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Grundy Coloring on Families of Tadpole Graphs

K. Annathurai, P. Periasamy and V. Sankar Raj

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Corresponding Author: K. Annathurai

Abstract A coloring of a graph G is a proper vertex coloring of G (whose colors, as usual, are positive integers) having the property that for every two colors i and j with $i < j$, every vertex colored j has a neighbor colored i. The current study establishes the Grundy coloring of some basic graphs. Also, the Grundy chromatic number is obtained for the families of tadpole graph such as line graph of a tadpole graph, middle graph of a tadpole graph, total graph of a tadpole graph and Mycielskian graph of a tadpole graph.

1 Introduction

Graph Theory is acknowledged as one of the most significant and lively fields in mathematics. The discipline of graph theory in mathematics was created in 1736 with Eulers' work on the long-pending Konigsberg bridge problem. Euler solved this particular problem and became known as the Father of Graph Theory. Since then, graph theory has expanded into a broad field with applications in a variety of fields. A graph is a symbolic representation of a set of things in which each pair of objects is connected by connections. The interrelated items are represented by mathematical abstractions known as vertices, and the ties that connect certain pairs of vertices are known as edges. Graph theory is used in a variety of fields, including social sciences, linguistics, physical sciences, communication engineering, and others. A detailed view about graph theory and their applications, one can refer [1, 4, 6, 11, 16].

A 1939 article by Patrick Michael Grundy (1917-1959) dealing with combinatorial games contained ideas that led to the concept of Grundy colorings of graphs. A Grundy coloring of a graph G is a proper vertex coloring of G (whose colors, asusual, are positive integers) having the property that for every two colors i and j with $i < j$, every vertex colored j has a neighbor colored i. Consequently, every Grundy coloring is a complete coloring. Greedy coloring c of a graph G is obtained from an ordering $\phi : v_1, v_2, ..., v_n$ of the vertices of G in some manner, by defining $c(v_1) = 1$, and once colors have been assigned to $v_1, v_2, ..., v_t$ for some integer t with $1 \leq t < n, c(v_{t+1})$ is defined as the smallest color not assigned to any neighbor of v_{t+1} belonging to the set $v_1, v_2, ..., v_t$. The coloring c so produced is then a Grundy coloring of G. That is, every greedy coloring is a Grundy coloring. The maximum positive integer k for which a graph G has a Grundy $k-$coloring is denoted by $\Gamma(G)$ and is called the Grundy chromatic number of G or more simply the Grundy number of G. If the Grundy number of a graph G is k, then in any Grundy $k-$coloring of G (using the colors $1, 2, \dots, k$), every vertex v of G colored k must be adjacent to a vertex colored i for each integer i with $1 \leq i \leq k$. Thus $\Delta(G) \geq deg(v) \geq (k-1)$ and so $\Gamma(G) \leq \Delta(G)+1$ for every graph G. Since every Grundy coloring of a graph G is a proper coloring, it follows that, $X(G) \leq \Gamma(G)$. C.A. Christen and S.M. Selkow [2] has obtained the results on some perfect coloring properties of graphs. Results on equitable coloring Helm graph, Gear graphs, corona of wheels and Sunlet families of graphs are obtained in [12, 13, 19]. G. J. Simmons [15] has analysed the chromatic number of a graph. M. Venkatachalam, J. Vernold Vivin and K. Kaliraj [17] has found a new results on Harmonious Coloring on Double Star Graph Families. Papers on Grundy coloring, we can refer these papers [3, 5, 8, 10, 20].

2 Basic Results

Definition 2.1. [13] The $n-$sunlet graph is the graph on $2n$ vertices obtained by attaching n pendant edges to a cycle graph C_n.

Definition 2.2. [9] The friendship graph (or Dutch windmill graph or n-fan) F_n is a planar, undirected graph with $2n+1$ vertices and $3n$ edges. The friendship graph F_n can be constructed by joining n copies of the cycle graph C_3 with a common vertex, which becomes a universal vertex for the graph.

Definition 2.3. [7] The *derived graph* of G, denoted by G^+ (also denoted by $[G]^\dagger$ in some works), is the graph with vertex set $V(G)$ such that two distinct vertices $u, v \in V(G)$ are adjacent in G^+ if and only if their distance in G is exactly 2.

Definition 2.4. [14] The (m, n)-tadpole graph is a special type of graph consisting of a cycle graph on m (at least 3) vertices and a path graph on n vertices, connected with a bridge. It is denoted as $T_{m,n}$.

Definition 2.5. [13] The Line Graph of a graph G denoted by $L(G)$ is a graph whose vertices are the edges of G and if $u, v \in E(G)$ then $u, v \in E(L(G))$ if u and v share a vertex in G.

Definition 2.6. [13] Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The middle graph of G is denoted by $M(G)$ is defined as follows: The vertex set of $M(G)$ is $V(G) \cup E(G)$ in which two vertices x, y are adjacent in $M(G)$ if the following condition holds:

- $x, y \in E(G)$ and x, y are adjacent in G
- $x \in V(G), y \in E(G)$ and they are incident in G.

Definition 2.7. [13] Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The total graph of G is denoted by $T(G)$ is defined as follows: The vertex set of $T(G)$ is $V(G) \cup E(G)$ in which two vertices x, y are adjacent in $T(G)$ if the following condition holds:

- $x, y \in V(G)$ and x is adjacent to y in G.
- $x, y \in E(G)$ and x, y are adjacent in G.
- $x \in V(G), y \in E(G)$ and they are incident in G.

Definition 2.8. [18] Let the n vertices of the graph G be $v_1, v_2, ..., v_n$. The Mycielski graph $\mu(G)$ contains G itself as a subgraph, together with $n+1$ additional vertices: a vertex u_i corresponding to each vertex v_i of G, and an extra vertex w. Each vertex u_i is connected by an edge to w, so that these vertices form a subgraph in the form of a star $K_{1,n}$. In addition, for each edge v_iv_j of G, the Mycielski graph includes two edges, u_iv_j and v_iu_j.

