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Analysing the Time Complexity of Geo Coloring Algorithm of Tree Graph Families

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Abstract A set S_c of vertices in G that is both geodetic and chromatic set is referred to as geo chromatic set and the minimum cardinality among all geo chromatic sets is called the geo chromatic number of G denoted by $\chi_{gc}(G)$. A total geo chromatic set S_c of vertices in G is defined as a geo chromatic set and the subgraph it induces has no isolated vertices. The total geo chromatic number is the minimum cardinality of the set S_c which is denoted by $\chi_{tg}(G)$. In this paper, we determine geo chromatic and total geo chromatic number of tree graph families such as centipede graph cp_n, m-star graph $S_{m,n}$, bistar graph $B_{m,n}$, banana tree graph $BT_{m,n}$ and firecracker graph $F_{m,n}$ and constructed an algorithm to find geo chromatic number and total geo chromatic number.

1 Introduction

Throughout this paper, we consider simple, finite, connected and undirected graphs. Let $V(G)$ and $E(G)$ denote the vertex and edge set of graph G respectively. The distance $d(x,y)$ between two vertices x and y is the length of the shortest $x-y$ path in the graph G. An $x-y$ geodesic is the $x-y$ path of length $d(x,y)$. If z is the internal vertex of $x-y$ geodesic Q, then it is said to lie on Q. The closed interval $I[x,y]$ consists of x,y and all vertices lying on some Q of G. Let $I[S] = \bigcup\limits_{x,y\in S} I[x,y]$, where $S \subseteq V$ is non empty. If $S = V(G)$, then S is said to be geodetic set and its cardinality is the geodetic number denoted by $g(G)$. The concept of geodetic number was introduced in [6, 7] and further studied in [9, 8]. A set $C \subseteq V(G)$ is called a chromatic set if C contains all k vertices of different colors in G. The minimum cardinality among all chromatic sets is called chromatic number and is denoted by $\chi(G)$.

The concept of geo chromatic number was introduced by *S. B. Samli* and *S. R. Chellathurai* [3] and further studied in [4, 5, 10, 15]. If the set S is both geodesic and chromatic set, then it is said to be geo chromatic set S_c of G. The minimum cardinality among all such sets is called geo chromatic number (GCN) and is denoted by $\chi_{gc}(G)$. Further, this concept was extended to total geo chromatic number which was introduced by *S. Annie Ajila and J. Robert Victor Edward* [1]. If the subgraph induced by the set S_c contains no isolated vertices, then the set S_c is said to be total geo chromatic set. The minimum cardinality of such a set is called total geo chromatic number (TGCN) and is denoted by $\chi_{tg}(G)$. In this paper, we have designed an algorithm for finding GCN and TGCN and also the geo chromatic and total geo chromatic number for tree graph families such as centipede graph cp_n, m-star graph $S_{m,n}$, bistar graph $B_{m,n}$, banana tree graph $BT_{m,n}$ and firecracker graph $F_{m,n}$ are determined.

2 Preliminaries

Definition 2.1. [11] A centipede graph cp_n is the graph obtained by taking a path P_n and n copies of K_1 and then joining the i-th vertex of P_n with an edge to every vertex in the i-th copy.

Definition 2.2. m-Star graph $S_{m,n}$ is a tree obtained from the $(m-1)$-star graph by adding a new pendent edge of the existing n pendant vertices. It has $mn+1$ vertices and mn edges.

Definition 2.3. [2] Bistar $B_{m,n}$ is the graph obtained from K_2 by joining m pendant edges to one end and n pendant edges to the other end of K_2.

Definition 2.4. [12] Banana tree $BT_{m,n}$ is a graph obtained by connecting one leaf of each m copies of a n star graph with a single root vertex.

Definition 2.5. [14] Firecracker graph $F_{m,n}$ is a graph obtained from m graphs S_n by connecting a leaf from each S_n through a path P_m.

