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A Study on $\alpha^* g^\# \psi$-Border, $\alpha^* g^\# \psi$-Frontier and $\alpha^* g^\# \psi$-Exterior and Their Relation with $\alpha^* g^\# \psi$-Interior and $\alpha^* g^\# \psi$-Closure in Topological Spaces

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Abstract: The purpose of this paper is to study the properties of $\alpha^* g^\# \psi$-border, $\alpha^* g^\# \psi$-frontier and $\alpha^* g^\# \psi$-exterior in topological spaces. A relation between $\alpha^* g^\# \psi$-Frontier and $\alpha^* g^\# \psi$-Border of a Set is given. Also the comparison between $\alpha^* g^\# \psi$-frontier and its $\alpha^* g^\# \psi$-interior, $\alpha^* g^\# \psi$-closure in topological spaces is derived.

1 Introduction

Levine[9, 10] developed generalised closed sets and semi-open sets in topological spaces. Njastad[11] introduced α- sets. S.P.Arya and T.Nour[1] introduced gs-closed sets. H.Maki et al.[6, 7] developed gα-closed sets and αg-closed sets. M.Vigneshwaran and R.Devi[17] developed the idea of *gα-closed sets. gsp-closed sets and gpr-closed sets were developed by Dontchev[2] and Gnanambal[4] respectively. Veerakumar[15] introduced ψ-closed sets. Kanimozhi, Balamani and Parvathi[5] introduced g$^\#\psi$-closed sets and T.Nandhini[12] introduced αg$^\#\psi$-closed sets. Gnanambal Ilango[3] has given some of the frontier, border and exterior sets in Intuitionistic Supra Topological Spaces. Jayakymar[8] and Senthilkumaran[13] has also given some new type of exterior and frontier in Topological Spaces. Thus the properties of $\alpha^* g^\# \psi$-border, $\alpha^* g^\# \psi$-frontier and $\alpha^* g^\# \psi$-exterior in topological spaces are derived. A relation between $\alpha^* g^\# \psi$-Frontier and $\alpha^* g^\# \psi$-Border of a Set is given. Also the comparison between $\alpha^* g^\# \psi$-frontier and its $\alpha^* g^\# \psi$-interior and $\alpha^* g^\# \psi$-closure in topological spaces is derived.

2 Preliminaries

Definition 2.1. For any subset A of X, then

(i) Border[3] of A is $Bd(A) = A \setminus \text{int}(A)$.

(ii) Frontier[16] of A $Fr(A) = \text{cl}(A) \setminus \text{int}(A)$.

(iii) Exterior[16] of A $Ex(A) = X \setminus \text{cl}(A)$.

(iv) Interior[14] of A is the union of all open sets contained in A
Iint(A)= $\bigcup$\{K : K is an OS in X and A $\supseteq$ K\}.

(v) Closure[18] of A is the intersection of all closed sets containing A
Icl(A)= $\bigcap$\{K : K is an CS in X and A $\subseteq$ K\}.

Definition 2.2. [13] Let X be a topological spaces and $x \in X$. Let $A \subseteq X$ be a neighborhood of x iff there exists an open set U in X such that $x \in U \subseteq A$.

Theorem 2.1. *[8] Let A be any subset of a topological space X. Then $\alpha^* g^\# \psi int(A)$, $\alpha^* g^\# \psi Ext(A)$ and $\alpha^* g^\# \psi Fr(A)$ are disjoint and $X = \alpha^* g^\# \psi int(A) \cup \alpha^* g^\# \psi Ext(A) \cup \alpha^* g^\# \psi Fr(A)$. Further $\alpha^* g^\# \psi Fr(A)$ is a $\alpha^* g^\# \psi$-closed set.*

Corollary 2.2. *[8]* $\alpha^* g^\# \psi Ext(A) = \alpha^* g^\# \psi cl(A')$.
i.e., $\alpha^* g^\# \psi int(A) = \alpha^* g^\# \psi Ext(A') = (\alpha^* g^\# \psi cl(A'))'$.

3 $\alpha^* g^\# \psi$-Border of a Set

Definition 3.1. For any subset A of X, $\alpha^* g^\# \psi$-border of A is defined by, $\alpha^* g^\# \psi Bd(A) = A \setminus \alpha^* g^\# \psi int(A)$.

Theorem 3.1. *In the topological space (X, τ), for any subset A of X, the following statements are hold:*

(i) $\alpha^* g^\# \psi Bd(\emptyset) = \alpha^* g^\# \psi Bd(X) = \emptyset$

(ii) $A = \alpha^* g^\# \psi int(A) \cup \alpha^* g^\# \psi Bd(A)$

(iii) $\alpha^* g^\# \psi int(A) \cap \alpha^* g^\# \psi Bd(A) = \emptyset$

(iv) $\alpha^* g^\# \psi int(A) = A \setminus \alpha^* g^\# \psi Bd(A)$

(v) $\alpha^* g^\# \psi int(\alpha^* g^\# \psi Bd(A)) = \emptyset$

(vi) *A is $\alpha^* g^\# \psi$-open iff* $\alpha^* g^\# \psi Bd(A) = \emptyset$

(vii) $\alpha^* g^\# \psi Bd(\alpha^* g^\# \psi int(A)) = \emptyset$

(viii) $\alpha^* g^\# \psi Bd(\alpha^* g^\# \psi Bd(A)) = \alpha^* g^\# \psi Bd(A)$

(ix) $\alpha^* g^\# \psi Bd(A) = A \cap \alpha^* g^\# \psi cl(X \setminus A)$

Proof :
Statements (i) to (iv) are obvious by the definition of $\alpha^* g^\# \psi$-border of A.
If possible, let $x \in \alpha^* g^\# \psi int(\alpha^* g^\# \psi Bd(A))$. Then $x \in \alpha^* g^\# \psi Bd(A)$,
since $\alpha^* g^\# \psi Bd(A) \subseteq A$, $x \in \alpha^* g^\# \psi int(\alpha^* g^\# \psi Bd(A)) \subseteq \alpha^* g^\# \psi int(A)$.
Therefore $x \in \alpha^* g^\# \psi int(A) \cap \alpha^* g^\# \psi Bd(A)$, which is contradiction to (iii).
Hence (v) is proved.
A is $\alpha^* g^\# \psi$-open iff $\alpha^* g^\# \psi int(A) = A$.
But $\alpha^* g^\# \psi Bd(A) = A \setminus \alpha^* g^\# \psi int(A)$ implies $\alpha^* g^\# \psi Bd(A) = \emptyset$.
This proves (vi) and (vii).
When $A = \alpha^* g^\# \psi Bd(A)$, then the definition of $\alpha^* g^\# \psi$-border of A becomes $\alpha^* g^\# \psi Bd(\alpha^* g^\# \psi Bd(A))$ $= \alpha^* g^\# \psi Bd(A) \setminus \alpha^* g^\# \psi int(\alpha^* g^\# \psi Bd(A))$.
By using (vii), we get the proof of (viii).
Now,

$$\begin{aligned}\alpha^* g^\# \psi Bd(A) &= A \backslash \alpha^* g^\# \psi int(A)\\ &= A \cap (X \backslash \alpha^* g^\# \psi int(A))\\ &= A \cap \alpha^* g^\# \psi cl(X \backslash A).\end{aligned}$$

Hence (ix) is proved.

