

# ALGORITHMIC ANALYSIS OF EQUITABLE COLORING IN CENTRAL GRAPHS

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**Abstract** The study emphasizes the relationship between central graph modifications and equitable chromatic characteristics in special snake graphs, including the triangular snake graph, alternate triangular snake graph, diamond triangular snake graph, alternate quadrilateral snake graph, double triangular snake graph, and double alternate triangular snake graph.

## 1 Introduction

Let  $H = (V, E)$  be a simple, undirected graph. A proper  $k$ -coloring of  $H$  is a mapping  $f : V \rightarrow 1, 2, \dots, k$  such that  $f(u) \neq f(v)$  for every edge  $uv \in E$ . An equitable  $k$ -coloring [9] is a proper  $k$ -coloring in which the sizes of any two color classes differ by at most one; that is, for all  $i, j \in 1, \dots, k$ ,  $||f^{-1}(i)| - |f^{-1}(j)|| \leq 1$ . Hajnal and Szemerédi [5] proved that every graph  $H$  satisfies  $\chi_{=}(H) \leq \Delta(H) + 1$ , settling a conjecture by Erdős. Meyer [9] introduced the concept of equitable coloring and proposed the Equitable Coloring Conjecture (ECC): if  $H$  is connected and  $H \notin K_n, C_{2n+1}$ , then  $\chi_{=}(H) \leq \Delta(H)$ . A strengthened version, the Equitable  $\Delta$ -Coloring Conjecture [3], excludes graphs of the form  $K_{2n+1, 2n+1}$  as well and asserts that such graphs are equitably  $\Delta$ -colorable. These conjectures have been verified for various graph classes: 1. Bipartite graphs:  $\chi_{=}(H) = 2$  [13], 2. Outerplanar graphs with  $\Delta(H) \geq 3$  [14], 3. Planar graphs with  $\Delta(H) \geq 13$  [15]. From a complexity standpoint, deciding whether  $\chi_{=}(H) \leq 3$  is NP-complete [7].

**Clique Bound for Equitable Coloring.** Let  $H$  be a graph and let  $\omega(H)$  denote its clique number, i.e., the size of the largest complete subgraph in  $H$ . Since all vertices in a clique must receive distinct colors in any proper coloring, and equitable coloring is a proper coloring with additional constraints, it follows that

$$\chi_{=}(H) \geq \omega(H)$$

## 2 Preliminaries

Let  $H = (V, E)$  be a simple undirected graph. The **central graph** [4], [12] of  $H$ , denoted by  $C(H)$ , is obtained as follows:

- For each edge  $uv \in E$ , insert a new vertex  $w_{uv}$  (unique for each edge) and replace the edge  $uv$  with the path  $u - w_{uv} - v$ .
- Then, for every pair of vertices  $u, v \in V$  such that  $uv \notin E$  and  $u \neq v$ , add the edge  $uv$ .

The **Triangular Snake Graph** [1], denoted as  $TS_y$ , is constructed from a path graph  $P_y$  with vertex set  $V(P_y) = u_1, u_2, \dots, u_y$  a new vertex  $w_{m'}$  is introduced for each edge  $u_m u_{m+1}$ , and edges  $w_{m'} u_m$  and  $w_{m'} u_{m+1}$  are added, forming  $y - 1$  triangular subgraphs.

An **Alternate Triangular Snake Graph** [11], denoted as  $ATS_y$ , is constructed from a path graph  $P_y$  with vertices  $u_1, u_2, \dots, u_y$ , the construction involves introducing a new vertex  $w_{m'}$  for every alternate edge  $u_m u_{m+1}$  and connecting it to both  $u_m$  and  $u_{m+1}$ , thereby forming a triangle. This results in a sequence of alternating triangles along the path.

