha S_5\}$*-decomposition of* H=$\{K_{11,11} \oplus I\} - \{K_{7,7} \oplus I\}$*,where* $\lambda \neq 1$.

Proof. Let $V(H)$=$\{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}\}$∪$\{y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, y_9, y_{10}, y_{11}\}$. The $(\lambda P_5, \alpha S_5) - decomposition$ of H graph is obtained using Constructions and , as follows..

(1)λ=0 and α=19. The required stars are
$(x_{11} : y_5y_6y_7y_{11})$, $(x_{10} : y_5y_6y_7y_{10})$, $(x_{11} : y_4y_3y_2y_1)$, $(x_{10} : y_4y_3y_2y_1)$ $(x_9 : y_5y_6y_7y_9)$ $(x_9 : y_1y_2y_3y_4)$ $(x_8 : y_5y_6y_7y_8)$ $(x_8 : y_1y_2y_3y_4)$ $(x_7 : y_8y_9y_{10}y_{11})$, $(x_6 : y_8y_9y_{10}y_{11})$, $(x_5 : y_8y_9y_{10}y_{11})$, $(x_4 : y_8y_9y_{10}y_{11})$, $(x_3 : y_8y_9y_{10}y_{11})$, $(x_2 : y_8y_9y_{10}y_{11})$, $(x_1 : y_8y_9y_{10}y_{11})$, $(x_8 : y_8y_9y_{10}y_{11})$, $(x_9 : y_8y_9y_{10}y_{11})$, $(x_{10} : y_8y_9y_{10}y_{11})$, $(x_{11} : y_8y_9y_{10}y_{11})$

(2)λ=2 and α=17. The required stars and paths are
$(x_{11} : y_8y_9y_{10}y_{11})$, $(x_9 : y_8y_9y_{10}y_{11})$, $(x_8 : y_8y_9y_{10}y_{11})$, $(x_{10} : y_8y_9y_{10}y_{11})$, $(x_1 : y_8y_9y_{10}y_{11})$, $(x_4 : y_8y_9y_{10}y_{11})$, $(x_5 : y_8y_9y_{10}y_{11})$, $(x_6 : y_8y_9y_{10}y_{11})$, $(x_7 : y_8y_9y_{10}y_{11})$, $(x_8 : y_1y_2y_3y_4)$, $(x_8 : y_5y_6y_7y_8)$, $(x_9 : y_1y_2y_3y_4)$, $(x_9 : y_5y_6y_7y_9)$, $(x_{10} : y_1y_2y_3y_4)$, $(x_{10} : y_5y_6y_7y_{10})$, $(x_{11} : y_1y_2y_3y_4)$, $(x_{11} : y_5y_6y_7y_{11})$, $y_8x_2y_9x_3y_{10}$, $y_8x_3y_{11}x_2y_{10}$

(3)λ=3 and α=16. The required stars and paths are

$(x_1 : y_8y_9y_{10}y_{11})$, $(x_2 : y_8y_9y_{10}y_{11})$, $(x_3 : y_8y_9y_{10}y_{11})$, $(x_4 : y_8y_9y_{10}y_{11})$, $(x_5 : y_8y_9y_{10}y_{11})$, $(x_6 : y_8y_9y_{10}y_{11})$, $(x_7 : y_8y_9y_{10}y_{11})$, $(x_8 : y_8y_9y_{10}y_{11})$, $(x_9 : y_8y_9y_{10}y_{11})$, $(x_{10} : y_8y_9y_{10}y_{11})$, $(x_{11} : y_8y_9y_{10}y_{11})$, $(x_8 : y_1y_2y_3y_4)$, $(x_9 : y_1y_2y_3y_4)$, $(x_{10} : y_1y_2y_3y_4)$, $(x_{11} : y_1y_2y_3y_4)$, $(x_{11} : y_5y_6y_7y_{11})$, $y_{10}x_{10}y_7x_8y_8$, $y_9x_9y_5x_8y_6$, $y_5x_{10}y_6x_9y_7$