4 $\alpha^* g^\# \psi$-Frontier of a Set

Definition 4.1. For any subset A of X, $\alpha^* g^\# \psi$-frontier of A is defined by,
$\alpha^* g^\# \psi Fr(A) = \alpha^* g^\# \psi cl(A) \setminus \alpha^* g^\# \psi int(A)$.

Theorem 4.1. *In the topological space (X, τ), for any subset A of X, the following statements are hold:*

(i) $\alpha^* g^\# \psi Fr(\emptyset) = \alpha^* g^\# \psi Fr(X) = \emptyset$

(ii) $\alpha^* g^\# \psi int(A) \cap \alpha^* g^\# \psi Fr(A) = \emptyset$

(iii) $\alpha^* g^\# \psi Fr(A) \subseteq \alpha^* g^\# \psi cl(A)$

(iv) $\alpha^*g^\#\psi int(A) \cup \alpha^*g^\#\psi Fr(A) = \alpha^*g^\#\psi cl(A)$

(v) $\alpha^*g^\#\psi int(A) = A \setminus \alpha^*g^\#\psi Fr(A)$

(vi) If A is $\alpha^*g^\#\psi$-closed, then $A = \alpha^*g^\#\psi int(A) \cup \alpha^*g^\#\psi Fr(A)$

(vii) $\alpha^*g^\#\psi Fr(\alpha^*g^\#\psi Fr(A)) = \alpha^*g^\#\psi Fr(A)$

(viii) If A is $\alpha^*g^\#\psi$-open, then $A \cap \alpha^*g^\#\psi Fr(A) = \emptyset$

(ix) $X = \alpha^*g^\#\psi cl(A) \cup \alpha^*g^\#\psi cl(X \setminus A)$

(x) If A is $\alpha^*g^\#\psi$-open, then $\alpha^*g^\#\psi Fr(\alpha^*g^\#\psi int(A)) \subseteq \alpha^*g^\#\psi Fr(A)$

(xi) If A is $\alpha^*g^\#\psi$-closed, then $\alpha^*g^\#\psi Fr(\alpha^*g^\#\psi cl(A)) \subseteq \alpha^*g^\#\psi Fr(A)$

(xii) A is $\alpha^*g^\#\psi$-open iff $\alpha^*g^\#\psi Fr(\alpha^*g^\#\psi int(A)) \cap \alpha^*g^\#\psi int(A) = \emptyset$

Proof :
Statement (i) to (vii) are obvious by the definition of $\alpha^*g^\#\psi$-frontier of A.
By remark 3.2, If A is $\alpha^*g^\#\psi$-open, $A = \alpha^*g^\#\psi int(A)$ and
by statement (ii), $A \cap \alpha^*g^\#\psi Fr(A) = \emptyset$.
Hence (viii) is proved.
Statement (ix) is obvious. Since $\alpha^*g^\#\psi int(A)$ is $\alpha^*g^\#\psi$-open, then

$$\alpha^*g^\#\psi int(A) = A, \textit{which implies}$$
$$\alpha^*g^\#\psi(\alpha^*g^\#\psi int(A)) \subseteq \alpha^*g^\#\psi Fr(A).$$

Similarly, (xi) can be proved.
By remark 3.2 and by statement of (ii), (xii) is straight forward.

5 $\alpha^*g^\#\psi$-Exterior of a Set

Definition 5.1. For any subset A of X, $\alpha^*g^\#\psi$-Exterior of A is defined by,
$\alpha^*g^\#\psi Ext(A) = X \setminus \alpha^*g^\#\psi cl(A)$.

Theorem 5.1. *In the topological space (X, τ), for any subset A of X, the following statements are hold:*

(i) $\alpha^*g^\#\psi Ext(X) = \emptyset$, $\alpha^*g^\#\psi Ext(\emptyset) = X$

(ii) $\alpha^*g^\#\psi Ext(A) \subseteq A'$

(iii) $\alpha^*g^\#\psi Ext(A) = \alpha^*g^\#\psi Ext(\alpha^*g^\#\psi Ext(A)')$

(iv) $A \subseteq B \Rightarrow \alpha^*g^\#\psi Ext(B) \subseteq \alpha^*g^\#\psi Ext(A)$

(v) $\alpha^*g^\#\psi int(A) \subseteq \alpha^*g^\#\psi Ext(\alpha^*g^\#\psi Ext(A))$

(vi) $\alpha^*g^\#\psi Ext(A \cup B) = \alpha^*g^\#\psi Ext(A) \cap \alpha^*g^\#\psi Ext(B)$

Proof :
(i) and (ii) Obvious
(iii)

$$\begin{aligned} \textit{Let } \alpha^*g^\#\psi Ext(\alpha^*g^\#\psi Ext(A)') &= \alpha^*g^\#\psi Ext[(\alpha^*g^\#\psi int(A'))'] \\ &= \alpha^*g^\#\psi int[\alpha^*g^\#\psi int(A)'] \\ &= \alpha^*g^\#\psi int(A') \\ &= \alpha^*g^\#\psi Ext(A) \end{aligned}$$

(iv)

$$\begin{aligned} A \subseteq B &\Rightarrow B' \subseteq A' \\ &\Rightarrow \alpha^* g^\# \psi int(B') \subseteq \alpha^* g^\# \psi int(A') \\ &\Rightarrow \alpha^* g^\# \psi Ext(B) \subseteq \alpha^* g^\# \psi Ext(A). \end{aligned}$$

(v)

$$\alpha^* g^\# \psi Ext(A) \subseteq A' \text{ [By (ii)]}$$

$$\text{By (iv), } \alpha^* g^\# \psi int(A) \subseteq \alpha^* g^\# \psi Ext[\alpha^* g^\# \psi Ext(A)]$$

$$\text{i.e., } \alpha^* g^\# \psi int(A) \subseteq \alpha^* g^\# \psi Ext[\alpha^* g^\# \psi Ext(A)]$$

(vi)

$$\begin{aligned} \alpha^* g^\# \psi Ext(A \cup B) &= \alpha^* g^\# \psi int(A \cup B)' \\ &= \alpha^* g^\# \psi int(A' \cap B') \\ &= \alpha^* g^\# \psi int(A') \cap \alpha^* g^\# \psi int(B') \\ &= \alpha^* g^\# \psi Ext(A) \cap \alpha^* g^\# \psi Ext(B) \end{aligned}$$