3 Main Results

Theorem 3.1. *Let H be the path graph with order $m \geq 3$ then,*

$$\Gamma(G^+(P_m)) = \begin{cases} 2 & \textit{if} \quad m \leq 6, \\ 3 & \textit{if} \quad m \geq 7. \end{cases}$$

Proof. Let $V(H) = \{v_i : 1 \leq i \leq m\}$ and $E(H) = \{e_i : 1 \leq i \leq m-1\}$, where e_i denotes the edge v_iv_{i+1}.

Now, we define

$$V(G^+(H)) = V(H),$$

and

$$E(G^+(H)) = \begin{cases} \{\, e_{(2i-1,\,2i+1)} : 1 \leq i \leq \left\lfloor \frac{m-1}{2} \right\rfloor, & \text{if } m \text{ is odd,} \\ \{\, e_{(2i,\,2(i+1))} : 1 \leq i \leq \left\lceil \frac{m-2}{2} \right\rceil, & \text{if } m \text{ is even,} \end{cases}$$

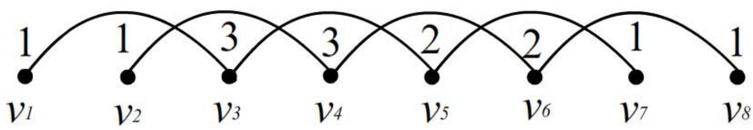

Figure 1. $\Gamma(G^+(P_8))$

where $e_{(i,j)}$ denotes the edge v_iv_j.

Finally, we define the mapping

$$\sigma : V(G^+(H)) \longrightarrow \mathbb{N}.$$

Case (i): Let c_1, c_2 be the distinct colors for the vertices P_m. We assign the color c_1 for v_i, when $i = 4k + 1$ and $4k + 2$ for some integer $k \in \mathbb{Z}++$. Assign the color c_2 for v_i, when $i = 4k + 3$ and $4(k + 1)$. The maximum degree of $V(G^+(H))$ is 2. By the definition of Grundy chromatic it should not be greater than 3 i.e., ($\Gamma(G^+(P_m)) \ngtr 3$). Hence $\Gamma(G^+(P_m)) = 2$.

Case (ii): Let c_1, c_2, c_3 be the distinct colors. If $m = 6k + 2, k = 1, 2, ...$ then we assign the color c_1 for v_i, when $i = 6k + 1$ and $6k + 2$. Assign the color c_2 for v_i when $i = 6k + 5$ and $6(k + 1)$. Assign the color c_3 for v_i, when $i = 6k + 3$ and $6k + 4$ for some integer $k \in \mathbb{Z}++$. If $m = 6k + 3$, $k = 1, 2, ...$ then we change the color c_2 only for v_i, where $i = m$. If $m = 6k + 4$, $k = 1, 2, ...$ then we change the color c_2 only for v_i where $i = m$ and $i = m - 1$. If $m = 6k + 5$, $k = 1, 2, ...$ then we change the color c_1 for v_i, when $i = m$ and c_2 only for v_i, when $i = m - 1$ and $i = m - 2$. If $m = 6(k + 1)$, $k = 1, 2, ...$ then we change the color c_1 for v_i, when $i = m$, $i = m - 1$ and c_2 only for v_i, when $i = m - 2$ and $i = m - 3$. If $m = 6k + 7$, $k = 1, 2, ...$ then we change the color c_1 for v_i, when $i = m$, $i = m - 1$ and c_2 only for v_i, when $i = m - 2$, $i = m - 3$ and c_i for v_i, when $i = m - 4$. The maximum degree of $V(G^+(H))$ is 2. By the definition of Grundy chromatic it should not be greater than 3 i.e., ($\Gamma(G^+(H)) \ngtr 3$). The Grundy chromatic is maximum. Hence $\Gamma(G^+(H) = 3$.

□

Theorem 3.2. *Let H be the cycle graph with order $m \geq 4$ then,*

$$\Gamma(G^+(H)) = \begin{cases} 2 & \textit{if m is divisible by 4}, \\ 3 & \textit{otherwise}. \end{cases}$$

Proof. Let C_m be a cycle graph with m vertices, $V(H) = \{v_i : 1 \leq i \leq m\}$, and edges $E(H) = \{e_i : 1 \leq i \leq m\}$, where e_i connects v_i and v_{i+1} for $1 \leq i \leq m - 1$ and e_m connects v_m and v_1.

Define the derived graph $G^+(H)$ as follows:

$$V(G^+(H)) = V(H),$$

$$E(G^+(C_m)) = \left\{ e_{(2i-1,2j+1)} \;\middle|\; 1 \leq i = j \leq \left\lfloor \frac{m}{2} \right\rfloor \right\} \cup \left\{ e_{(2i,2(j+1))} \;\middle|\; 1 \leq i = j \leq \left\lfloor \frac{m}{2} \right\rfloor \right\}$$

where $e_{(2i-1,2j+1)}$ and $e_{(2i,2(j+1))}$ are edges between vertices v_{2i-1} and v_{2j+1}, and v_{2i} and $v_{2(j+1)}$, respectively, effectively connecting every second vertex in the cycle.

For boundary cases, if $i = m - 1$, include $e_{i,1}$; if $i = m$, include $e_{i,2}$.

Now, define a mapping $\sigma : V(G^+(H)) \to \mathbb{N}$, which assigns a natural number to each vertex of $G^+(H)$.

Case (i): Let c_1, c_2 be the distinct colors for the vertices c_m. We assign the color c_1 for v_i, when $i = 4k + 2$ and $i = 4k + 3$. And we assign the color c_2 for v_i, when $i = 4k + 1$ and $4(k + 1)$ for some integer $k \in \mathbb{Z}++$. The maximum degree of $V(G^+(H))$ is 2. By the definition of Grundy chromatic it should not be greater than 3 i.e., $\Gamma(G^+(H)) \ngtr 3$. Hence $\Gamma(G^+(H)) = 2$.