(4)λ=4 and α=15. The required stars and paths are
$(x_{11} : y_8y_9y_{10}y_{11})$, $(x_9 : y_8y_9y_{10}y_{11})$, $(x_8 : y_8y_9y_{10}y_{11})$, $(x_{10} : y_8y_9y_{10}y_{11})$, $(x_5 : y_8y_9y_{10}y_{11})$, $(x_6 : y_8y_9y_{10}y_{11})$, $(x_7 : y_8y_9y_{10}y_{11})$, $(x_8 : y_1y_2y_3y_4)$, $(x_8 : y_5y_6y_7y_8)$, $(x_9 : y_1y_2y_3y_4)$, $(x_9 : y_5y_6y_7y_9)$, $(x_{10} : y_1y_2y_3y_4)$, $(x_{10} : y_5y_6y_7y_{10})$, $(x_{11} : y_1y_2y_3y_4)$, $(x_{11} : y_5y_6y_7y_{11})$, $y_8x_2y_9x_3y_{10}$, $y_8x_3y_{11}x_2y_{10}$, $y_{11}x_4y_{10}x_1y_9$, $y_9x_4y_8x_1y_{11}$

(5)λ=5 and α=14. The required stars and paths are
$(x_3 : y_8y_9y_{10}y_{11})$, $(x_4 : y_8y_9y_{10}y_{11})$, $(x_5 : y_8y_9y_{10}y_{11})$, $(x_6 : y_8y_9y_{10}y_{11})$, $(x_7 : y_8y_9y_{10}y_{11})$, $(x_8 : y_8y_9y_{10}y_{11})$, $(x_9 : y_8y_9y_{10}y_{11})$, $(x_{10} : y_8y_9y_{10}y_{11})$, $(x_{11} : y_8y_9y_{10}y_{11})$, $(x_8 : y_1y_2y_3y_4)$, $(x_9 : y_1y_2y_3y_4)$, $(x_{10} : y_1y_2y_3y_4)$, $(x_{11} : y_1y_2y_3y_4)$, $(x_{11} : y_5y_6y_7y_{11})$, $y_{10}x_{10}y_7x_8y_8$, $y_9x_9y_5x_8y_6$, $y_5x_{10}y_6x_9y_7$, $y_8x_1y_9x_2y_{10}$, $y_8x_2y_{11}x_1y_{10}$

(6)λ=6 and α=13. The required stars and paths are
$(x_{11} : y_8y_9y_{10}y_{11})$ $(x_9 : y_8y_9y_{10}y_{11})$ $(x_8 : y_8y_9y_{10}y_{11})$ $(x_{10} : y_8y_9y_{10}y_{11})$ $(x_7 : y_8y_9y_{10}y_{11})$ $(x_8 : y_1y_2y_3y_4)$ $(x_8 : y_5y_6y_7y_8)$ $(x_9 : y_1y_2y_3y_4)$ $(x_9 : y_5y_6y_7y_9)$ $(x_{10} : y_1y_2y_3y_4)$ $(x_{10} : y_5y_6y_7y_{10})$ $(x_{11} : y_1y_2y_3y_4)$ $(x_{11} : y_5y_6y_7y_{11})$ $y_8x_2y_9x_3y_{10}$ $y_8x_3y_{11}x_2y_{10}$ $y_{11}x_4y_{10}x_1y_9$ $y_9x_4y_8x_1y_{11}$ $y_8x_5y_9x_6y_{10}$ $y_{10}x_5y_{11}x_6y_8$

(7)λ=7 and α=12.the required stars and paths are
$(x_5 : y_8y_9y_{10}y_{11})$ $(x_6 : y_8y_9y_{10}y_{11})$ $(x_7 : y_8y_9y_{10}y_{11})$ $(x_8 : y_8y_9y_{10}y_{11})$ $(x_9 : y_8y_9y_{10}y_{11})$ $(x_{10} : y_8y_9y_{10}y_{11})$ $(x_{11} : y_8y_9y_{10}y_{11})$ $(x_8 : y_1y_2y_3y_4)$ $(x_9 : y_1y_2y_3y_4)$ $(x_{10} : y_1y_2y_3y_4)$ $(x_{11} : y_1y_2y_3y_4)$ $(x_{11} : y_5y_6y_7y_{11})$ $y_{10}x_{10}y_7x_8y_8$ $y_9x_9y_5x_8y_6$ $y_5x_{10}y_6x_9y_7$ $y_8x_1y_9x_2y_{10}$ $y_8x_2y_{11}x_1y_{10}$ $y_8x_3y_9x_4y_{10}$ $y_8x_4y_{11}x_3y_{10}$