6 Relation between $\alpha^* g^\# \psi$-Frontier and $\alpha^* g^\# \psi$-Border of a Set

Theorem 6.1. *In the topological space (X, τ), for any subset A of X, the following statements are hold:*

(i) $\alpha^* g^\# \psi Bd(A) \setminus \alpha^* g^\# \psi Fr(A) = \emptyset$

(ii) $\alpha^* g^\# \psi Bd(A) \subseteq \alpha^* g^\# \psi Fr(A)$

(iii) $\alpha^* g^\# \psi Fr(\alpha^* g^\# \psi Bd(A)) = \alpha^* g^\# \psi Bd(A)$

(iv) $\alpha^* g^\# \psi Bd(\alpha^* g^\# \psi Fr(A)) = \alpha^* g^\# \psi Fr(A)$

(v) *If A is $\alpha^* g^\# \psi$-open, then* $\alpha^* g^\# \psi Fr(A) \cup \alpha^* g^\# \psi Bd(A) = \alpha^* g^\# \psi Fr(A)$

(vi) $\alpha^* g^\# \psi Fr(A) \cap \alpha^* g^\# \psi Bd(A) = \alpha^* g^\# \psi Bd(A)$

(vii) $[\alpha^* g^\# \psi Fr(A)]' \cup [\alpha^* g^\# \psi Bd(A)]' = [\alpha^* g^\# \psi Bd(A)]'$

(viii) $[\alpha^* g^\# \psi Fr(A)]' \cap [\alpha^* g^\# \psi Bd(A)]' = [\alpha^* g^\# \psi Fr(A)]'$

Proof : *Statements (i) to (iv) are obvious by the definitions of $\alpha^* g^\# \psi$-border and $\alpha^* g^\# \psi$-frontier of a set.*
Since A is $\alpha^* g^\# \psi$-open, then we have a statement from $\alpha^* g^\# \psi$-border of a set,

$$\begin{aligned} &\alpha^* g^\# \psi Bd(A) = \emptyset \\ &\Rightarrow \alpha^* g^\# \psi Fr(A) \cup \emptyset = \alpha^* g^\# \psi Fr(A). \end{aligned}$$

Hence (v) is proved.
We know from statement (ii),

$$\begin{aligned} &\alpha^* g^\# \psi Bd(A) \subseteq \alpha^* g^\# \psi Fr(A) \\ &\Rightarrow \alpha^* g^\# \psi Fr(A) \cap \alpha^* g^\# \psi Bd(A) = \alpha^* g^\# \psi Bd(A). \end{aligned}$$

It gives the proof of (vi).
By the above statement,

$$[\alpha^* g^\# \psi Fr(A)]' = \alpha^* g^\# \psi Fr(A)$$
$$= [\alpha^* g^\# \psi Bd(A)]',$$

By using De Morgan's law,

$$[\alpha^* g^\# \psi Fr(A) \cap \alpha^* g^\# \psi [Bd(A)]]' = [\alpha^* g^\# \psi Fr(A)]' \cup [\alpha^* g^\# \psi [Bd(A)]]'$$

It gives the proof of (vii). Similarly we can prove for (viii).

7 Relation between $\alpha^* g^\# \psi$-Interior, $\alpha^* g^\# \psi$-Closure and $\alpha^* g^\# \psi$-Frontier

Definition 7.1. Let X be a topological spaces and $x \in X$. Let $A \subseteq X$ be a $\alpha^* g^\# \psi$ neighborhood of x if there exists an $\alpha^* g^\# \psi$-open set U in X such that $x \in U \subseteq A$.

Definition 7.2. Let A be a subset of the topological space (X, τ). A point $x \in A$ is said to be a $\alpha^* g^\# \psi$ interior point of A if A is a $\alpha^* g^\# \psi$ neighborhood of x.

Remark 7.1. Let A be a subset of the topological space (X, τ). Then the set of all $\alpha^* g^\# \psi$-interior points of A is called the $\alpha^* g^\# \psi$ interior of A.

Theorem 7.2. *If A be a subset of the topological space (X, τ). Then, $\alpha^* g^\# \psi$ int(A) = $\cup$ {U: U is $\alpha^* g^\# \psi$ open set and $U \subseteq A$}.*
Proof : *Let A be a subset of the topological space (X, τ). Then,*

$$x \in \alpha^* g^\# \psi int(A) \Leftrightarrow x \text{ is a } \alpha^* g^\# \psi\text{-interior point of A.}$$
$$\Leftrightarrow A \text{ is a } \alpha^* g^\# \psi \text{ neighborhood of point x.}$$
$$\Leftrightarrow \exists\ \alpha^* g^\# \psi \text{ open set G such that } x \in G \subseteq A.$$
$$\Leftrightarrow x \in \cup\{U : U\ is\ \alpha^* g^\# \psi \text{ open}, G \in A.\}$$
$$Hence\ \alpha^* g^\# \psi\ int(A) = \cup\{U : U\ is\ \alpha^* g^\# \psi \text{ open set and } G \subseteq A.\}$$

Theorem 7.3. *Let A and B be any two subsets of the topological space (X, τ). Then,*

(i) $\alpha^* g^\# \psi int(X) = X$ *and* $\alpha^* g^\# \psi int(\emptyset) = \emptyset$

(ii) $\alpha^* g^\# \psi int(A) \subseteq A$

(iii) If B is any $\alpha^* g^\# \psi$-open set contained in A, then $B \subseteq \alpha^* g^\# \psi int(A)$

(iv) If $A \subseteq B$, then $\alpha^* g^\# \psi int(A) \subseteq \alpha^* g^\# \psi int(B)$

(v) $\alpha^* g^\# \psi int(\alpha^* g^\# \psi int(A)) = \alpha^* g^\# \psi int(A)$

Proof : *(i) Since X and $\emptyset$ are $\alpha^* g^\# \psi$-open sets, then by the above theorem*

$$\alpha^* g^\# \psi cl(X) = \cup\{U : U \text{ is } \alpha^* g^\# \psi\text{-open set and } H \subseteq X\}$$
$$= X \cup \{all\ \alpha^* g^\# \psi\text{-open sets}\}$$
$$= X.$$
$$i.e., \alpha^* g^\# \psi cl(X) = X.$$

Since $\emptyset$ is the only $\alpha^* g^\# \psi$-open set contained in $\emptyset$,

$$\Rightarrow \alpha^* g^\# \psi cl(\emptyset) = \emptyset.$$

(ii) Let $x \in \alpha^* g^\# \psi int(A) \Rightarrow x$ *is a* $\alpha^* g^\# \psi$*-interior point of* A.