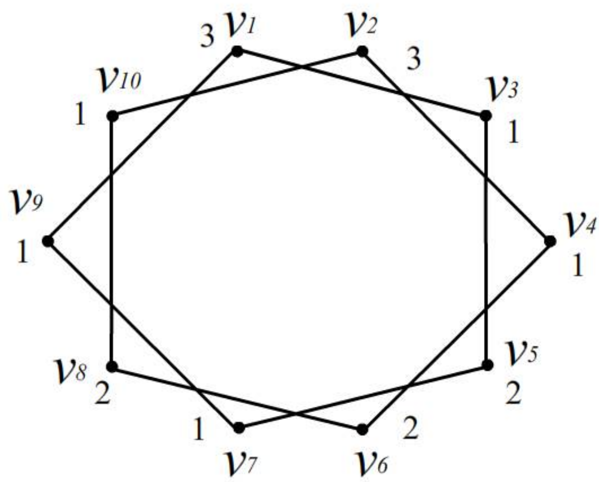

Figure 2. $\Gamma(G^+(C_{10}))$

Case (ii): Let c_1, c_2, c_3 be the distinct colors for the vertices $V(G^+(H))$. We assign the color c_1 for v_i, when $i = 3$ in e_i $i < i+1$ and also assign if $v_i = c_2$ in e_i. Assign the color c_2, if $v_i = 1$ in e_i. And assign the color c_3 for v_i, when $i = 1$ in $e_i i < i+1$. The maximum degree of $V(G^+(H))$ is 2. By the definition of Grundy chromatic it should not be greater than 3 i.e., $\Gamma(G^+(H)) \not> 3$. Assigning color c_3, it satisfies the Grundy coloring definition and Grundy chromatic is maximum. Hence $\Gamma(G^+(H)) = 3$.

□

Theorem 3.3. *Let H be the friendship graph with order $m \geq 2$ then,*

$$\Gamma(G^+(H)) = m.$$

Proof. Let $V(H) = v_i : 1 \leq i \leq 2m+1$ and

$$E(H) = \begin{cases} e_{(i,j)} & \text{i=1}, \quad 2 \leq j \leq 2m+1 \\ e_{(2i,2i+1)} & 1 \leq i \leq m, \end{cases},$$

where $e_{i,j}$ is the edge of $v_i v_{i+1}$. Now, we define $V(G^+(H) = V(H) = (v_1) \cup (v_i : 2 \leq i \leq 2m)$, where v_1 is the common vertex, v_i is the outer vertices. $E(G^+(H) = e_{i,j}, i = n, 2 \leq n \leq 2m+1$, for all j except $(i+1)$ for even and $(i-1)$ for odd, where $e_{(2i,2(i+1))}$ is edge of $v_i v_{i+2}$.
Now, we define the mapping $\sigma : V(G^+(H)) \to \mathbb{N}$. Let c_i, $1 \leq i \leq m$ be the colors for $V(G^+(H))$. Assign the color c_1 for v_i, when $i = 1$. Assign the color c_m for v_i, when $i = 2m$ and $i = 2m+1$. And assign the color c_{m-n} for v_i and v_{i+1} when $i = 2(m-n), 1 \leq n < m$ The maximum degree of $V(G^+(H))$ is $2(m-1)$. So, $\Gamma(G^+(H)) \not> 2(m-1)$. Suppose we assign the colors $m < c_i \leq 2(m-1)$ for the vertices v_i and v_j has same color. It contradicts the proper coloring. So $\Gamma(G^+(H))$ will be less than or equal to m. (i.e) $\Gamma(G^+(H)) \leq m$. But by the definition of Grundy chromatic its maximum Hence, $\Gamma(G^+(H)) = m$. □

Theorem 3.4. *Let H be the sunlet graph with order $m \geq 3$ then,*

$$\Gamma(G^+(H)) = \begin{cases} 3, & \text{if} \quad m \leq 4, \\ 4, & \text{otherwise.} \end{cases}$$

Proof. Let S_m be defined on $2m$ vertices as follows:

$$V(H) = \{v_i : 1 \leq i \leq m\} \cup \{v_j : m+1 \leq j \leq 2m\},$$
$$E(H) = \{v_i v_{i+1} : 1 \leq i \leq m-1\} \cup \{v_m v_1\} \cup \{v_i v_{m+i} : 1 \leq i \leq m\}$$

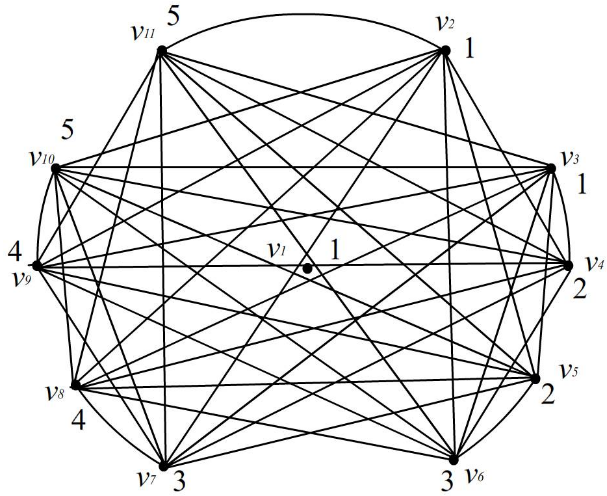

Figure 3. $\Gamma(G^+(F_5))$

where $\{v_i : 1 \leq i \leq m\}$ are the vertices of the cycle C_m, and each v_i is connected to a unique pendant vertex v_{m+i} for $1 \leq i \leq m$.

We now define the derived graph $G^+(H)$ as:

$$V(G^+(H)) = V(H) = \{v_i : 1 \leq i \leq m\} \cup \{v_j : m+1 \leq j \leq 2m\}$$

where v_i are the cycle vertices, and v_{m+i} are their corresponding pendant vertices.

The edge set of $G^+(H)$ is then:

$$E(G^+(H)) = \{v_i v_{i+1} : 1 \leq i \leq m-1\} \cup \{v_m v_1\} \cup \{v_i v_{m+i} : 1 \leq i \leq m\}$$

That is, $G^+(H)$ consists of a cycle on m vertices, each having one pendant edge.

Now, define a mapping $\sigma : V(G^+(H)) \to \mathbb{N}$.