Theorem 2.6. *[1, 3] For any path with k vertices,*

$$\chi_{gc}(P_k) = \begin{cases} 3, & if\ k\ is\ odd \\ 2, & if\ k\ is\ even \end{cases}$$

$$\chi_{tg}(P_k) = \begin{cases} k, & if k = 2,3,4 \\ 4, & if k \geq 5 \end{cases}$$

Theorem 2.7. *[1, 3] For a star graph $K_{1,k-1}$ of order $k > 2$, $\chi_{gc}(K_{1,k-1}) = \chi_{tg}(K_{1,k-1}) = k$.*

Theorem 2.8. *[3] For a double star $S_{a,b}$, $a+b \geq 4$ and $a, b < k$, $\chi_{gc}(S_{a,b}) = a+b$.*

3 Geo Coloring Algorithm

In this section, we design an algorithm for finding geo chromatic and total geo chromatic number and also included the time complexity of this algorithm. This algorithm includes the idea of Greedy algorithm in finding chromatic number and similarity matrix which has been used in Coloring Pair(CP) algorithm [13]. Further, it has been extended to find the geo chromatic and total geo chromatic number using chromatic and adjacency matrix. The major difference between chromatic and geo chromatic number is that we have some addition restrictions to form both geodetic and chromatic set. While, in total geo chromatic, addition to geo chromatic condition it must have non isolated vertices in the subgraph induced.

3.1 Construction of Geo Coloring Algorithm

Step 01: Construct a n x n similarity matrix.

$$S = (s_{ab}) = \begin{cases} 0, & a = b\ or\ v_a\ is\ adjacent\ to\ v_b \\ d_G(v_a) + d_G(v_b), & Otherwise \end{cases}$$

Step 02: Frame the set S_c by selecting all the columns that have the fewest zeros.

Step 03: Initially assign no colors to the vertices and define connected sequential order for coloring the nodes.

Step 04: Select the next node from the graph based on the defined strategy.

Step 05: For the selected node, assign a smallest color that hasn't been used by any of its neighboring nodes.

Step 06: If all the colors have been used then assign a new color.

Step 07: Repeat the steps 5 and 6 until all the nodes in the graph have been colored.

Step 08: The total number of colors used in the graph constitutes a chromatic number of G.

Step 09: Frame a chromatic matrix X

$$X = (x_{ab}) = \begin{cases} 1, & a = b \text{ or } v_b \text{ has a color } c_a; 1 \leq a \leq \chi(G); 1 \leq b \leq n \\ 0, & Otherwise \end{cases}$$

Step 10: Check if there is at least one vertex in the column associated with the entry '1' in the row that is in set S_c. If not, include that vertex in S_c and form a new set S_c'.

Step 11: Cardinality of S_c' represents the geo chromatic number $\chi_{gc}(G)$.

Step 12: Frame the adjacency matrix A with vertices of the set S_c' as row and vertices of G as columns.

$$A = (a_{ab}) = \begin{cases} 1, & a = b \text{ or } v_a \text{ is adjacent to } v_b \\ 0, & Otherwise \end{cases}$$

Step 13: Check if each vertex in S_c' is adjacent to some other vertex in the same set. If exists, move to step 15.

Step 14: Check matrix A for any vertex adjacent to isolated vertices. If exists, include that vertex in S_c' and form a set S_c''; if there are multiple such vertices, choose one.

Step 15: Cardinality of S_c'' represents total geo chromatic number $\chi_{tg}(G)$.

3.2 Time Complexity

In this section, we have examined the computational complexity involved in determining the geo chromatic and total geo chromatic numbers using the proposed algorithm, employing the essential concept of 'Big O notation' in algorithmic analysis. According to this algorithm, while constructing similarity matrix, adjacency matrix and finding chromatic number we have to iterate over the number of vertices, so the time complexity is $O(n \wedge 2)$ times. Whereas, to find the geo chromatic number we iterate over the total number of vertices n and chromatic number $\chi(G)$, the time complexity is $O(\chi(G) * n)$. Thus, we conclude that the overall time complexity of an algorithm is $O(\chi(G) * n)$.

4 GCN and TGCN of Some Tree Graphs

In this section, by applying above algorithm, we obtain the geo chromatic and total geo chromatic number of tree graph families such as centipede graph cp_n, m-star graph $S_{m,n}$, bistar graph $B_{m,n}$, banana tree graph $BT_{m,n}$ and firecracker graph $F_{m,n}$.