$\Rightarrow A$ *is a* $\alpha^* g^\# \psi$ *neighborhood of* x

$. \Rightarrow x \in A.$

$Thus\ x \in \alpha^* g^\# \psi int(A) \Rightarrow x \in A.$

Hence $\alpha^* g^\# \psi int(A) \subseteq A.$

(iii) Let B *be any* $\alpha^* g^\# \psi$*-open set, such that* $B \subseteq A$.
Let $x \in B$.
Since B *is a* $\alpha^* g^\# \psi$*-open set contained in* A*, then* x *is a* $\alpha^* g^\# \psi$*-interior point of* A.
(i.e.,) $x \in \alpha^* g^\# \psi int(A)$.
Hence $B \subseteq \alpha^* g^\# \psi int(A)$.

(iv) Let A *and* B *be any two subsets of the topological space* (X, τ) *such that* $A \subseteq B$.
Let $x \in \alpha^* g^\# \psi int(A)$.
Then x *is a* $\alpha^* g^\# \psi$*-interior point of* A *and so* A *is a* $\alpha^* g^\# \psi$*-neighborhood of* x.
Since $B \supset A$*,* B *is also a* $\alpha^* g^\# \psi$*-neighborhood of* x.
This implies that $x \in \alpha^* g^\# \psi cl(B)$.
Thus $x \in \alpha^* g^\# \psi int(A) \Rightarrow x \in \alpha^* g^\# \psi cl(B)$.
Hence $\alpha^* g^\# \psi int(A) \subseteq \alpha^* g^\# \psi cl(B)$.

(v) Let A *be any subset of the topological space* (X, τ).
Then by the definition of $\alpha^* g^\# \psi$*-interior,* $\alpha^* g^\# \psi int(A)$ *is* $\alpha^* g^\# \psi$*-open and hence* $\alpha^* g^\# \psi int(\alpha^* g^\# \psi int(A))$ $= \alpha^* g^\# \psi int(A)$.

Theorem 7.4. *If a subset* A *of the topological space* (X, τ) *is* $\alpha^* g^\# \psi$*-open, then* $\alpha^* g^\# \psi int(A) = A$.
Proof : *Let* A *be a* $\alpha^* g^\# \psi$*-open subset of* (X, τ).
We know that $\alpha^* g^\# \psi int(A) \subseteq A$.
Also, A *is an* $\alpha^* g^\# \psi$*-open set contained in* A.
Then by the above theorem, statement (iii)
we have $A \subseteq \alpha^* g^\# \psi int(A)$.
Hence $\alpha^* g^\# \psi int(A) = A$.

Theorem 7.5. *If* A *and* B *be any two subsets of the topological space* (X, τ)*, then* $\alpha^* g^\# \psi int(A) \cup \alpha^* g^\# \psi int(B) \subseteq \alpha^* g^\# \psi int(A \cup B)$.
Proof : *We know that* $A \subseteq A \cup B$ *and* $B \subseteq A \cup B$.
We have, by a theorem, statement (iv),
"If $A \subseteq B$*, then* $\alpha^* g^\# \psi int(A) \subseteq \alpha^* g^\# \psi int(B)$*"*

$$\alpha^* g^\# \psi int(A) \subseteq \alpha^* g^\# \psi int(A \cup B) and$$

$$\alpha^* g^\# \psi int(B) \subseteq \alpha^* g^\# \psi int(A \cup B),$$

$$\Rightarrow \alpha^* g^\# \psi int(A) \cup \alpha^* g^\# \psi int(B) \subseteq \alpha^* g^\# \psi int(A \cup B).$$

Theorem 7.6. *If* A *and* B *be any two subsets of the topological space* (X, τ)*, then* $\alpha^* g^\# \psi int(A \cap B) = \alpha^* g^\# \psi int(A) \cap \alpha^* g^\# \psi int(B)$.
Proof : *We know that* $A \cap B \subseteq A$ *and* $A \cap B \subseteq B$.
We have, by a theorem, statement (iv),
"If $A \subseteq B$*, then* $\alpha^* g^\# \psi int(A) \subseteq \alpha^* g^\# \psi int(B)$*"*

$$\alpha^* g^\# \psi int(A \cap B) \subseteq \alpha^* g^\# \psi int(A),\ and$$

$$\alpha^* g^\# \psi int(A \cap B) \subseteq \alpha^* g^\# \psi int(B).$$

$$\Rightarrow \alpha^* g^\# \psi int(A \cap B) \subseteq \alpha^* g^\# \psi int(A) \cap \alpha^* g^\# \psi int(B). \ \rightarrow \mathbf{(1)}$$

Again,

$$\textit{Let } x \in \alpha^* g^{\#}\psi int(A) \cap \alpha^* g^{\#}\psi int(B).$$
$$\textit{Then } x \in \alpha^* g^{\#}\psi int(A) \textit{ and}$$
$$x \in \alpha^* g^{\#}\psi int(B).$$

Hence x is a $\alpha^* g^{\#}\psi$-interior point of each sets A and B.
It follows that A and B are $\alpha^* g^{\#}\psi$-neighborhoods of x, so that their intersection $A \cap B$ is also a $\alpha^* g^{\#}\psi$-neighborhoods of x.

$$\textit{Hence } x \in \alpha^* g^{\#}\psi int(A \cap B).$$
$$\textit{Thus } x \in \alpha^* g^{\#}\psi int(A) \cap \alpha^* g^{\#}\psi int(B)$$
$$\Rightarrow x \in \alpha^* g^{\#}\psi int(A \cap B).$$

Therefore

$$\alpha^* g^{\#}\psi int(A) \cap \alpha^* g^{\#}\psi int(B) \subseteq \alpha^* g^{\#}\psi int(A \cap B). \rightarrow \textbf{(2)}$$

*From **(1)** and **(2)**, we get $\alpha^* g^{\#}\psi int(A \cap B) = \alpha^* g^{\#}\psi int(A) \cap \alpha^* g^{\#}\psi int(B)$.*

Theorem 7.7. *If A and B be any two subsets of the topological space (X, τ). Then,*

(i) $\alpha^* g^{\#}\psi cl(X) = X$ *and* $\alpha^* g^{\#}\psi cl(\emptyset) = \emptyset$

(ii) $A \subseteq \alpha^* g^{\#}\psi cl(A)$

(iii) If B is any $\alpha^* g^{\#}\psi$-closed set containing A, then $\alpha^* g^{\#}\psi cl(A) \subseteq B$

(iv) If $A \subseteq B$, then $\alpha^* g^{\#}\psi cl(A) \subseteq \alpha^* g^{\#}\psi cl(B)$

(v) $\alpha^* g^{\#}\psi cl(\alpha^* g^{\#}\psi cl(A)) = \alpha^* g^{\#}\psi cl(A)$.