The **Diamond Snake Graph** [8], [2], denoted as  $DTS_y$ , is constructed from a path graph  $P_y$  by systematically replacing each edge with a quadrilateral cycle  $C_4$ . Specifically, given a path graph  $P_y = u_1, u_2, \dots, u_{y+1}$  each edge  $u_m u_{m+1}$  is extended into a cycle by introducing two additional vertices  $u'_{m'}, u''_{m'} : 1 \leq m' \leq y$ , such that the resulting set of edges forms a **sequence of interconnected cycles** in diamond shape. Each  $C_4$  forms the structure:  $u_m - u'_m - u_{m+1} - u''_m - u_m$ , for  $1 \leq m \leq y$

An **Alternate Quadrilateral Snake Graph** [10], denoted as  $A(QS_y)$ , is obtained from a path  $P_y$  with vertices  $u_1, u_2, \dots, u_y$  by modifying its structure as follows: Each vertex  $u_m$  and  $u_{m+1}$  (alternatively) is connected to new vertices  $u'_{m'}$  and  $u'_{m'+1}$ , respectively; Each pair  $u'_{m'}, u'_{m'+1}$  is also connected. Thus, every alternate edge of the path is replaced by a cycle  $C_4$ , forming a repeating quadrilateral pattern along the path.

A **Double Triangular Snake Graph** [2], denoted by  $D_y$ , is obtained from a path  $P_y$  consisting of  $y$  vertices  $u_1, u_2, \dots, u_y$ , where  $y \geq 2$  by modifying its structure as follows: For each internal edge  $u_m u_{m+1}$  of the path, where  $m$  is odd and  $1 \leq m \leq y - 1$ , introduce two new vertices:  $w_{m'}$  (top) and  $w'_{m'}$  (bottom). Now add edges  $w_{m'} u_m$  and  $w_{m'} u_{m+1}$ , forming triangle  $\Delta w_{m'} u_m u_{m+1}$  and add edges  $w'_{m'} u_m$  and  $w'_{m'} u_{m+1}$ , forming triangle  $\Delta w'_{m'} u_m u_{m+1}$ .

A **Double Alternate Triangular Snake Graph** [6], denoted as  $DATS_y$ , is obtained from a path graph  $P_y$  with vertices  $u_1, u_2, \dots, u_y$ , where  $y \geq 2$ , as follows: For every alternate edge  $u_m u_{m+1}$  of the path, introduce two new vertices:  $w_{m'}$  (top) and  $w'_{m'}$  (bottom). Now add edges  $w_{m'} u_m$  and  $w_{m'} u_{m+1}$ , forming triangle  $\Delta w_{m'} u_m u_{m+1}$  and add edges  $w'_{m'} u_m$  and  $w'_{m'} u_{m+1}$ , forming triangle  $\Delta w'_{m'} u_m u_{m+1}$ .

### 3 Results

**Theorem 3.1.** *The equitable chromatic number of central graph associated with a triangular snake graph, denoted as  $\chi = [C(TS_y)]$ ,  $y \geq 4$  is determined for all cases and is given by*

$$\chi = [C(TS_y)] = \left\lceil \frac{(y-1)^2 + 1}{y} \right\rceil, \forall y \geq 4$$

*Proof.* By the definition of the triangular snake graph, the vertex set is given by  $V(TS_y) = \{u_m, w_{m'} \mid 1 \leq m \leq y, 1 \leq m' \leq y - 1\}$ . Applying the central graph operation, the vertex set of  $C(TS_y)$  becomes

$$V[C(TS_y)] = V(TS_y) \cup \{e_{m'}, e'_{m''} \mid 1 \leq m' \leq y - 1, 1 \leq m'' \leq 2y - 2\},$$

where  $e_{m'}$  represents the subdivision vertices corresponding to edges  $u_m u_{m+1}$  for  $1 \leq m \leq y - 1$  and each  $e'_{m''}$  represents the subdivision vertices of edges  $u_m w_{m'}$  respectively.