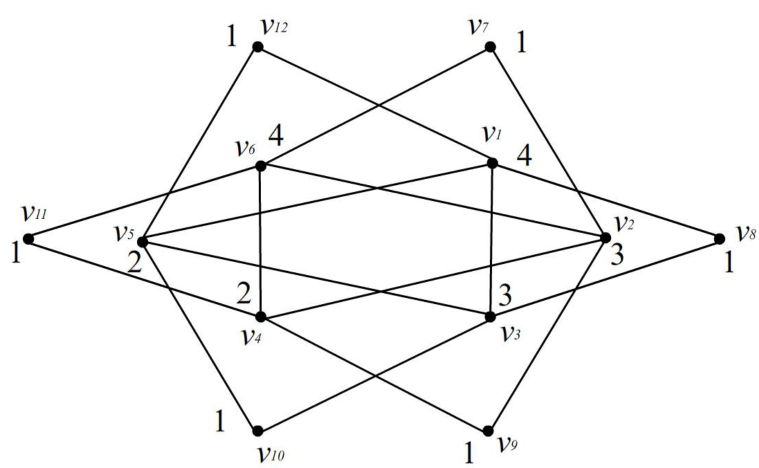

Figure 4. $\Gamma(G^+(S_6))$

Case (i): Let c_1, c_2, c_3 be the distinct colors for the vertices $v(G^+(H))$. In this proof, we assign the colors by two cases, first if m is odd then we assign the color c_1 for v_i, when $i = 3k+1$. Assign the color c_2 for v_i, when $i = 3k+2$. And assign the color c_3 for v_i, when $i = 3(k+1)$. If m is even then we assign the color c_1 for v_i, when $i = 4k+2$. Assign the color c_2 for v_i, when $i = 4(k+1)$. And assign the color c_3 for v_i, when $i = 1, 3, 5, 7$ for some integer $k \in \mathbb{Z}++$. The maximum degree of $V(G^+(H))$ is $m-1$. By the definition of Grundy chromatic it should not be greater than m here $3 \leq m \leq 4$. (i.e.) $\Gamma(G^+(H)) \ngtr m$. Suppose, we assign 4 color then there doesn't exist $v_iv_j, i<j$ for $v_i, 1 \leq i \leq 3$. The only possible way is $\Gamma(G^+(H)) \leq 3$. So, we assign color 3, which satisfies the Grundy coloring. By the definition of Grundy chromatic $\Gamma(G^+(H)) = 3$.

Case (ii): Let c_1, c_2, c_3, c_4 be the distinct colors. If $m = 4k+1$ then we assign the colors by c_1 for v_i, when $m+1 \leq i \leq 2m$. Assign color c_2 for v_i, when $i = 4k$ and $i = 4k+1$. Assign the color c_3 for v_i, when $i = 4k-1$ and $i = 4k-2$. And assign the color c_4 for v_i, when $i = 1$. If $m = 4k+2$ then only change the color c_4 for v_i, when $i = 1$ and $i = m$. If $m = 4k+3$ then assign the color c_1 for v_i, when $m+1 \leq i \leq 2m$. Assign color c_2 for v_i, when $i = 4k-2$ and $i = 4k+1$. Assign the color c_3 for v_i, when $i = 4k-1$ and $i = 4k$. And assign the color c_4 for v_i, when $i = 1$. If $m = 4k$ (where $k = 2, 3, ...$) then we assign the color c_1 for v_i, when $m+1 \leq i \leq 2m$. Assign color c_2 for v_i, when $i = 6k-1$ and $i = 4k-2$. Assign the color c_3 for v_i, when $i = 4k-1$ and $i = 4k$. And assign the color c_4 for v_i, when $i = 1$ and $i = m-3$ for some integer $k \in \mathbb{N}$. The maximum degree of $V(G^+(H))$ is 4. By the definition of Grundy chromatic, it should not be greater than 5 (i.e.) $\Gamma(G^+(H)) \ngtr 5$. Suppose we assign color 5 then there doesn't exist $v_iv_j, i<j$ for the vertex set 5. So, $\Gamma(G^+(H)) \neq 5$. Now, we assign the color 4 which satisfies the Grundy coloring. The Grundy chromatic is maximum. Hence, $\Gamma(G^+(H)) = 4$.

□

Theorem 3.5. *Let H be the Tadpole graph with order $(m \geq 3, n \geq 1)$ then,*

$$\Gamma(G^+(H)) = \begin{cases} 2, & \text{if } m=3 \text{ and } \quad n<5, \\ 3, & \text{if } m \text{ is divisible by } 4, \\ 4, & \quad\quad \text{otherwise}. \end{cases}$$

Proof. Let $V(H) = (v_i : 1 \leq i \leq m) \cup (u_i : 1 \leq j \leq n)$ and $V(G^+(T_{m,n}) = V(H)$, where v_i is the vertices of cycle, u_i is the vertices of path. Let the path be connected to the cycle by the vertex u_1. By the definition of derived graph we connect the vertices where the distances is 2 between the vertices.
Now we define the mapping $\sigma : V(G^+(H)) \to \mathbb{N}$.

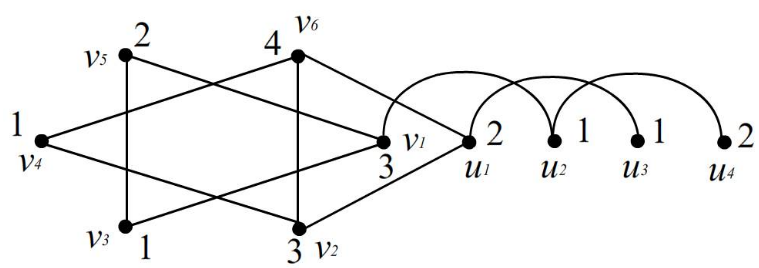

Figure 5. $\Gamma(G^+(T_{6,4}))$

Case (i): Let c_1, c_2 be the distinct colors. If $m = 3$ and $n < 5$ then we assign the colors by c_1 for v_i and u_i, where $1 \leq i \leq 3$ and $3 \leq i \leq 4$. Assign the color c_2 for u_i, where $2 \leq i \leq 3$. The maximum degree is 2 for this case. Suppose, we assign $d+1$ colors then there doesn't exist $v_i > v_j$ for the color 2 in v_2. So,$\Gamma(G^+(H)) = 2$.