Proof :
(i) By the definition of $\alpha^* g^{\#}\psi$-closure,
X is the only $\alpha^* g^{\#}\psi$-closed set containing X.
Therefore

$$\begin{aligned} \alpha^* g^{\#}\psi cl(X) &= \textit{ Intersection of all the } \alpha^* g^{\#}\psi\textit{-closed sets containing X} \\ &= \cap\{X\} \\ &= X. \\ \textit{i.e.,}\alpha^* g^{\#}\psi cl(X) &= X. \end{aligned}$$

By the definition of $\alpha^* g^{\#}\psi$-closure,

$$\begin{aligned} \alpha^* g^{\#}\psi cl(\emptyset) &= \textit{ Intersection of all the } \alpha^* g^{\#}\psi\textit{-closed sets containing } \emptyset \\ &= \emptyset \cap \textit{ any } \alpha^* g^{\#}\psi\textit{-closed sets containing } \emptyset \\ &= \emptyset. \\ \textit{i.e.,}\alpha^* g^{\#}\psi cl(\emptyset) &= \emptyset. \end{aligned}$$

(ii) By the definition of $\alpha^* g^{\#}\psi$-closure of A, it is obvious that

$$A \subseteq \alpha^* g^{\#}\psi cl(A).$$

(iii) Let B be any $\alpha^* g^{\#}\psi$-closed set containing A.
Since $\alpha^* g^{\#}\psi cl(A)$ is the intersection of all $\alpha^* g^{\#}\psi$-closed sets containing A,
$\alpha^* g^{\#}\psi cl(A)$ is contained in every $\alpha^* g^{\#}\psi$-closed set containing A.

Hence

$$\alpha^* g^\# \psi cl(A) \subseteq B.$$

(iv) Let A and B be any two subsets of the topological space (X, τ), such that $A \subseteq B$.
By the definition of $\alpha^* g^\# \psi$-closure,

$$\alpha^* g^\# \psi cl(B) = \cap\{U : B \subseteq U \in \alpha^* g^\# \psi - C(X)\}.$$

$$\text{If } B \subseteq U \in \alpha^* g^\# \psi - C(X), \text{then } \alpha^* g^\# \psi cl(B) \subseteq U.$$

$$\begin{aligned} \text{Since } A \subseteq B, A \subseteq B \subseteq U &\in \alpha^* g^\# \psi - C(X), \\ \text{we have } \alpha^* g^\# \psi cl(A) &\subseteq U. \\ \text{Therefore } \alpha^* g^\# \psi cl(A) &\subseteq \cap U : B \subseteq U \in \alpha^* g^\# \psi - C(X) \\ &= \alpha^* g^\# \psi cl(B). \\ \text{Hence } \alpha^* g^\# \psi cl(A) &\subseteq \alpha^* g^\# \psi cl(B). \end{aligned}$$

(v) Let B be a $\alpha^* g^\# \psi$-closed set containing A. Then by the definition,

$$\alpha^* g^\# \psi cl(A) \subseteq B.$$

Since B is $\alpha^* g^\# \psi$-closed set and contains $\alpha^* g^\# \psi$cl(A) which contained in every $\alpha^* g^\# \psi$-closed set containing A.
It follows that

$$\begin{aligned} \alpha^* g^\# \psi cl(\alpha^* g^\# \psi cl(A)) &\subseteq \alpha^* g^\# \psi cl(A). \\ \text{Therefore } \alpha^* g^\# \psi cl(\alpha^* g^\# \psi cl(A)) &= \alpha^* g^\# \psi cl(A). \end{aligned}$$

Theorem 7.8. *If a subset A of the topological space (X, τ) is $\alpha^* g^\# \psi$-closed, then it is a $\alpha^* g^\# \psi cl(A)$.*
Proof : *Let A be $\alpha^* g^\# \psi$-closed subset of the topological space (X, τ).*
We know that $A \subseteq \alpha^* g^\# \psi cl(A)$.
Also, $A \subseteq A$ and A is $\alpha^* g^\# \psi$-closed.
Then by the above theorem, statement (iii),
"If B is any $\alpha^* g^\# \psi$-closed set containing A, then $\alpha^* g^\# \psi cl(A) \subseteq B$"
we have $\alpha^* g^\# \psi cl(A) \subseteq A$.
Hence $\alpha^* g^\# \psi cl(A) = A$.

Theorem 7.9. *If A and B be any two subsets of the topological space (X, τ), then $\alpha^* g^\# \psi cl(A \cap B) \subseteq \alpha^* g^\# \psi cl(A) \cap \alpha^* g^\# \psi cl(B)$.*
Proof : *Let A and B be any two subsets of the topological space (X, τ).*
Clearly, $A \cap B \subseteq A$ and $A \cap B \subseteq B$.
We have, by a theorem, statement (iv),
"If $A \subseteq B$, then $\alpha^* g^\# \psi cl(A) \subseteq \alpha^* g^\# \psi cl(B)$",

$$\begin{aligned} \alpha^* g^\# \psi cl(A \cap B) &\subseteq \alpha^* g^\# \psi cl(A) \text{ and} \\ \alpha^* g^\# \psi cl(A \cap B) &\subseteq \alpha^* g^\# \psi cl(B). \\ \text{Hence } \alpha^* g^\# \psi cl(A \cap B) &\subseteq \alpha^* g^\# \psi cl(A) \cap \alpha^* g^\# \psi cl(B). \end{aligned}$$

Theorem 7.10. *If A and B be any two subsets of the topological space (X, τ), then $\alpha^* g^\# \psi cl(A \cup B) = \alpha^* g^\# \psi cl(A) \cup \alpha^* g^\# \psi cl(B)$.*
Proof : *Let A and B be any two subsets of the topological space (X, τ). Clearly,*

$$\begin{aligned} A &\subseteq A \cup B \text{ and} \\ B &\subseteq A \cup B. \\ \text{Hence } \alpha^* g^\# \psi cl(A) \cup \alpha^* g^\# \psi cl(B) &\subseteq \alpha^* g^\# \psi cl(A \cup B). \rightarrow \textbf{(1)} \end{aligned}$$

Now to prove

$$\alpha^* g^\# \psi cl(A \cup B) \subseteq \alpha^* g^\# \psi cl(A) \cup \alpha^* g^\# \psi cl(B).$$
$$\textit{Let } x \in \alpha^* g^\# \psi cl(A \cup B) \textit{ and}$$
$$\textit{suppose } x \notin \alpha^* g^\# \psi cl(A) \cup \alpha^* g^\# \psi cl(B).$$

Then there exists $\alpha^* g^\# \psi$-closed sets A' and B' with

$$A \subseteq A',$$
$$B \subseteq B' \textit{and}$$
$$x \notin A' \cup B'.$$
$$\textit{We have } A \cup B \subseteq A' \cup B' \textit{ and}$$

$A' \cup B'$ is $\alpha^* g^\# \psi$-closed set,
since we know, "The union of two $\alpha^* g^\# \psi$-closed subsets of (X, τ) is also $\alpha^* g^\# \psi$-closed subset of (X, τ)", such that $x \notin A' \cup B'$.