Furthermore, let the vertex cardinality of  $[C(TS_y)]$  be defined as  $5y - 4$ ,  $\forall y \geq 4$ . Since the order of the adjacency matrix of  $C(TS_y)$ , which corresponds to the number of vertices in the graph, is given by:  $(5y - 4) \times (5y - 4)$ ,  $\forall y \geq 3$ . Then, the adjacency matrix representation for the first vertex  $w_1$  can be expressed as:

$$w_1 \begin{bmatrix} u_1 & u_2 & u_3 & \cdots & u_y & w_1 & w_2 & \cdots & w_{y-1} \\ 0 & 0 & 1 & \cdots & 1 & 0 & 1 & \cdots & 1 \\ e_1 & e_2 & e_3 & \cdots & e_{y-1} & e'_1 & e'_2 & \cdots & e'_{2y-2} \\ 0 & 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 0 \end{bmatrix}$$

From the adjacency matrix, we systematically derive the **neighbourhood set**. The neighbourhood sets are listed as follows:

$$\begin{aligned}
 N(u_1) &= \{e_1, e'_1, w_{m'+1}, u_{m+2} : 1 \leq m, m' \leq y-2\} \\
 N(u_2) &= \{e_1, e_2, e'_2, e'_3, u_{m+3}, w_{m'+2} : 1 \leq m, m' \leq y-3\} \\
 &\vdots \\
 N(u_y) &= \{e_{y-1}, e'_{2y-2}, u_m, w_{m'} : 1 \leq m \leq y-2, 1 \leq m' \leq y-2\} \\
 \\ 
 N(w_1) &= \{e'_1, e'_2, w_{m'+1}, u_{m+2} : 1 \leq m, m' \leq y-2\} \\
 N(w_2) &= \{e'_3, e'_4, w_1, w_{m'+1}, u_1, u_{m+2} : 2 \leq m, m' \leq y-2\} \\
 &\vdots \\
 N(w_{y-1}) &= \{e'_{2y-1}, e'_{2y-2}, w_{m'}, u_m : 1 \leq m, m' \leq y-2\}
 \end{aligned}$$

$$\begin{array}{c|c}
 \begin{array}{l}
 N(e_1) = \{u_1, u_2\} \\
 N(e_2) = \{u_2, u_3\} \\
 \vdots \\
 N(e_{y-1}) = \{u_{y-1}, u_y\}
 \end{array} &
 \begin{array}{l}
 N(e'_1) = \{u_1, w_1\} \\
 N(e'_2) = \{w_1, u_2\} \\
 \vdots \\
 N(e'_{2y-2}) = \{w_{y-1}, u_y\}
 \end{array}
 \end{array}$$

We implement an algorithmic approach using a C++ procedure to assign colors optimally while satisfying equitable coloring constrains.

**Algorithm: Equitable Chromatic Number of Central Graph of Triangular Snake Graph  $TS_y$**

```

Require: Integer  $y \geq 4$ 
Ensure: Equitable color sets  $A_u, B_w, C_e, D_{e'}$  and chromatic number  $\chi_{=}(C[TS_y])$ 
1: repeat
2:   Prompt user: "Enter the size of Triangular Snake Graph ( $y \geq 4$ ):"
3:   Read  $y$ 
4: until  $y \geq 4$ 
5: Define arrays:  $u[0 \dots y-1], w[0 \dots y-2], e[0 \dots y-2]$ 
6:  $k \leftarrow 6 + 2(y-4)$ ; Define  $e'[0 \dots k-1]$ 
7: for  $i = 0$  to  $y-1$  do
8:    $u[i] \leftarrow \max(i, 1)$ 
9: end for
10: for  $i = 0$  to  $y-2$  do
11:    $w[i] \leftarrow i+1$ ;  $e[i] \leftarrow (i < y-2) ? i+2 : 1$ 
12: end for
13: for  $i = 0$  to  $\frac{k}{2} - 2$  do
14:    $e'[2i] \leftarrow i+2$ ;  $e'[2i+1] \leftarrow i+2$ 
15: end for
16:  $e'[k-2] \leftarrow 1$ ;  $e'[k-1] \leftarrow 1$ 
17:  $\ell \leftarrow \lceil \frac{(y-1)^2+1}{y} \rceil$ 
18: Output: "The Equitable Chromatic Number  $\chi_{=}[C[TS_y]] = \ell$ "
19: Output: Print arrays  $u[0 \dots y-1], w[0 \dots y-2], e[0 \dots y-2], e'[0 \dots k-1]$ 
    
```