(8)λ=8 and α=11. The required stars and paths are
$(x_{11} : y_8y_9y_{10}y_{11})$ $(x_9 : y_8y_9y_{10}y_{11})$ $(x_8 : y_8y_9y_{10}y_{11})$ $(x_{10} : y_8y_9y_{10}y_{11})$ $(x_7 : y_8y_9y_{10}y_{11})$ $(x_8 : y_5y_6y_7y_8)$ $(x_9 : y_5y_6y_7y_9)$ $(x_{10} : y_1y_2y_3y_4)$ $(x_{10} : y_5y_6y_7y_{10})$ $(x_{11} : y_1y_2y_3y_4)$ $(x_{11} : y_5y_6y_7y_{11})$ $y_8x_2y_9x_3y_{10}$ $y_8x_3y_{11}x_2y_{10}$ $y_{11}x_4y_{10}x_1y_9$ $y_9x_4y_8x_1y_{11}$ $y_8x_5y_9x_6y_{10}$ $y_{10}x_5y_{11}x_6y_8$ $y_3x_9y_2x_8y_1$ $y_1x_9y_4x_8y_3$

(9)λ=9 and α=10. The required stars and paths are
$(x_7 : y_8y_9y_{10}y_{11})$ $(x_8 : y_8y_9y_{10}y_{11})$ $(x_9 : y_8y_9y_{10}y_{11})$ $(x_{10} : y_8y_9y_{10}y_{11})$ $(x_{11} : y_8y_9y_{10}y_{11})$ $(x_8 : y_1y_2y_3y_4)$ $(x_9 : y_1y_2y_3y_4)$ $(x_{10} : y_1y_2y_3y_4)$ $(x_{11} : y_1y_2y_3y_4)$ $(x_{11} : y_5y_6y_7y_{11})$ $y_{10}x_{10}y_7x_8y_8$ $y_9x_9y_5x_8y_6$ $y_5x_{10}y_6x_9y_7$ $y_8x_1y_9x_2y_{10}$ $y_8x_2y_{11}x_1y_{10}$ $y_8x_3y_9x_4y_{10}$ $y_8x_4y_{11}x_3y_{10}$ $y_8x_5y_9x_6y_{10}$ $y_{10}x_5y_{11}x_6y_8$

(10)λ=10 and α=9. The required stars and paths are
$(x_{11} : y_8y_9y_{10}y_{11})$, $(x_9 : y_8y_9y_{10}y_{11})$, $(x_8 : y_8y_9y_{10}y_{11})$, $(x_{10} : y_8y_9y_{10}y_{11})$, $(x_7 : y_8y_9y_{10}y_{11})$, $(x_8 : y_5y_6y_7y_8)$, $(x_9 : y_5y_6y_7y_9)$, $(x_{10} : y_5y_6y_7y_{10})$, $(x_{11} : y_5y_6y_7y_{11})$, $y_8x_2y_9x_3y_{10}$, $y_8x_3y_{11}x_2y_{10}$, $y_{11}x_4y_{10}x_1y_9$, $y_9x_4y_8x_1y_{11}$, $y_8x_5y_9x_6y_{10}$, $y_{10}x_5y_{11}x_6y_8$, $y_3x_9y_2x_8y_1$, $y_1x_9y_4x_8y_3$, $y_1x_{10}y_2x_{11}y_3$, $y_1x_{11}y_4x_{10}y_3$