$$\textit{Thus } x \notin \alpha^* g^\# \psi cl(A \cup B) \textit{ which is a contradiction to } x$$
$$\in \alpha^* g^\# \psi cl(A \cup B).$$
$$\textit{Hence } \alpha^* g^\# \psi cl(A \cup B) \subseteq \alpha^* g^\# \psi cl(A) \cup \alpha^* g^\# \psi cl(B) \rightarrow \textbf{(2)}$$

*From **(1)** and **(2)**, we get $\alpha^* g^\# \psi cl(A \cup B) = \alpha^* g^\# \psi cl(A) \cup \alpha^* g^\# \psi cl(B)$.*

Theorem 7.11. *For any $x \in X$, $x \in \alpha^* g^\# \psi cl(A)$ iff $U \cap A \neq \emptyset$ for every $\alpha^* g^\# \psi$-open sets U containing x.*
Proof : *Let $x \in X$ and $x \in \alpha^* g^\# \psi cl(A)$.*
To prove $U \cap A \neq \emptyset$ for every $\alpha^* g^\# \psi$-open set U containing x.
Suppose there exists a $\alpha^* g^\# \psi$-open set U containing x such that
$U \cap A = \emptyset$.
Then $A \subseteq X - U$ and $X - U$ is $\alpha^* g^\# \psi$-closed.
We have $\alpha^* g^\# \psi cl(A) \subseteq X - U$.
This shows that $x \notin \alpha^* g^\# \psi cl(A)$, which is a contradiction.
Hence $U \cap A \neq \emptyset$ for every $\alpha^* g^\# \psi$-open set U containing x.

Conversely, let $U \cap A \neq \emptyset$ for every $\alpha^* g^\# \psi$-open set U containing x.
To prove $x \in \alpha^* g^\# \psi cl(A)$.
Suppose $x \notin \alpha^* g^\# \psi cl(A)$.
Then there exists a $\alpha^* g^\# \psi$-closed subset V containing U such that $x \notin V$.
Then $x \in X - V$ and $X - V$ is $\alpha^* g^\# \psi$-open.
Also $(X - V) \cap A = \emptyset$, which is a contradiction.
Hence $x \in \alpha^* g^\# \psi cl(A)$.

Theorem 7.12. *Let A be any subset of the topological space (X, τ). Then*

(i) $[\alpha^* g^\# \psi int(A)]' = \alpha^* g^\# \psi cl(A')$

(ii) $\alpha^* g^\# \psi int(A) = [\alpha^* g^\# \psi cl(A')]'$

(iii) $\alpha^* g^\# \psi cl(A) = [\alpha^* g^\# \psi int(A')]'$.

Proof : *(i) Let $x \in [\alpha^* g^\# \psi int(A)]'$. Then $x \in \alpha^* g^\# \psi int(A)$.*
That is every $\alpha^* g^\# \psi$-open set U containing x is such that $U \nsubseteq A$.
That is every $\alpha^* g^\# \psi$-open set U containing x is such that $U \cap A' \neq \emptyset$.
By the above theorem,
$x \in \alpha^* g^\# \psi cl(A')$ and therefore $[\alpha^* g^\# \psi\text{-}int(A)]' \subseteq \alpha^* g^\# \psi cl(A')$.
Conversely, let $x \in \alpha^* g^\# \psi cl(A')$.
Then by the above theorem,

every $\alpha^* g^\# \psi$-open set U containing x such that $U \cap A' \neq \emptyset$.
i.e., every $\alpha^* g^\# \psi$-open set U containing x such that $U \nsubseteq A$.
$\Rightarrow \alpha^* g^\# \psi$-interior of A, $x \notin \alpha^* g^\# \psi int(A)$.
i.e., $x \in [\alpha^* g^\# \psi cl(A)]'$ and $\alpha^* g^\# \psi cl(A') \subseteq [\alpha^* g^\# \psi cl(A)]'$.
Thus $[\alpha^* g^\# \psi cl(A)]' = \alpha^* g^\# \psi cl(A')$.
(ii) By taking complement in (i) we get,
$\alpha^* g^\# \psi int(A) = [\alpha^* g^\# \psi cl(A')]'$.
(iii) By replacing A by A' in (i), we get
$\alpha^* g^\# \psi cl(A) = [\alpha^* g^\# \psi int(A')]'$.

Theorem 7.13. *Let X be a topological space and let $A \subseteq X$. Then*
$\alpha^* g^\# \psi cl(A) = \alpha^* g^\# \psi int(A) \cup \alpha^* g^\# \psi Fr(A)$
Proof : $[\alpha^* g^\# \psi\text{-}cl(A)]' = \alpha^* g^\# \psi Ext(A)$.
Taking complements,

$$\begin{aligned}\alpha^* g^\# \psi cl(A) &= (\alpha^* g^\# \psi Ext(A))' \\ &= \alpha^* g^\# \psi int(A) \cup \alpha^* g^\# \psi Fr(A).\end{aligned}$$

Note : $\alpha^* g^\# \psi Fr(A) \subseteq \alpha^* g^\# \psi cl(A)$

Theorem 7.14. *Let X be a topological space and let $A \subseteq X$. Then*
$\alpha^* g^\# \psi cl(A) = A \cup \alpha^* g^\# \psi Fr(A)$
Proof :

$$\begin{aligned} A \subseteq \alpha^* g^\# \psi cl(A) \textit{ and} \\ \alpha^* g^\# \psi Fr(A) \subseteq \alpha^* g^\# \psi cl(A).\end{aligned}$$

Hence

$$A \cup \alpha^* g^\# \psi Fr(A) \subseteq \alpha^* g^\# \psi cl(A).$$

Also

$$\begin{aligned}\alpha^* g^\# \psi Fr(A) &= \alpha^* g^\# \psi int(A) \alpha^* g^\# \psi Ext(A)' \\ &= (\alpha^* g^\# \psi int(A))' \alpha^* g^\# \psi Ext(A')\end{aligned}$$

Again, since

$$\begin{aligned}\alpha^* g^\# \psi int(A) \subseteq A \textit{ and} \\ \alpha^* g^\# \psi cl(A) = \alpha^* g^\# \psi int(A) \cup \alpha^* g^\# \psi Fr(A).\end{aligned}$$

It follows that

$$\alpha^* g^\# \psi cl(A) \subseteq A \cup \alpha^* g^\# \psi Fr(A).$$

Hence

$$\alpha^* g^\# \psi cl(A) = A \cup \alpha^* g^\# \psi Fr(A).$$

Note : *Every $\alpha^* g^\# \psi$ closed subset is the disjoint union of its $\alpha^* g^\# \psi$-interior and $\alpha^* g^\# \psi$-frontier.*

Theorem 7.15. *Let X be a topological space and let A and B be subsets of X. Then*

(i) $\alpha^* g^\# \psi Fr(A) = \alpha^* g^\# \psi cl(A) \cap \alpha^* g^\# \psi cl(A') = \alpha^* g^\# \psi cl(A) - \alpha^* g^\# \psi int(A)$.