Theorem 4.1. *For any positive integer $n \geq 2$, the geo chromatic and total geo chromatic number of centipede graph cp_n is n and $2n$ respectively.*

Proof. Let $V = \{v_q : 1 \leq q \leq 2n\}$ and $E = \{e_r : 1 \leq r \leq 2n-1\}$ be the vertices and edges of centipede graph cp_n respectively. The minimum degree of cp_n is 1 and the maximum degree is
$$\Delta(cp_n) = \begin{cases} 2, & n = 2 \\ 3, & n > 2 \end{cases}.$$
For $n \geq 2$, the set $S = \{v_x : n+1 \leq x \leq 2n\}$ is the geodetic set of cp_n, which is minimum. By assigning the proper colors to cp_n, the vertices in S belongs to different color class, say c_1, c_2. Thus, the set S is a chromatic set. That is $g(cp_n) = |S|$ and $\chi(cp_n) = |C| = |S|$. Hence, $|S| = \chi_{gc}(cp_n) = |S_c| = n$. Since the subgraph induced by S_c contains isolated vertices, we frame a set $S_c' = S_c \cup N(S_c)$. Thus $S_c' = \{v_z : 1 \leq z \leq 2n\}$. Therefore, $|S_c'| = \chi_{tg}(cp_n) = 2n$.

Alternatively, the geo chromatic and total geo chromatic number can also be obtained by proposed algorithm as follows.

Step 01: From the similarity matrix S, we obtain the set $S_c = \{v_g : n+1 \leq g \leq 2n\}$ which yields a minimum geodetic set.

Step 02: After coloring all the vertices by connected sequential strategy, we obtain the chromatic number as 2. i.e., $\chi(cp_n) = 2$.

Step 03: In the matrix X, exactly n vertices receives color c_1 and n vertices receives c_2.

Step 04: From that, we find that when n is even, $\{v_{n+2}, v_{n+4}, \ldots, v_{2n}\}$ and $\{v_{n+1}, v_{n+3}, \ldots, v_{n+(n-1)}\}$ vertices receives color c_1 and c_2 respectively. When n is odd, $\{v_{n+2}, v_{n+4}, \ldots, v_{n+(n-1)}\}$ and $\{v_{n+1}, v_{n+3}, \ldots, v_{2n}\}$ vertices receives color c_1 and c_2 respectively. Thus we have $S'_c = \{v_h : n+1 \leq h \leq 2n\}$. Hence $\chi_{gc}(cp_n) = n$. In Figure 1, $\diamond$ denotes the elements of S'_c which represents the geo chromatic set.

Step 05: We observe that all the vertices in the set S'_c are isolated. To make it non isolated we include the adjacent vertices $\{v_z : 1 \leq z \leq n\}$ in the set S'_c. Thus, $S''_c = \{v_t : 1 \leq t \leq 2n\}$. Hence, $\chi_{tg}(cp_n) = 2n$. In Figure 1, both $\diamond$ and $\circ$ leads to S''_c that constitutes the total geo chromatic set.

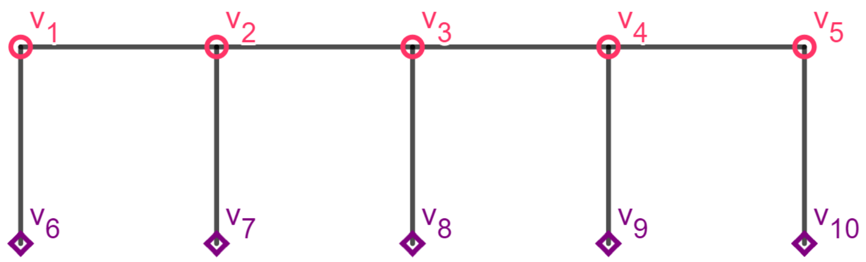

Figure 1. $\chi_{gc}(cp_5) = 5$; $\chi_{tg}(cp_5) = 10$

For example: When $n = 5$, $V = \{v_q : 1 \leq q \leq 10\}$ and $E = \{e_r : 1 \leq r \leq 9\}$ are the vertices and edges of cp_5 respectively. Let $\delta = 1$ and $\Delta = 3$.