**Example of  $\chi_{=}(C[TS_7])$**

**Input:** Enter the size of Triangular Snake Graph ( $y \geq 4$ ): 7  
**Output:** The Equitable Chromatic Number of  $C[TS_7]$  is = 6

| $A_u$      | $B_w$      | $C_e$      | $D_{e'}$    | $D_{e'}$     |
|------------|------------|------------|-------------|--------------|
| $u[1] = 1$ | $w[1] = 1$ | $e[1] = 2$ | $e'[1] = 2$ | $e'[8] = 5$  |
| $u[2] = 1$ | $w[2] = 2$ | $e[2] = 3$ | $e'[2] = 2$ | $e'[9] = 6$  |
| $u[3] = 2$ | $w[3] = 3$ | $e[3] = 4$ | $e'[3] = 3$ | $e'[10] = 6$ |
| $u[4] = 3$ | $w[4] = 4$ | $e[4] = 5$ | $e'[4] = 3$ | $e'[11] = 1$ |
| $u[5] = 4$ | $w[5] = 5$ | $e[5] = 6$ | $e'[5] = 4$ | $e'[12] = 1$ |
| $u[6] = 5$ | $w[6] = 6$ | $e[6] = 1$ | $e'[6] = 4$ | —            |
| $u[7] = 6$ | —          | $e[7] = 2$ | $e'[7] = 5$ | —            |

By the above algorithm, we derive the upper bound:

$$\chi_{=}[C(TS_y)] \leq \left\lceil \frac{(y-1)^2+1}{y} \right\rceil, \forall y \geq 4.$$

Upon examining the structural properties of  $C(TS_y)$ , we observe that the vertex subset  $V_r = \{w_{m'} : 1 \leq m' \leq y-1\}$  forms a **clique** of order  $\left\lceil \frac{(y-1)^2+1}{y} \right\rceil, \forall y \geq 4$ . It is well-known that for any graph  $H$ , the equitable chromatic number satisfies  $\chi_=(H) \geq \omega(H)$ . Therefore, we obtain the lower bound as

$$\chi_=[C(TS_y)] \geq \left\lceil \frac{(y-1)^2+1}{y} \right\rceil, \forall y \geq 4$$

Combining the upper and lower bounds, we conclude:

$$\chi_=[C(TS_y)] = \left\lceil \frac{(y-1)^2+1}{y} \right\rceil, \forall y \geq 4.$$

□

**Theorem 3.2.** *The equitable chromatic number of the central graph associated with an alternate triangular snake graph, denoted as  $\chi_=[C(AT S_y)]$ , for  $y \geq 3$ , is determined for all cases and is given by:*

$$\chi_=(C[AT S_y]) = \begin{cases} 3; & y = 3 \\ \lfloor \frac{y}{2} \rfloor + y \bmod 2; & y \geq 4 \end{cases}$$

*Proof.* By the definition of an alternate triangular snake graph, we have  $V(AT S_y) = \{u_m \mid 1 \leq m \leq y\} \cup \{w_{m'} \mid 1 \leq m' \leq \lfloor y/2 \rfloor\}$ . Applying the central graph operation, the vertex set of  $C(AT S_y)$  becomes

$$V[C(AT S_y)] = V(AT S_y) \cup \{e_{m''} \mid 1 \leq m'' \leq y-1\} \cup \left\{ \begin{array}{l} e'_{m'''} \mid 1 \leq m''' \leq y-1, \text{ if } y \text{ is odd,} \\ e'_{m'''} \mid 1 \leq m''' \leq y, \text{ if } y \text{ is even} \end{array} \right\}$$

where  $e_{m''}$  be the sub divided vertices of  $u_m u_{m+1}$  for  $1 \leq m \leq y-1$  and  $e'_{m'''}$  be the sub-divided vertices of each edge  $u_m w_{m'}$  respectively. Since the vertices are defined with two cases for  $\{e'_{m'''}\}$  as odd and even, this leads to two different sizes of vertex sets:

- If  $y$  is **odd**:  $|V(C(AT S_y))| = 8 + 7k'$ ,  $k' = \frac{y-3}{2}$ ,  $y \geq 3$ .
- If  $y$  is **even**:  $|V(C(AT S_y))| = 13 + 7k$ ,  $k = \frac{y-4}{2}$ ,  $y \geq 3$ .

The order of the adjacency matrix of  $C(AT S_y)$ , which corresponds to the number of vertices in the graph, is given by:

$$|V(C(AT S_y))| = \begin{cases} (8 + 7k') \times (8 + 7k'), & \text{if } y \text{ is odd,} \\ (13 + 7k) \times (13 + 7k), & \text{if } y \text{ is even.} \end{cases} \quad (3.1)$$

Then, the adjacency matrix representation for the first vertex  $u_1$  for an odd case of  $y$  can be expressed as:

$$u_1 \begin{bmatrix} u_1 & u_2 & u_3 \cdots & u_y & w_1 & w_2 & \cdots & w_{\lfloor y/2 \rfloor} \\ 0 & 0 & 1 \cdots & 1 & 0 & 1 & \cdots & 1 \\ e_1 & e_2 & \cdots & e_{y-1} & e'_1 & e'_2 & \cdots & e'_{y-1} \\ 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \end{bmatrix}$$

From the adjacency matrix, we systematically derive the **neighbourhood set**. The neighbourhood sets are listed as follows:

$$N(u_1) = \{e_1, e'_1, \{u_{m+2}\}_{m=1}^{y-2}, \{w_{m'}\}_{m'=2}^{\lfloor \frac{y}{2} \rfloor}\}$$

$$N(u_2) = \{e_1, e'_2, \{u_{m+3}\}_{m=1}^{y-3}, \{w_{m'}\}_{m'=1}^{\lfloor \frac{y}{2} \rfloor}\}$$

⋮

$$N(u_y) = \begin{cases} \{e_{y-1}, \{w_{m'}\}_{m'=1}^{\lfloor \frac{y}{2} \rfloor}, \{u_m\}_{m=1}^{y-2}\}, & \text{if } y \text{ is odd,} \\ \{e_{y-1}, e'_y, \{w_{m'}\}_{m'=1}^{\lfloor \frac{y}{2} \rfloor - 1}, \{u_m\}_{m=1}^{y-2}\}, & \text{if } y \text{ is even.} \end{cases}$$

$$N(w_1) = \{e'_1, e'_2, \{w_{m'}\}_{m'=2}^{\lfloor \frac{y}{2} \rfloor}, \{u_m\}_{m=3}^y\}$$

$$\begin{aligned}
N(w_2) &= \{e'_3, e'_4, u_1, u_2, w_1, \{w_{m'}\}_{m'=3}^{\lfloor \frac{y}{2} \rfloor}, \{u_m\}_{m=5}^y\} \\
&\vdots \\
N(w_{\lfloor y/2 \rfloor}) &= \begin{cases} \{e'_{y-2}, e'_{y-1}, u_y, \{w_{m'}\}_{m'=1}^{\lfloor \frac{y}{2} \rfloor - 1}, \{u_m\}_{m=1}^{y-3}\}, & \text{if } y \text{ is odd,} \\ \{e'_{y-1}, e'_y, \{w_{m'}\}_{m'=1}^{\lfloor \frac{y}{2} \rfloor - 1}, \{u_m\}_{m=1}^{y-2}\}, & \text{if } y \text{ is even.} \end{cases} \\
N(e_1) &= \{u_1, u_2\} & N(e'_1) &= \{u_1, w_1\} \\
N(e_2) &= \{u_2, u_3\} & N(e'_2) &= \{u_2, w_1\} \\
&\vdots & & \vdots \\
N(e_{y-1}) &= \{u_{y-1}, u_y\} & N(e'_y) &= \{w_{\lfloor y/2 \rfloor}, u_y\}, \text{ if } y \text{ is even} \\
& & N(e'_y) &= \{w_{\lfloor y/2 \rfloor}, u_{y-1}\}, \text{ if } y \text{ is odd}
\end{aligned}$$