(ii) $\alpha^* g^\# \psi int(A) = A - \alpha^* g^\# \psi Fr(A)$.

(iii) $(\alpha^* g^\# \psi Fr(A))' = \alpha^* g^\# \psi int(A) \cup \alpha^* g^\# \psi int(A')$.

(iv) $\alpha^* g^\# \psi Fr(\alpha^* g^\# \psi int(A)) \subseteq \alpha^* g^\# \psi Fr(A)$.

(v) $\alpha^* g^\# \psi Fr(\alpha^* g^\# \psi cl(A)) \subseteq \alpha^* g^\# \psi Fr(A)$.

(vi) $\alpha^* g^\# \psi Fr(A \cup B) \subseteq \alpha^* g^\# \psi Fr(A) \cup \alpha^* g^\# \psi Fr(B)$.

(vii) $\alpha^* g^\# \psi Fr(A \cap B) \subseteq \alpha^* g^\# \psi Fr(A) \cup \alpha^* g^\# \psi Fr(B)$.

Proof :
(i)

$$\begin{aligned}\alpha^* g^\# \psi Fr(A) &= (\alpha^* g^\# \psi int(A) \cup \alpha^* g^\# \psi Ext(A))' \\ &= (\alpha^* g^\# \psi int(A))' \cap (\alpha^* g^\# \psi Ext(A))' \\ &= (\alpha^* g^\# \psi cl(A') \cap \alpha^* g^\# \psi cl(A).\end{aligned}$$

Now

$$\begin{aligned}(\alpha^* g^\# \psi cl(A) \cap \alpha^* g^\# \psi cl(A') &= \alpha^* g^\# \psi cl(A) - \alpha^* g^\# \psi cl(A'))' \\ &= \alpha^* g^\# \psi cl(A) - \alpha^* g^\# \psi int(A).\end{aligned}$$

Hence

$$\alpha^* g^\# \psi Fr(A) = \alpha^* g^\# \psi cl(A) - \alpha^* g^\# \psi int(A).$$

(ii)

$$\begin{aligned}A - \alpha^* g^\# \psi Fr(A) &= A - (\alpha^* g^\# \psi cl(A) - \alpha^* g^\# \psi int(A)) \\ &= \alpha^* g^\# \psi int(A).\end{aligned}$$

(iii)

$$\begin{aligned}(\alpha^* g^\# \psi Fr(A))' &= (\alpha^* g^\# \psi cl(A) \cap \alpha^* g^\# \psi cl(A'))'. \\ &= (\alpha^* g^\# \psi cl(A))' \cup (\alpha^* g^\# \psi cl(A'))'.\end{aligned}$$

By Corollary

$$\begin{aligned}(\alpha^* g^\# \psi cl(A'))' &= \alpha^* g^\# \psi int(A) \\ \textit{So } (\alpha^* g^\# \psi cl(A))' &= \alpha^* g^\# \psi int(A')\end{aligned}$$

Hence

$$(\alpha^* g^\# \psi Fr(A))' = \alpha^* g^\# \psi int(A) \cup \alpha^* g^\# \psi int(A').$$

(iv)

$$\begin{aligned}\alpha^* g^\# \psi Fr(\alpha^* g^\# \psi int(A)) &= \alpha^* g^\# \psi cl(\alpha^* g^\# \psi int(A)) \cap \alpha^* g^\# \psi cl(\alpha^* g^\# \psi int(A))' \quad \textit{by (i).} \\ &= \alpha^* g^\# \psi cl(\alpha^* g^\# \psi int(A)) \cap \alpha^* g^\# \psi cl(\alpha^* g^\# \psi cl(A')) \cup (\alpha^* g^\# \psi cl(A) \\ &\qquad \cap \alpha^* g^\# \psi cl(A') \\ &= \alpha^* g^\# \psi Fr(A).\end{aligned}$$

(v)

$$\begin{aligned}\alpha^* g^\# \psi Fr(\alpha^* g^\# \psi cl(A)) &= \alpha^* g^\# \psi cl(\alpha^* g^\# \psi cl(A)) \cap \alpha^* g^\# \psi cl(\alpha^* g^\# \psi cl(A))' \\ &= \qquad \alpha^* g^\# \psi cl(A)) \cap \alpha^* g^\# \psi cl(\alpha^* g^\# \psi cl(A))'\end{aligned}$$

Hence

$$\alpha^* g^\# \psi Fr(\alpha^* g^\# \psi cl(A)) \subseteq \alpha^* g^\# \psi cl(A)) \cap \alpha^* g^\# \psi cl(A') = \alpha^* g^\# \psi Fr(A).$$

(vi)

$$\begin{aligned}\alpha^* g^\# \psi Fr(A \cup B) &= \alpha^* g^\# \psi cl(A \cup B) \cap \alpha^* g^\# \psi cl(A \cup B)' \\ &= (\alpha^* g^\# \psi cl(A) \cup \alpha^* g^\# \psi cl(B)) \cap \alpha^* g^\# \psi cl(A' \cap B') \\ &\subseteq (\alpha^* g^\# \psi cl(A) \cup \alpha^* g^\# \psi cl(B)) \cap (\alpha^* g^\# \psi cl(A') \cap \alpha^* g^\# \psi cl(B')) \\ &= \alpha^* g^\# \psi cl(A) \cap (\alpha^* g^\# \psi cl(A') \cap \alpha^* g^\# \psi cl(B')) \cup \alpha^* g^\# \psi cl(B) \\ &\qquad \cap (\alpha^* g^\# \psi cl(A') \cap \alpha^* g^\# \psi cl(B')). \\ &= (\alpha^* g^\# \psi cl(A) \cap \alpha^* g^\# \psi cl(A') \cap \alpha^* g^\# \psi cl(B')) \cup (\alpha^* g^\# \psi cl(B) \\ &\qquad \cap \alpha^* g^\# \psi cl(B') \cap \alpha^* g^\# \psi cl(A')). \\ &= (\alpha^* g^\# \psi Fr(A) \cap \alpha^* g^\# \psi cl(B')) \cup (\alpha^* g^\# \psi Fr(B) \cap \alpha^* g^\# \psi cl(A')). \\ &\subseteq \alpha^* g^\# \psi Fr(A) \cup \alpha^* g^\# \psi Fr(B).\end{aligned}$$

Hence

$$\alpha^* g^\# \psi Fr(A \cup B) \subseteq \alpha^* g^\# \psi Fr(A) \cup \alpha^* g^\# \psi Fr(B).$$