Case (ii): Let c_1, c_2, c_3 be the distinct colors. If m is divisible by 4 then we assign the color c_1 for v_i, when $i = 4(k+1)$ and $i = 4k-1$ (for some integer $k \in \mathbb{N}$). Assign the color c_2 for v_i when $i = 4k+1$ and $i = 4k+2$. Assign the color c_1 for u_i, when $i = 4k+3$ and $i = 4k+2$. Assign the color c_2 for u_i, when $i = 4(k+1)$ and $i = 4k+5$. c_3 for u_i, when

$i = 3$. If $n = 6k + 3$ then assign c_3 for u_i when $i = 6k + 1$ and $i = 6k + 2$ for some integer $k \in \mathbb{Z}++$. The maximum degree in this case is 3. So, first we assign $d + 1 = c_4$ color then there doesn't exist $v_i > v_j$ for the color c_2 and c_3. So,$\Gamma(G^+(H)) \neq 4$. Now we assign color 3 which satisfies the Grundy coloring. By the definition of Grundy chromatic $\Gamma(G^+(H)) = 3$.

Case (iii): Let c_1, c_2, c_3, c_4 be the distinct colors. We assign the color c_1 for u_i, when $i = 4k+1$ and $i = 4k + 2$. We assign the color c_2 for u_i, when $i = 4k + 3$ and $i = 4(k + 1)$ for some integer $k \in \mathbb{Z}++$. For m is odd we assign the color c_1 for v_i, when $i = 4k - 1$ and $i = 4k - 2$. We assign the color c_2 for v_i, when $i = 4k$ and $i = 4k - 3$. We assign the color c_3 for v_i, when $i = 2$. We assign the color c_4 for v_i, when $i = m$. For m is even, we assign the color c_1 for v_i, when $i = 4k + 1$ and $i = 4k + 2$. We assign the color c_2 for v_i, when $i = 4k$ and $i = 4k - 1$. We assign the color c_3 for v_i, when $i = 1, 2$. We assign the color c_4 for v_i, when $i = m$ for some integer $k \in \mathbb{N}$. The maximum degree for this case is 3. So, $\Gamma(G^+(H)) \ngtr 4$. We assign color 4, which satisfies the Grundy coloring. By the definition of Grundy chromatic $\Gamma(G^+(H)) = 4$.

□

Remark 3.6. (i) if m=6 and n=1,2 then $\Gamma(G^+(H)) = 3$,

(ii) if m=3 and $n \geq 5$ then $\Gamma(G^+(H)) = 3$.

Theorem 3.7. *The Grundy chromatic number of line graph of a tadpole graph $T_{m,n}$ is*

(i) 3 when n=1

(ii) 3 when m is even, $n \geq 2$

(iii) 4 when m is odd, $n \geq 2$.

Proof. Let $T_{m,n}$ be a tadpole graph with vertices $V(T_{m,n}) = \{v_i : 1 \leq i \leq m\} \cup \{u_j : 1 \leq j \leq n\}$ and the edges $E(T_{m,n}) = \{e_i : 1 \leq i \leq m\} \cup \{e'_j : 1 \leq j \leq n\}$, where e_i's edges of $v_i v_{i+1}$, e_m is the edge of $v_m v_1$, e'_1 edge of $v_m u_1$ and e'_j edge of $u_j u_{j+1}$, $(2 \leq j \leq n)$. Let $V(L(T_{m,n})) = E(T_{m,n}) = \{e_i : 1 \leq i \leq m\} \cup \{e_j : 1 \leq j \leq n\}$.

Case (i): Let $n = 1$
We assign the color c_3 to the vertex e_{m-1} and the color c_2 to the vertex e_m as well to the vertex e_i if i is even $(1 \leq i \leq m - 2)$. The vertex e'_1 is assigned the color c_1. Assume $\Gamma(L(T_{m,n})) > 3$. Let c_1, c_2, c_3, c_4 be distinct colors. Suppose, we assign the color c_4 to vertex e_{m-1} and the neighborhood vertices are assigned the colors c_3, c_2, c_1. But the property that for every two colors i and j with $i < j$, every vertex colored j has a neighbor colored i is not satisfied which contradicts the definition of Grundy coloring. Therefore $\Gamma(L(T_{m,n})) = 3 \ \forall m, n = 1$.

Case (ii): Let m is even, $n \geq 2$.
We assign the color c_3 to the vertex e_{m-1}. The vertex e_i is colored c_2 when i is even $(i = mand1 \leq i \leq m - 2)$. Similarly the vertex e_i is colored c_1 when i is odd $(1 \leq i \leq m - 2)$. Whereas the vertex e'_j is colored c_1, when j is odd and c_2, when j is even. Assume $\Gamma(L(T_{m,n})) > 3$. Let c_1, c_2, c_3, c_4 be distinct colors. Suppose we assign the color c_4 to vertex e_{m-1} and the neighborhood vertices are assigned the colors c_3, c_2, c_1. But the property that for every two colors i and j with $i < j$, every vertex colored j has a neighbor colored i is not satisfied which contradicts the definition of Grundy coloring. Therefore $\Gamma(L(T_{m,n})) = 3$ when m is even, $n \geq 2$.