(11)λ=11 and α=8. The required stars and paths are
$(x_9 : y_8y_9y_{10}y_{11})$, $(x_{10} : y_8y_9y_{10}y_{11})$, $(x_{11} : y_8y_9y_{10}y_{11})$, $(x_8 : y_1y_2y_3y_4)$, $(x_9 : y_1y_2y_3y_4)$, $(x_{10} : y_1y_2y_3y_4)$, $(x_{11} : y_1y_2y_3y_4)$, $(x_{11} : y_5y_6y_7y_{11})$, $y_{10}x_{10}y_7x_8y_8$, $y_9x_9y_5x_8y_6$, $y_5x_{10}y_6x_9y_7$, $y_8x_1y_9x_2y_{10}$, $y_8x_2y_{11}x_1y_{10}$, $y_8x_3y_9x_4y_{10}$, $y_8x_4y_{11}x_3y_{10}$, $y_8x_5y_9x_6y_{10}$, $y_{10}x_5y_{11}x_6y_8$, $y_{10}x_8y_9x_7$, y_8 $y_{10}x_7y_{11}x_8y_8$

(12)λ=12 and α=7. The required stars and paths are
$(x_{11} : y_8y_9y_{10}y_{11})$, $(x_9 : y_8y_9y_{10}y_{11})$, $(x_8 : y_8y_9y_{10}y_{11})$, $(x_{10} : y_8y_9y_{10}y_{11})$, $(x_7 : y_8y_9y_{10}y_{11})$, $(x_{10} : y_5y_6y_7y_{10})$, $(x_{11} : y_5y_6y_7y_{11})$, $y_8x_2y_9x_3y_{10}$, $y_8x_3y_{11}x_2y_{10}$, $y_{11}x_4y_{10}x_1y_9$, $y_9x_4y_8x_1y_{11}$, $y_8x_5y_9x_6y_{10}$, $y_{10}x_5y_{11}x_6y_8$, $y_3x_9y_2x_8y_1$,$y_1x_9y_4x_8y_3$,

$y_1x_{10}y_2x_{11}y_3$, $y_1x_{11}y_4x_{10}y_3$ $y_9x_9y_7x_8y_6$, $y_8x_8y_5x_9y_6$

(13)λ=13 and α=6. The required stars and paths are
$(x_{11}: y_8y_9y_{10}y_{11})$, $(x_8: y_1y_2y_3y_4)$, $(x_9: y_1y_2y_3y_4)$, $(x_{10}: y_1y_2y_3y_4)$, $(x_{11}: y_1y_2y_3y_4)$, $(x_{11}: y_5y_6y_7y_{11})$, $y_{10}x_{10}y_7x_8y_8$, $y_9x_9y_5x_8y_6$, $y_5x_{10}y_6x_9y_7$, $y_8x_1y_9x_2y_{10}$, $y_8x_2y_{11}x_1y_{10}$, $y_8x_3y_9x_4y_{10}$, $y_8x_4y_{11}x_3y_{10}$, $y_8x_5y_9x_6y_{10}$, $y_{10}x_5y_{11}x_6y_8$, $y_{10}x_8y_9x_7$, y_8 $y_{10}x_7y_{11}x_8y_8$, $y_8x_9y_9x_{10}y_{10}$, $y_8x_{10}y_{11}x_9y_{10}$

(14)λ=14 and α=5. The required stars and paths are
$(x_{11}: y_8y_9y_{10}y_{11})$, $(x_9: y_8y_9y_{10}y_{11})$, $(x_8: y_8y_9y_{10}y_{11})$, $(x_{10}: y_8y_9y_{10}y_{11})$, $(x_7: y_8y_9y_{10}y_{11})$, $y_8x_2y_9x_3y_{10}$, $y_8x_3y_{11}x_2y_{10}$, $y_{11}x_4y_{10}x_1y_9$, $y_9x_4y_8x_1y_{11}$, $y_8x_5y_9x_6y_{10}$, $y_{10}x_5y_{11}x_6y_8$, $y_3x_9y_2x_8y_1$, $y_1x_9y_4x_8y_3$, $y_1x_{10}y_2x_{11}y_3$, $y_1x_{11}y_4x_{10}y_3$, $y_9x_9y_7x_8y_6$, $y_8x_8y_5x_9y_6$, $y_{11}x_{11}y_9x_{10}y_6$, $y_{10}x_{10}y_5x_{11}y_6$