Step 01: The 10 x 10 similarity matrix of cp_5 as follows

$$S = \begin{bmatrix} 0 & 0 & 5 & 5 & 4 & 0 & 3 & 3 & 3 & 3 \\ 0 & 0 & 0 & 6 & 5 & 4 & 0 & 4 & 4 & 4 \\ 5 & 0 & 0 & 0 & 5 & 4 & 4 & 0 & 4 & 4 \\ 5 & 6 & 0 & 0 & 0 & 4 & 4 & 4 & 0 & 4 \\ 4 & 5 & 5 & 0 & 0 & 3 & 3 & 3 & 3 & 0 \\ 0 & 4 & 4 & 4 & 3 & 0 & 2 & 2 & 2 & 2 \\ 3 & 0 & 4 & 4 & 3 & 2 & 0 & 2 & 2 & 2 \\ 3 & 4 & 0 & 4 & 3 & 2 & 2 & 0 & 2 & 2 \\ 3 & 4 & 4 & 0 & 3 & 2 & 2 & 2 & 0 & 2 \\ 3 & 4 & 4 & 4 & 0 & 2 & 2 & 2 & 2 & 0 \end{bmatrix}$$

Step 02: From the similarity matrix we obtain $S_c = \{v_g : 6 \leq g \leq 10\}$

Step 03: Chromatic number $\chi(cp_5) = 2$.

Step 04:

$$X = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

From this we find that 5 vertices receives color c_1 and 5 vertices receives color c_2.

Step 05: We find that the vertices v_7, v_9 receives color c_1 and v_6, v_8, v_{10} receives color c_2.

$$S'_c = \{v_g : 6 \leq g \leq 10\}. \text{ Thus, } \chi_{gc}(cp_5) = 5$$

Step 06:

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

To make the vertices in the set S'_c non isolated we include $\{v_z : 1 \leq z \leq 5\}$ vertices. Thus,

$$S''_c = \{v_g : 1 \leq g \leq 10\}. \text{ Thus, } \chi_{tg}(cp_5) = 10$$

□

Theorem 4.2. *For any pair of positive integers $m, n \geq 2$, the geo chromatic and total geo chromatic number of m-star graph $S_{m,n}$ are $n+1$ and $2n$, respectively.*

Proof. Let $V = \{v_q : 1 \leq q \leq mn+1\}$ and $E = \{e_r : 1 \leq r \leq mn\}$ be the vertices and edges of m-star graph $S_{m,n}$ respectively. The minimum degree of $S_{m,n}$ is 1 and the maximum degree is n.
For $m, n \geq 2$, the set of all leaves $S = \{v_x : ((m-1)n) + 2 \leq x \leq mn+1\}$ is the minimum geodetic set for $S_{m,n}$. But S is not a chromatic set. Choose another vertex from $S_{m,n}$ which belongs to different color class. If $v_{(m-1)n+1}$ is in S, then the set becomes $S_c = S \cup v_{((m-1)n)+1}$. Hence, $\chi_{gc}(S_{m,n}) \leq n+1$. Since $\chi_{gc}(S_{m,n}) < n+1$ is not possible. Therefore, $\chi_{gc}(S_{m,n}) = n+1$. Since the subgraph induced by S_c contains isolated vertices, we frame a set $S'_c = \{S_c \cup N(v_i) : v_i \in [((m-1)n)+2, mn+1]\}$. Thus, $|S'_c| = \chi_{tg}(S_{m,n}) = 2n$.

We can also obtain the geo-chromatic number and the total geo-chromatic number using the proposed algorithm.

Step 01: From the similarity matrix S, we obtain the set $S_c = \{v_g : ((m-1)n)+2 \leq g \leq mn+1\}$ which yields a minimum geodetic set.

Step 02: After coloring all the vertices by connected sequential strategy, we obtain the chromatic number as 2. i.e., $\chi(S_{m,n})$ = 2.

Step 03: In the matrix X, $(\lceil \frac{m-1}{2} \rceil)n + 1$ vertices receives color c_1 and $(\lfloor \frac{m+1}{2} \rfloor)n$ vertices receives color c_2.