Case (iii): Let m is odd, $n \geq 2$.
We assign the color c_4 to the vertex e_{m-1}. The vertex e_m is colored c_3. The color c_2 is assigned to the vertex e_i if i is even and the color c_1 is assigned if i is odd $(1 \leq i \leq m-2)$. The vertex e'_j is assigned the color c_1 when j is even and c_2 when j is odd. Assume $\Gamma(L(T_{m,n})) > 4$. Let c_1, c_2, c_3, c_4, c_5 be the distinct colors. Suppose, we assign color c_5 to vertex e_{m-1} and the neighborhood vertex are assigned colors c_4, c_3, c_2, c_1. But the property that for every two colors i and j with $i < j$, every vertex colored j has a neighbor colored i is

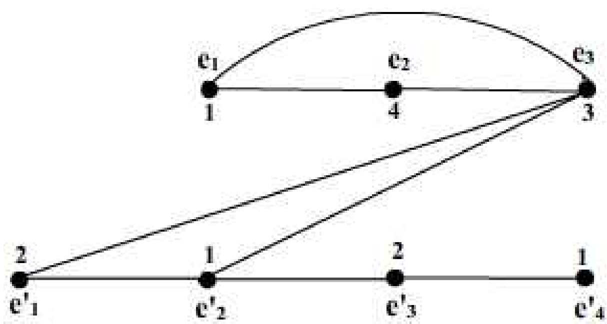

Figure 6. $\Gamma[L(T_{3,4})]$

not satisfied which contradicts the definition of Grundy coloring. Therefore $\Gamma(L(T_{m,n})) = 4$ when m is odd, $n \geq 2$.

□

Theorem 3.8. *The Grundy chromatic number of middle graph of a tadpole graph is*

(i) 4 when n=1

(ii) 5 when $n \geq 2$.

Proof. Let $T_{m,n}$ be a tadpole graph with vertices $V(T_{m,n}) = \{v_i : 1 \leq i \leq m\} \cup \{u_j : 1 \leq j \leq n\}$ and the edges $E(T_{m,n}) = \{e_i : 1 \leq i \leq m\} \cup \{e'_j : 1 \leq j \leq n\}$, where e_i's edges of v_iv_{i+1}, e_m is the edge of v_mv_1, e'_1 edge of v_mu_1 and e'_j edge of u_ju_{j+1}, $(2 \leq j \leq n)$. By the definition of middle graph on $T_{m,n}$ each edge v_iv_{i+1} $(1 \leq i \leq m)$ are subdivided by the e_i $(1 \leq i \leq m-1)$ and the edgev_mv_1 by e_m.Each edge u_ju_{j+1} is subdivided by e'_{j+1} $(1 \leq j \leq n)$and the edge v_1u_1 by e'_1. Clearly, $V(M(T_{m,n})) = \{v_i : 1 \leq i \leq m\} \cup \{u_j : 1 \leq j \leq n\} \cup \{e_i : 1 \leq i \leq m\} \cup \{e'_j : 1 \leq j \leq n\}$.

Case (i): Let $n = 1$

We assign the color c_4 to the vertex e_m. The vertex e_i is assigned the color c_3 if i is even and the color c_2 if i is odd $(1 \leq i \leq m-1)$. The color c_1 is assigned to the vertices $v_i, (1 \leq i \leq m)$ and $u_j, (1 \leq j \leq m)$. The vertex e'_1 is assigned the color c_2. Assume $\Gamma(M(T_{m,n})) > 4$. Let c_1, c_2, c_3, c_4, c_5 be distinct colors. Suppose we assign the color c_5 to vertex e_m.and the neighborhood vertices are assigned the colors c_4, c_3, c_2, c_1. But the property that for every two colors i and j with $i < j$,every vertex colored j has a neighbor colored i is not satisfied which contradicts the definition of Grundy coloring. Therefore $\Gamma(M(T_{m,n})) = 4$, $\forall m,\ n = 1$.

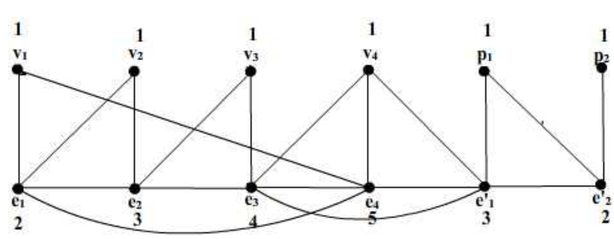

Figure 7. $\Gamma[M(T_{4,2})]$

Case (ii): Let $n \geq 2$.

We assign the color c_5 to the vertex e_m and the color c_4 to the vertex v_{m-1}. The vertex e_i is assigned the color c_3 if i is even and the color c_2 if i is odd $(1 \leq i \leq m-2)$. The color c_1 is assigned to the vertices $v_i, (1 \leq i \leq m)$ and $u_j, (1 \leq j \leq m)$. The vertex e'_j

is assigned the color c_2 if j is odd and the color c_1 if j is even $(1 \le j \le n)$. Assume $\Gamma(M(T_{m,n})) > 5$. Let $c_1, c_2, c_3, c_4, c_5, c_6$ be the distinct colors. Suppose, we assign the color c_6 to the vertex e_m and the neighborhood vertex are assigned colors c_5, c_4, c_3, c_2, c_1. But the property that for every two colors i and j with $i < j$,every vertex colored j has a neighbor colored i is not satisfied which contradicts the definition of Grundy coloring. Therefore $\Gamma(M(T_{m,n})) = 5,\ \forall m,\ n \ge 2$.

□

Theorem 3.9. *The Grundy chromatic number of total graph of a tadpole graph is*

(i) 5 when $m = 3, n \in \mathbb{N}$

(ii) 6 when $m \ge 4, n \ge 2$.

Proof. Let $T_{m,n}$ be a tadpole graph with vertices $V(T_{m,n}) = \{v_i : 1 \le i \le m\} \cup \{u_j : 1 \le j \le n\}$ and the edges $E(T_{m,n}) = \{e_i : 1 \le i \le m\} \cup \{e'_j : 1 \le j \le n\}$, where e_i's edges of $v_i v_{i+1}$, e_m is the edge of $v_m v_1$, e'_1 edge of $v_m u_1$ and e'_j edge of $u_j u_{j+1}$, $(2 \le j \le n)$.
By the definition of total graph on $T_{m,n}$ We have $V[T(T_{m,n})] = V(T_{m,n}) \cup E(T_{m,n}) = \{v_i : 1 \le i \le m\} \cup \{u_j : 1 \le j \le n\} \cup \{e_i : 1 \le i \le m\} \cup \{e'_j : 1 \le j \le n\}$.