(vii)

$$\begin{aligned}\alpha^* g^\# \psi Fr(A \cap B) &= \alpha^* g^\# \psi cl(A \cap B) \cap \alpha^* g^\# \psi cl(A \cap B)' \\ &= (\alpha^* g^\# \psi cl(A) \cap \alpha^* g^\# \psi cl(B)) \cap \alpha^* g^\# \psi cl(A' \cup B') \\ &\subseteq \alpha^* g^\# \psi cl(A) \cap \alpha^* g^\# \psi cl(B) \cap (\alpha^* g^\# \psi cl(A') \cup \alpha^* g^\# \psi cl(B')) \\ &= (\alpha^* g^\# \psi cl(A) \cap \alpha^* g^\# \psi cl(B) \cap \alpha^* g^\# \psi cl(A')) \cup (\alpha^* g^\# \psi cl(A) \\ &\qquad \cap \alpha^* g^\# \psi cl(B) \cap \alpha^* g^\# \psi cl(B')) \\ &= (\alpha^* g^\# \psi cl(A) \cap \alpha^* g^\# \psi cl(A') \cap \alpha^* g^\# \psi cl(B)) \cup (\alpha^* g^\# \psi cl(A) \\ &\qquad \cap \alpha^* g^\# \psi cl(B) \cap \alpha^* g^\# \psi cl(B')). \\ &= (\alpha^* g^\# \psi Fr(A) \cap \alpha^* g^\# \psi cl(B)) \cup (\alpha^* g^\# \psi cl(A) \cap \alpha^* g^\# \psi Fr(B)). \\ &\subseteq \alpha^* g^\# \psi Fr(A) \cup \alpha^* g^\# \psi Fr(B).\end{aligned}$$

Hence

$$\alpha^* g^\# \psi Fr(A \cap B) \subseteq \alpha^* g^\# \psi Fr(A) \cup \alpha^* g^\# \psi Fr(B).$$

Theorem 7.16. *Let X be a topological space and let $A \subseteq X$. Then*

(i) If A is open, then $\alpha^* g^\# \psi Fr(A) = \alpha^* g^\# \psi cl(A) - A$.

(ii) $\alpha^* g^\# \psi Fr(A) = \emptyset$ iff A is $\alpha^* g^\# \psi$ open as well as $\alpha^* g^\# \psi$-closed.

(iii) A is $\alpha^* g^\# \psi$-open iff $A \cap \alpha^* g^\# \psi Fr(A) = \emptyset$.

(iv) A is $\alpha^* g^\# \psi$-closed iff $\alpha^* g^\# \psi Fr(A) \subseteq A$.

Proof :

(i) By theorem 7.6

$$\alpha^* g^{\#}\psi Fr(A) = \alpha^* g^{\#}\psi cl(A) - \alpha^* g^{\#}\psi int(A).$$

$$A \text{ is open } \Rightarrow A \text{ is } \alpha^* g^{\#}\psi\text{-open}$$
$$\Rightarrow \alpha^* g^{\#}\psi int(A) = A.$$

Hence

$$\alpha^* g^{\#}\psi Fr(A) = \alpha^* g^{\#}\psi cl(A) - A.$$

(ii)

$$\text{Let } \alpha^* g^{\#}\psi Fr(A) = \emptyset$$
$$\text{Then } \alpha^* g^{\#}\psi Fr(A) = \emptyset$$
$$\Rightarrow \alpha^* g^{\#}\psi cl(A) - \alpha^* g^{\#}\psi int(A) = \emptyset$$
$$\Rightarrow \alpha^* g^{\#}\psi cl(A) = \alpha^* g^{\#}\psi int(A) = A.$$
$$\Rightarrow A \text{ is } \alpha^* g^{\#}\psi \text{ open as well as closed.}$$

Conversely, let A be $\alpha^* g^{\#}\psi$-open as well as closed.

$$\alpha^* g^{\#}\psi Fr(A) = \alpha^* g^{\#}\psi cl(A) - \alpha^* g^{\#}\psi int(A).$$

Since A is $\alpha^* g^{\#}\psi$-closed,

$$\alpha^* g^{\#}\psi cl(A) = A.$$

Since A is $\alpha^* g^{\#}\psi$-open,

$$\alpha^* g^{\#}\psi int(A) = A.$$

Hence $\alpha^* g^{\#}\psi Fr(A) = \emptyset$.
(iii) We know that

$$\alpha^* g^{\#}\psi Fr(A) = \alpha^* g^{\#}\psi cl(A) \cap \alpha^* g^{\#}\psi cl(A').$$

Let A be $\alpha^* g^{\#}\psi int(A)$-open. Hence A' is $\alpha^* g^{\#}\psi$-closed.

$$\alpha^* g^{\#}\psi cl(A') = A'.$$
$$\begin{aligned}\text{Now, } A \cap \alpha^* g^{\#}\psi Fr(A) &= A \cap (\alpha^* g^{\#}\psi cl(A) \cap \alpha^* g^{\#}\psi cl(A')) \\ &= A \cap (\alpha^* g^{\#}\psi cl(A) \cap (A")) \\ &= (A \cap \alpha^* g^{\#}\psi cl(A)) \cap A' \\ &= A \cap A' \\ &= \emptyset\end{aligned}$$

Conversely, Let $A \cap \alpha^* g^{\#}\psi Fr(A) = \emptyset$.

$$\Rightarrow A \cap (\alpha^* g^{\#}\psi cl(A) \cap (\alpha^* g^{\#}\psi cl(A')) = \emptyset.$$
$$\Rightarrow \alpha^* g^{\#}\psi cl(A') = A',$$
$$\Rightarrow A' \text{is } \alpha^* g^{\#}\psi\text{-closed}$$
$$\Rightarrow A \text{is } \alpha^* g^{\#}\psi\text{-open.}$$

(iv) Let A be $\alpha^* g^{\#}\psi$-closed. Then $\alpha^* g^{\#}\psi cl(A) = A$.

$$\textit{Hence } \alpha^* g^{\#}\psi Fr(A) = \alpha^* g^{\#}\psi cl(A) \cap \alpha^* g^{\#}\psi cl(A')$$
$$= A \cap \alpha^* g^{\#}\psi cl(A').$$
$$\textit{So that } \alpha^* g^{\#}\psi Fr(A) \subseteq A.$$
$$\textit{Conversely, let } \alpha^* g^{\#}\psi Fr(A) \subseteq A.$$
$$\textit{Then } A \cup \alpha^* g^{\#}\psi Fr(A) = A.$$
$$\textit{But } A \cup \alpha^* g^{\#}\psi Fr(A) = \alpha^* g^{\#}\psi cl(A).$$
$$\textit{(i.e) } A = \alpha^* g^{\#}\psi cl(A).$$
$$\textit{Hence A is } \alpha^* g^{\#}\psi\textit{-closed}.$$

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