(15)λ=15 and α=4. The required stars and paths are
$(x_{11}: y_8y_9y_{10}y_{11})$, $(x_{10}: y_1y_2y_3y_4)$, $(x_{11}: y_1y_2y_3y_4)$, $(x_{11}: y_5y_6y_7y_{11})$, $y_{10}x_{10}y_7x_8y_8$, $y_9x_9y_5x_8y_6$, $y_5x_{10}y_6x_9y_7$, $y_8x_1y_9x_2y_{10}$, $y_8x_2y_{11}x_1y_{10}$, $y_8x_3y_9x_4y_{10}$, $y_8x_4y_{11}x_3y_{10}$, $y_8x_5y_9x_6y_{10}$, $y_{10}x_5y_{11}x_6y_8$, $y_{10}x_8y_9x_7$ y_8, $y_{10}x_7y_{11}x_8y_8$, $y_8x_9y_9x_{10}y_{10}$, $y_8x_{10}y_{11}x_9y_{10}$, $y_1x_8y_2x_9y_3$, $y_1x_9y_4x_8y_3$

(16)λ=16 and α=3. The required stars and paths are
$(x_{11}: y_8y_9y_{10}y_{11})$, $(x_{10}: y_8y_9y_{10}y_{11})$, $(x_7: y_8y_9y_{10}y_{11})$, $y_8x_2y_9x_3y_{10}$, $y_8x_3y_{11}x_2y_{10}$, $y_{11}x_4y_{10}x_1y_9$, $y_9x_4y_8x_1y_{11}$, $y_8x_5y_9x_6y_{10}$, $y_{10}x_5y_{11}x_6y_8$, $y_3x_9y_2x_8y_1$, $y_1x_9y_4x_8y_3$, $y_1x_{10}y_2x_{11}y_3$, $y_1x_{11}y_4x_{10}y_3$, $y_9x_9y_7x_8y_6$, $y_8x_8y_5x_9y_6$, $y_{11}x_{11}y_9x_{10}y_6$, $y_{10}x_{10}y_5x_{11}y_6$, $y_8x_8y_9x_9y_{10}$, $y_8x_9y_{11}x_8y_{10}$

(17)λ=17 and α=2. The required stars and paths are
$(x_{11}: y_8y_9y_{10}y_{11})$, $(x_{11}: y_5y_6y_7y_{11})$, $y_{10}x_{10}y_7x_8y_8$, $y_9x_9y_5x_8y_6$, $y_5x_{10}y_6x_9y_7$, $y_8x_1y_9x_2y_{10}$, $y_8x_2y_{11}x_1y_{10}$, $y_8x_3y_9x_4y_{10}$, $y_8x_4y_{11}x_3y_{10}$, $y_8x_5y_9x_6y_{10}$, $y_{10}x_5y_{11}x_6y_8$, $y_{10}x_8y_9x_7$ y_8, $y_{10}x_7y_{11}x_8y_8$, $y_8x_9y_9x_{10}y_{10}$, $y_8x_{10}y_{11}x_9y_{10}$, $y_1x_8y_2x_9y_3$, $y_1x_9y_4x_8y_3$, $y_1x_{11}y_4x_{10}y_3$, $y_1x_{10}y_2x_{11}y_3$

(18)λ=18 and α=1. The required stars and paths are
$(x_7: y_8y_9y_{10}y_{11})$, $y_8x_2y_9x_3y_{10}$, $y_8x_3y_{11}x_2y_{10}$, $y_{11}x_4y_{10}x_1y_9$, $y_9x_4y_8x_1y_{11}$, $y_8x_5y_9x_6y_{10}$, $y_{10}x_5y_{11}x_6y_8$, $y_3x_9y_2x_8y_1$, $y_1x_9y_4x_8y_3$, $y_1x_{10}y_2x_{11}y_3$, $y_1x_{11}y_4x_{10}y_3$, $y_9x_9y_7x_8y_6$, $y_8x_8y_5x_9y_6$, $y_{11}x_{11}y_9x_{10}y_6$, $y_{10}x_{10}y_5x_{11}y_6$, $y_8x_8y_9x_9y_{10}$, $y_8x_9y_{11}x_8y_{10}$, $y_8x_{10}y_9x_{11}y_{10}$, $y_8x_{11}y_{11}x_{10}y_{10}$