Step 04: From that, we find that when m is odd, $\{v_x : ((m-1)n)+2 \leq x \leq mn+1\}$ receives color c_1 and there is no vertices with color c_2. So, we include the vertex $v_{((m-1)n)+1}$ that has color c_2. When m is even, all the vertices in S_c receives color c_2 and so we must include a vertex $v_{((m-1)n)+1}$ that has color c_1. Thus, we have $S'_c = \{v_h : ((m-1)n)+1 \leq h \leq mn+1\}$. Hence $\chi_{gc}(S_{m,n}) = n+1$.

Step 05: We observe that all the vertices in the set S'_c are isolated except $v_{((m-1)n)+1}$ and v_{mn+1}. To make it non isolated we include the adjacent vertices $\{v_z : ((m-2)n)+2 \leq z \leq ((m-1)n)\}$ in the set S'_c. Thus, $S''_c = \{v_t : ((m-2)n)+2 \leq t \leq mn+1\}$. Hence, $\chi_{tg}(S_{m,n})$=$2n$.

□

Theorem 4.3. *For any pair of positive integers $m, n \geq 2$, the geo chromatic and total geo chromatic number of bistar graph $B_{m,n}$ are $m+n$ and $m+n+2$ respectively.*

Proof. Let $V = \{v_q : 1 \leq q \leq m+n+2\}$ and $E = \{e_r : 1 \leq r \leq m+n+1\}$ be the vertices and edges of bistar graph $B_{m,n}$ respectively. The minimum degree of $B_{m,n}$ is 1. The maximum degree is, $\Delta(B_{m,n}) = \begin{cases} n+1, & m \leq n \\ m+1, & m > n \end{cases}$.
By theorem 2.3, $\chi_{gc}(B_{m,n}) = m+n$. Let $S_c = \{v_x : 3 \leq x \leq m+n+2\}$ is both geodetic and chromatic set of $B_{m,n}$. Since the subgraph induced by S_c contains isolated vertices, we frame a set $S'_c = S_c \cup N(S_c)$. Thus, $S'_c = \{v_z : 1 \leq z \leq m+n+2\}$. Hence, $|S'_c| = \chi_{tg}(B_{m,n}) = m+n+2$.

Alternatively, the geo chromatic and total geo chromatic number can also be obtained by proposed algorithm as follows.

Step 01: From the similarity matrix S, we obtain the set $S_c = \{v_g : 3 \leq g \leq m+n+2\}$ which yields a minimum geodetic set.

Step 02: After coloring all the vertices by connected sequential strategy, we obtain the chromatic number as 2. i.e., $\chi(B_{m,n})$ = 2.

Step 03: In the matrix X, exactly $n+1$ vertices receives color c_1 and $m+1$ vertices receives c_2.

Step 04: From that, we find that vertices $\{v_x : m+3 \leq m+n+2\}$ and $\{v_y : 3 \leq y \leq m+2\}$ receives color c_1 and c_2 respectively. Thus we have $S'_c = \{v_h : 3 \leq h \leq m+n+2\}$. Hence $\chi_{gc}(B_{m,n}) = m+n$ (by theorem 2.3).

Step 05: We observe that all the vertices in the set S'_c are isolated. To make it non isolated, we include the adjacent vertices v_1 and v_2 in the set S'_c. Thus, $S''_c = \{v_t : 1 \leq t \leq m+n+2\}$. Hence, $\chi_{tg}(B_{m,n})$=$m+n+2$.