Case (i): Let $m = 3$

We assign the color c_5 to the vertex v_m. The vertex v_{m-1} is assigned the color c_3 and the color c_1 is assigned to vertex to v_{m-2}. For the vertex e_m we assign the color c_4. The vertex e_{m-1} is assigned the color c_1 and the color c_2 is assigned to vertex to e_{m-2}.The vertex u_j is assigned the color c_2 if $j = 3k + 1,\ (k = 0, 1, 2, ...)$, if $j = 3k - 1,\ (k = 1, 2, 3, ...)$ we assign the color c_3 and c_1 if $j = 3k,\ (k = 1, 2, 3, ...)$. The vertex e'_j is assigned the color c_3 if $j = 3k + 1,\ (k = 0, 1, 2, ...)$, if $j = 3k - 1,\ (k = 1, 2, 3, ...)$ we assign the color c_1 and c_2 if $j = 3k,\ (k = 1, 2, 3, ...)$. Assume $\Gamma(T(T_{m,n})) > 5$.

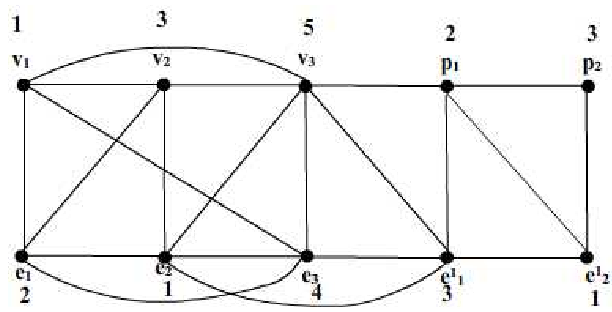

Figure 8. $\Gamma[T(T_{3,3})]$

Let $c_1, c_2, c_3, c_4, c_5, c_6$ be distinct colors. Suppose we assign the color c_6 to vertex v_m and the neighborhood vertices are assigned the colors c_5, c_4, c_3, c_2, c_1. But the property that for every two colors i and j with $i < j$,every vertex colored j has a neighbor colored i is not satisfied which contradicts the definition of Grundy coloring. Therefore $\Gamma(T(T_{m,n})) = 5,\ m = 3,\ \forall n$.

Case (ii): Let $m \ge 4, n \ge 2$.

Subcase (a): Let $m = 4, n \ge 2$.

We assign the color c_6 to the vertex v_m. The vertex v_{m-1} is assigned the color c_1, the color c_3 is assigned to vertex to v_{m-2} and the color c_2 is assigned to the vertex v_{m-3}. For the vertex e_m we assign the color c_5. The vertex e_{m-1} is assigned the color c_4 , the color c_2 is assigned to vertex to e_{m-2} and the color c_1 to the vertex e_{m-3}. The vertex u_j is assigned the color c_1 if $j = 3k+1,\ k = 0, 1, 2, ...$, if $j = 3k-1,(\ k = 1, 2, 3, ...)$ we assign the color c_3 and c_2 if $j = 3k,\ (k = 1, 2, 3, ...)$. The vertex e'_j is assigned the color c_3 if $j = 3k + 1,\ k = 0, 1, 2, ...$, if $j = 3k - 1,\ (k = 1, 2, 3, ...)$ we assign the color c_2 and c_1 if $j = 3k,\ (k = 1, 2, 3, ...)$.

Subcase (b): Let $m = 3q - 1, q = 2, 3, ..., n \geq 2$.
We assign the color c_6 to the vertex v_m. The vertex v_{m-1} is assigned the color c_2, the color c_3 is assigned to vertex to v_{m-2}, the color c_4 is assigned to the vertex v_2 and v_1 is assigned the color c_2. The vertex v_i is assigned the color c_3 if $j = 3k,\ (k = 1, 2, 3, ...)$, if $j = 3k + 1,\ k = 0, 1, 2, ...$ we assign the color c_2 and c_1 if $j = 3k - 1, (\ k = 1, 2, 3, ...)$. For the vertex e_m we assign the color c_5. The vertex e_{m-1} is assigned the color c_4, the color c_1 is assigned to vertex to e_{m-2}, the color c_2 to the vertex e_2 and e_1 is assigned the color c_1. The vertex e_i is assigned the color c_1 if $j = 3k,\ (k = 1, 2, 3, ...)$, if $j = 3k + 1,\ k = 0, 1, 2, ...$ we assign the color c_3 and c_2 if $j = 3k - 1, (\ k = 1, 2, 3, ...)$. The vertex u_j is assigned the color c_1 if $j = 3k + 1,\ k = 0, 1, 2, ...,$ if $j = 3k - 1, (\ k = 1, 2, 3, ...)$ we assign the color c_3 and c_2 if $j = 3k,\ (k = 1, 2, 3, ...)$. The vertex e'_j is assigned the color c_3 if $j = 3k + 1,\ k = 0, 1, 2, ...,$ if $j = 3k - 1,\ (k = 1, 2, 3, ...)$ we assign the color c_2 and c_1 if $j = 3k,\ (k = 1, 2, 3, ...)$.

Subcase (c): Let $m = 3q, q = 2, 3, ..., n \geq 2$.
We assign the color c_6 to the vertex v_m. The vertex v_{m-1} is assigned the color c_2, the color c_3 is assigned to vertex to v_{m-2}, the color c_4 is assigned to the vertex v_2 and v_1 is assigned the color c_2. The vertex v_i is assigned the color c_1 if $j = 3k,\ (k = 1, 2, 3, ...)$, if $j = 3k + 1,\ k = 0, 1, 2, ...$ we assign the color c_3 and c_2 if $j = 3k - 1, (\ k = 1, 2, 3, ...)$. For the vertex e_m we assign the color c_5. The vertex e_{m-1} is assigned the color c_4, the color c_1 is assigned to vertex to e_{m-2}, the color c_3 to the vertex e_2 and e_1 is assigned the color c_1. The vertex e_i is assigned the color c_2 if $j = 3k,\ (k = 1, 2, 3, ...)$, if $j = 3k + 1,\ k = 0, 1, 2, ...$ we assign the color c_1 and c_3 if $j = 3k - 1, (\ k = 1, 2, 3, ...)$. The vertex u_j is assigned the color c_1 if $j = 3k + 1,\ k = 0, 1, 2, ...,$ if $j = 3k - 1, (\ k = 1, 2, 3, ...)$ we assign the color c_3 and c_2 if $j = 3k,\ (k = 1, 2, 3, ...)$. The vertex e'_j is assigned the color c_3 if $j = 3k + 1,\ k = 0, 1, 2, ...,$ if $j = 3k - 1,\ (k = 1, 2, 3, ...)$ we assign the color c_2 and c_1 if $j = 3k,\ (k = 1, 2, 3, ...)$.