(19)λ=19 and α=0. The required stars and paths are
$x_8y_1x_9y_2x_{10}$, $x_8y_2x_{11}y_1x_{10}$, $x_8y_4x_{11}y_3x_{10}$, $x_8y_3x_9y_4x_{10}$, $x_8y_6x_{11}y_5x_{10}$, $x_8y_5x_9y_6x_{10}$, $x_8y_8x_{11}y_7x_{10}$, $x_8y_7x_9y_8x_{10}$, $x_1y_8x_2y_9x_3$, $x_1y_9x_4y_8x_3$, $x_1y_{11}x_4y_{10}x_3$, $x_1y_{10}x_2y_{11}x_3$, $y_9x_6y_{11}x_{10}y_{10}$, $y_9x_{10}y_{10}x_{11}y_{11}$, $y_{11}x_7y_9x_9y_{10}$, $x_8y_{11}x_5y_8x_6$, $x_8y_8x_7y_{10}x_6$, $x_5y_9x_9y_{11}x_{11}$, $x_{11}y_9x_8y_{10}x_5$

□ □

Lemma 3.4. *There exists a $\{\lambda P_5, \alpha S_5\}$-decomposition of $K_{4,4}$, where $\lambda \neq 1$.*

Proof. Let $V(K_{4,4})$=$\{x_1, x_2, x_3, x_4\} \cup \{y_1, y_2, y_3, y_4\}$. The $\{\lambda P_5, \alpha S_5\}$-decomposition $K_{4,4}$ is obtained using Constructions and , as follows.

(i) λ=0 and α=4. The required stars are
$(x_1: y_1y_2y_3y_4)$, $(x_2: y_1y_2y_3y_4)$, $(x_3: y_1y_2y_3y_4)$, $(x_4: y_1y_2y_3y_4)$

(ii) λ=2 and α=2. The required paths and stars are
$y_2x_1y_1x_2y_3$, $y_2x_2y_4x_1y_3$, $(x_3: y_1y_2y_3y_4)$, $(x_4: y_1y_2y_3y_4)$

(iii) λ=3 and α=1. The required paths and stars are
$y_1x_1y_4x_3y_3$, $y_2x_1y_4x_3y_3$, $y_3x_1y_2x_2y_4$, $(x_4 : y_1y_2y_3y_4)$

(iv) λ=4 and α=0. The required paths are
$y_1x_1y_4x_3y_3$, $y_2x_1y_4x_3y_3$, $y_1x_4y_2x_2y_4$, $y_2x_1y_3x_4y_4$
□

□

4 Sufficient Condition

Lemma 4.1. *Let $m = 4t + 3$ for some integer $t \geq 1$. The graph $K_{m,m} \oplus I$, which can be expressed as $K_{7,7} \oplus I \cup (t-1)(t-2)K_{4,4} \cup (t-1)H$, admits a $(\lambda P_5, \alpha S_5)$-decomposition if and only if the following conditions hold:*

$$\lambda + \alpha = \frac{m(m+1)}{4} \quad \textit{and} \quad \lambda \neq 1.$$

Furthermore, the decomposition can be constructed based on the value of λ using the following cases:

(i) $0 < \lambda \leq 14$ *and* $\lambda \neq 1$,

(ii) $14 < \lambda \leq 4(t-1)(t-2)$,

(iii) $4(t-1)(t-2) < \lambda \leq \frac{m^2+m}{4}$.

Proof. Let $m = 4t+3$ and $K_{m,m} \oplus I = K_{7,7} \oplus I \cup (t-1)(t-2)K_{4,4} \cup (t-1)H$. We construct the required decomposition in three cases.
Case(i).$0 < \lambda \leq 14$ and $\lambda \neq 1$.
By Lemma 3.2, we have λP_5 and $(14-\lambda)S_5$ from the graph $K_{7,7} \oplus I$. By Lemma 3.3 and Lemma 3.4 both the graphs H and $K_{4,4}$ have a S_5-decomposition.
Case(ii). $14 < \lambda \leq 4(t-1)(t-2)$.
By Lemma 3.2, we have $14P_5$ from the graph $K_{7,7} \oplus I$. By Lemma 3.4, we have λP_5 and $(4(t-1)(t-2)-\lambda)S_5$ from the graph $(t-1)(t-2)K_{4,4}$. By Lemma 3.3 the graph H has a S_5-decomposition.
Case(iii).$4(t-1)(t-2) < \lambda \leq \frac{m^2+m}{4}$.
By Lemma 3.2 and Lemma 3.4 , we have $4(t-1)(t-2)P_5$ from the graphs $K_{7,7} \oplus I$ and $(t-1)(t-2)K_{4,4}$. By Lemma 3.3 , we have required copies of P_5 from the graph $(t-1)H$. Hence by Remark 1, the graph $K_{m,m} \oplus I = K_{7,7} \oplus I \cup (t-1)(t-2)K_{4,4} \cup (t-1)H$ has the desired decomposition. □ □