□

Theorem 4.4. *For any positive integers $m \geq 2, n \geq 4$, the geo chromatic and total geo chromatic number of banana tree graph $BT_{m,n}$ is $m(n-2)+1$ and $m(n-1)$ respectively.*

Proof. Let $V = \{v_q : 1 \leq q \leq mn+1\}$ and $E = \{e_r : 1 \leq r \leq mn\}$ be the vertices and edges of banana tree graph $B_{m,n}$ respectively. The minimum degree of $B_{m,n}$ is 1. The maximum degree is, $\Delta(B_{m,n}) = \begin{cases} m, & m \geq n \\ n-1, & m < n \end{cases}$.
If $m \geq 2, n \geq 4$, the set $S = \{v_x : 1 \leq x \leq m(n-2)\}$ is the minimum geodetic set of $BT_{m,n}$. By assigning the proper color to $BT_{m,n}$, the vertices in S belongs to same color class, say c_1. Then the set S is not a chromatic set of $BT_{m,n}$. Since, $\chi(BT_{m,n}) = 2$. Choose another vertex from $BT_{m,n}$ which belongs to the distinct color class. Let $v_{m(n-2)+1} \in c_j, j \neq 1$. Then the set becomes $S_c = S \cup \{v_{m(n-2)+1}\}$ is a geodetic set as well as chromatic set of $BT_{m,n}$. Hence, $\chi_{tg}(BT_{m,n}) \leq m(n-2)+1$. Since $\chi_{gc}(BT_{m,n}) < m(n-2)+1$ is not possible. Therefore, $|S_c| = \chi_{gc}(BT_{m,n}) = m(n-2)+1$. Since the subgraph induced by S_c contains isolated vertices, we frame a set $S'_c = \{S_c \cup N(v_i) : v_i \in [n-1, m(n-2)]\}$. Hence, $|S'_c| = \chi_{tg}(BT_{m,n}) = m(n-1)$.

The geo-chromatic number and the total geo-chromatic number are derived through the implementation of the proposed algorithm.

Step 01: From the similarity matrix S, we obtain the set $S_c = \{v_g : 1 \leq g \leq m(n-2)\}$ which yields a minimum geodetic set.

Step 02: After coloring all the vertices by connected sequential strategy, we obtain the chromatic number as 2. i.e., $\chi(BT_{m,n})$ = 2.

Step 03: In the matrix X, exactly $m(n-1)$ vertices receives color c_1 and $m+1$ vertices receives c_2.

Step 04: From that, we find that all the vertices receives color c_1. There is no vertices that receive color c_2. Thus, we choose the vertex $v_{m(n-2)+1}$ and we have S'_c = $\{v_h : 1 \leq h \leq m(n-2)+1\}$. Hence $\chi_{gc}(BT_{m,n})$ = $m(n-2)+1$.

Step 05: We observe that all the vertices in the set s'_c are isolated except $v_1, v_2, \ldots, v_m$ and $v_{m(n-2)+1}$ vertices. To make it non isolated, we include the adjacent vertices $\{v_z : m(n-2)+2 \leq z \leq m(n-1)\}$ in the set S'_c. Thus, $S''_c = \{v_t : 1 \leq t \leq m(n-1)\}$. Hence, $\chi_{tg}(BT_{m,n})$=$m(n-1)$.

□

Corollary 4.5. *For any positive integers $m \geq 2, 2 \leq n \leq 3$, theorem 4.4 becomes $\chi_{gc}(BT_{m,n})$ = $\chi_{gc}(K_{1,m,m})$ = $\chi_{gc}(K_{1,m,m,m})$ = $m+1$ and $\chi_{tg}(BT_{m,n})$ = $\chi_{tg}(K_{1,m,m})$ = $\chi_{gc}(K_{1,m,m,m})$ = $2m$.*

Theorem 4.6. *For any positive integers $m \geq 2, n \geq 3$, the geo chromatic and total geo chromatic number of firecracker graph $F_{m,n}$ is $m(n-2)$ and $m(n-1)$ respectively.*

Proof. Let $V = \{v_q : 1 \leq q \leq mn\}$ and $E = \{e_r : 1 \leq r \leq mn-1\}$ be the vertices and edges of firecracker graph $F_{m,n}$ respectively. The minimum degree of $F_{m,n}$ is 1. The maximum degree is, $\Delta(F_{m,n}) = \begin{cases} 3, & m \geq 3, 3 \leq n \leq 4 \\ n-1, & m \geq 3, n > 4 \end{cases}$.
For $m \geq 2, n \geq 3$, the set $s = \{v_x : 2m+1 \leq x \leq mn\}$ is the geodetic set of $F_{m,n}$, which is minimum. By assigning the proper colors to $F_{m,n}$ the vertices in S belongs to different color class, say c_1, c_2. Thus, the set S is a chromatic set. That is $g(F_{m,n}) = |S|$ and $\chi(c_n) = |C| = |S|$. Hence, $|S| = \chi_{gc}(F_{m,n}) = |S_c| = m(n-2)$. Since, the subgraph induced by S_c contains isolated vertices. We frame a set $S'_c = S_c \cup N(S_c)$. Thus $S'_c = \{v_z : m+1 \leq z \leq mn\}$. Therefore, $|S'_c| = \chi_{tg}(c_n) = m(n-1)$.