Subcase (d): Let $m = 3q + 1, q = 2, 3, ..., n \geq 2$.
We assign the color c_6 to the vertex v_m. The vertex v_{m-1} is assigned the color c_2, the color c_3 is assigned to vertex to v_{m-2}, the color c_3 is assigned to the vertex v_2 and v_1 is assigned the color c_2. The vertex v_i is assigned the color c_2 if $j = 3k,\ (k = 1, 2, 3, ...)$, if $j = 3k + 1,\ k = 0, 1, 2, ...$ we assign the color c_1 and c_3 if $j = 3k - 1, (\ k = 1, 2, 3, ...)$. For the vertex e_m we assign the color c_5. The vertex e_{m-1} is assigned the color c_4, the color c_1 is assigned to vertex to e_{m-2}, the color c_4 to the vertex e_2 and e_1 is assigned the color c_1. The vertex e_i is assigned the color c_3 if $j = 3k,\ (k = 1, 2, 3, ...)$, if $j = 3k + 1,\ k = 0, 1, 2, ...$ we assign the color c_2 and c_1 if $j = 3k - 1, (\ k = 1, 2, 3, ...)$. The vertex u_j is assigned the color c_1 if $j = 3k + 1,\ k = 0, 1, 2, ...,$ if $j = 3k - 1, (\ k = 1, 2, 3, ...)$ we assign the color c_3 and c_2 if $j = 3k,\ (k = 1, 2, 3, ...)$. The vertex e'_j is assigned the color c_3 if $j = 3k + 1,\ k = 0, 1, 2, ...,$ if $j = 3k - 1,\ (k = 1, 2, 3, ...)$ we assign the color c_2 and c_1 if $j = 3k,\ (k = 1, 2, 3, ...)$.

Assume $\Gamma(T(T_{m,n})) > 6$. Let $c_1, c_2, c_3, c_4, c_5, c_6, c_7$ be the distinct colors. Suppose we assign the color c_7 to the vertex v_m and the neighborhood vertex are assigned colors $c_6, c_5, c_4, c_3, c_2, c_1$ .But the property that for every two colors i and j with $i < j$,every vertex colored j has a neighbor colored i is not satisfied which contradicts the definition of Grundy coloring. Therefore $\Gamma(T(T_{m,n})) = 6,\ m \geq 4,\ n \geq 2$.

□

Theorem 3.10. *The Grundy chromatic number of Mycielskian graph of a tadpole graph $T_{m,n}$ is four.*

Proof. Let $T_{m,n}$ be tadpole graph with vertices $V(T_{m,n}) = \{v_i : 1 \leq i \leq m\} \cup \{u_j : 1 \leq j \leq n\}$ and the edges $E(T_{m,n}) = \{e_i : 1 \leq i \leq m\} \cup \{e'_j : 1 \leq j \leq n\}$ where e_i are the edges between $v_i v_{i+1}$, e'_1 is the of $v_m u_1$ and e'_j is the of $u_j u_{j+1}, (2 \leq j \leq n)$.
Let $V(\mu(T_{m,n})) = V'(T_{m,n}) = \{v_i : 1 \leq i \leq m\} \cup \{v'_i : 1 \leq 1 \leq m\} \cup \{u_j : 1 \leq j \leq n\} \cup \{p'_j : 1 \leq j \leq n\} \cup \{w\}$.
We assign the color c_4 to the vertex v_m. The vertex v_i is assigned the color c_3 when i is odd and c_2 when i is even $(1 \leq i \leq m - 1)$. The vertex v'_i is assigned the color c_1 $(\forall i)$.

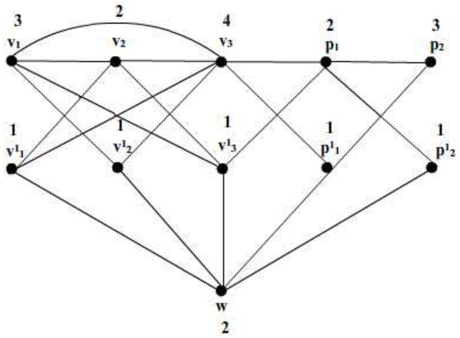

Figure 9. $\Gamma[\mu(T_{3,2})]$

Assume $\Gamma(\mu(T_{m,n})) > 3$. Let $\Gamma(\mu(T_{m,n})) = 4$. Let c_1, c_2, c_3, c_4 be distinct colors. Suppose, we assign the color c_4 to vertex v_m and the neighborhood vertices are assigned the colors c_3, c_2, c_1. But the property that for every two colors i and j with $i < j$, every vertex colored j has a neighbor colored i is not satisfied which contradicts the definition of Grundy coloring. Therefore $\Gamma(\mu(T_{m,n})) = 4, \ \forall m, n$. □

Conclusion

In this paper we discussed about the Grundy coloring of Path, Cycle, Friendship, Sunlet, Tadpole, line graph of tadpole graph, middle graph of tadpole graph, total graph of tadpole graph and Mycielskian graph of tadpole graph. This paper motivates the reader to do further research in grundy coloring of some other graphs and also other colorings of different types of graphs.

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Author information

K. Annathurai, Department of Mathematics, Thiruvalluvar College, Papanasam, Tirunelveli-627425, Tamilnadu., India.
E-mail: `kannathuraitvcmaths@gmail.com`

P. Periasamy, Research Scholar, Register Number 21124012091019,, Manonmaniam Sundaranar University, Tirunelveli-627012, Tamilnadu., India.
E-mail: `ppskmttc2013@gmail.com`

V. Sankar Raj, Department of Mathematics, Manonmaniam Sundaranar University, Tirunelveli-627012, Tamilnadu., India.
E-mail: `sankarrajv@gmail.com`