5 Conclusion

In this paper, we investigated about the existence of a fully $\{\lambda P_5, \alpha S_5\}$-decomposition. Also, we established necessary and sufficient conditions for the existence of fully $\{\lambda P_5, \alpha S_5\}$-decomposition of $K_{m,m} \oplus I$.

Theorem 5.1. *Let $m \in \mathbb{N}$ with $m \equiv 3 \pmod 4$. Then the graph $K_{m,m} \oplus I$ admits a $(\lambda P_5, \alpha S_5)$-decomposition if and only if λ and α are non-negative integers satisfying $\lambda + \alpha = \dfrac{m(m+1)}{4}$ and $\lambda \neq 1$. This provides a complete characterization of such decompositions based on edge count and vertex degree constraints.* □

References

[1] *A.A.Abueida, M.Daven, Multidesign for graph-pairs of order 4 and 5,* Graphs Combin., **19**, *433–447, (2003).*

[2] *A.A.Abueida, M.Daven, K.J.Roblee, Multidesigns of the $\lambda - fold$ complete graph-pairs of orders 4 and 5,* Australs. J. combin.*,* **32***, 125–136, (2005).*

[3] *A.A.Abueida, M.Daven, Multidecompositions of the complete graph,* Ars Combin.*,* **72***, 17–22, (2004).*

[4] *G.Chartrand, L.Lesniak, Graphs and Digraphs, 2nd edn,* Wadsworth, Belmont*, (1986).*

[5] *M.Ilayaraja, K.Sowndhariya, and A.Muthusamy, Decomposition of complete bipartite graphs into paths and stars on five vertices,* AKCE Int. J. Graphs and Comb.*,* **17***, 777–783, (2020).*

[6] *J.A.Bondy, U.R.S.Murry,* Graph Theory with Applications. *(The Macmillan Press Ltd, New York, 1976).*

[7] *Jenq-Jong Lin, Hung-Chih Lee, Decomposition of the complete bipartite graph with a 1-factor removed into paths and stars,* Contrib.Discrete.Math.*,* **13***, 63–73, (2018).*

[8] *S. Jeevadoss, A. Muthusamy, Decomposition of complete bipartite graphs into paths and cycles,* Discrete Math.*,* **331***, 98–108, (2014).*

[9] *V.Nalini, S. Jeevadoss, Decomposition of the complete bipartite graphs plus a $1-$factor $K_{m,m} \oplus I$ into paths and cycles of length four,* AIP Conf. Proc.*,* **2850***, 060010, (2024).*

[10] *Pauline Cain, Decomposition of complete bipartite graphs into stars,* Bull.Austral.Math.Soc*,* **10***, 23-30, (1974).*

[11] *T.W.Shyu, Decomposition of complete graphs into paths and stars,* Discrete.Math.*,* **310***, 2164-2169, (2010).*

[12] *T.W.Shyu, Decomposition of complete bipartite graphs into paths and stars with same number of edges,* Discrete.Math. ,**313(7)***, 865–871, (2013).*

[13] *T.W.Shyu, Decompositions of the complete graphs and complete bipartite graphs into paths, stars, and cycles with four edge each,* Discuss.Math.Graph Theory.*,* **41(2)***, 451–468, (2021).*

[14] *Yang Zhao, Baoyindureng Wu, Star decomposition of graphs,* Discrete Math. Algorithms Appl.*,* **7(2)***, 1550016-1 - 1550016-9, (2015).*

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