We can also obtain the geo-chromatic number and the total geo-chromatic number using the proposed algorithm.

Step 01: From the similarity matrix S, we obtain the set $S_c = \{v_g : 2m+1 \leq g \leq mn\}$ which yields a minimum geodetic set.

Step 02: After coloring all the vertices by connected sequential order, we obtain the chromatic number as 2. i.e., $\chi(F_{m,n})$ = 2.

Step 03: In the matrix X, exactly $(n-1)\lceil \frac{m}{2} \rceil + [m - \lceil \frac{m}{2} \rceil]$ receives color c_1 and $(n-1)[m - \lceil \frac{m}{2} \rceil] + [m - \lfloor \frac{m}{2} \rfloor]$ receives color c_2.

Step 04: From that, we find that $\{v_x : 2(m+k(n-2))+1 \leq x \leq 2(m+k(n-2))+1+(n-3)\}$ $where$ $0 \leq k \leq \lceil \frac{m}{2} \rceil$ vertices receives color c_1 and $\{v_y : 2(m+k(n-2))+(n-1) \leq y \leq 2(m+k(n-2))\}$ $where$ $0 \leq k \leq \lfloor \frac{m}{2} \rfloor$ vertices receives color c_2. Thus we have $S'_c = \{v_h : 2m+1 \leq h \leq mn\}$. Hence $\chi_{gc}(F_{m,n})$ = $m(n-2)$. In Figure 2, ◇, ◇ denotes the elements of S'_c which represents the geo chromatic set.

Step 05: We observe that all the vertices in the set S'_c are isolated. To make it non isolated, we include the adjacent vertices $\{v_z : m+1 \leq z \leq 2m\}$ in the set S'_c. Thus, $S''_c = \{v_t : m+1 \leq t \leq mn\}$. Hence, $\chi_{tg}(F_{m,n}) = m(n-1)$. In Figure 2, ◇, ◇ and ○, ○ leads to S''_c that constitutes the total geo chromatic set.

□

Corollary 4.7. *For any positive integers $m \geq 2, n = 2$, $\chi_{gc}(F_{m,n}) = \chi_{gc}(cp_n) = n$ and $\chi_{tg}(F_{m,n}) = \chi_{tg}(cp_n) = 2n$.*

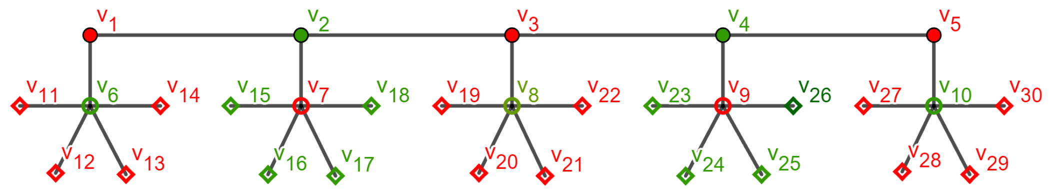

Figure 2. $\chi_{gc}(F_{5,6}) = 20$; $\chi_{tg}(F_{5,6}) = 25$

5 Conclusion

In this study, the geo chromatic number and total geo-chromatic number of several tree graph families have been determined, and an algorithm to compute GCN and TGCN has been developed. This concept has practical applications in network optimization, traffic flow management, sensor networks, urban planning, telecommunication tower placement, and resource allocation, where efficient, conflict-free assignments are crucial. Furthermore, this work can be extended to determine the double geo-chromatic number, connected geo-chromatic number, edge geo-chromatic number, and other related parameters for tree graph families.

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