We implement an algorithmic approach using a C++ procedure to assign colors optimally while satisfying equitable coloring constrains.

**Algorithm: Equitable Chromatic Number of Central Graph of Alternate Triangular Snake Graph  $ATS_y$**

**Require:** Integer  $y \geq 3$   
**Ensure:** Arrays  $u, w, e, e'$ , and equitable chromatic number  $l$

- 1: **repeat**
- 2:   Prompt: "Enter the size of Alternate Triangular Snake Graph ( $y \geq 3$ ):"
- 3:   Read  $y$
- 4: **until**  $y \geq 3$
- 5: Define arrays:  $u[0 \dots y-1], w[0 \dots y-1], e[0 \dots y-2], e'[0 \dots y-1]$
- 6:  $limit \leftarrow (y \geq 5 \wedge y \bmod 2 = 0) ? y/2 : \lceil y/2 \rceil$
- 7: **for**  $i = 1$  to  $limit$  **do**
- 8:    $u[2i-2] \leftarrow i; u[2i-1] \leftarrow i$
- 9: **end for**
- 10: **for**  $i = 0$  to  $\lfloor y/2 \rfloor - 1$  **do**
- 11:    $w[i] \leftarrow i + 1$
- 12: **end for**
- 13: **if**  $y = 4$  **then**
- 14:   **for**  $i = 0$  to  $3$  **do**
- 15:      $e'[i] \leftarrow 2; w[i] \leftarrow 1; u[2i], u[2i+1] \leftarrow 1; e[i] \leftarrow 2$
- 16:   **end for**
- 17: **else if**  $y = 3$  **then**
- 18:   **for**  $i = 0$  to  $2$  **do**
- 19:      $e'[i] \leftarrow 3; w[i] \leftarrow 1; e[i] \leftarrow i + 2$
- 20:   **end for**
- 21:    $u[0], u[1] \leftarrow 1; u[2] \leftarrow 2$
- 22: **else**
- 23:   **if**  $y \geq 6 \wedge y \bmod 2 = 0$  **then**
- 24:     **for**  $i = 0$  to  $y/2 - 2$  **do**
- 25:        $e'[2i+2], e'[2i+3] \leftarrow i + 1$
- 26:     **end for**
- 27:      $e'[0], e'[1] \leftarrow y/2$
- 28:   **else if**  $y = 5$  **then**
- 29:     **for**  $i = 0$  to  $1$  **do**
- 30:        $e'[2i], e'[2i+1] \leftarrow i + 2$
- 31:     **end for**
- 32:   **else if**  $y = 7$  **then**
- 33:     **for**  $i = 0$  to  $2$  **do**
- 34:        $e'[2i+2], e'[2i+3] \leftarrow i + 1$
- 35:     **end for**
- 36:      $e'[0], e'[1] \leftarrow \lfloor y/2 \rfloor$
- 37:   **else**
- 38:     **for**  $i = 1$  to  $\lfloor y/2 \rfloor$  **do**
- 39:        $e'[2i], e'[2i+1] \leftarrow i$
- 40:     **end for**
- 41:      $e'[0], e'[1] \leftarrow \lfloor y/2 \rfloor$

```

42: end if
43: if  $y = 6$  then
44:    $pattern \leftarrow [3, 3, 1, 1, 2]$ 
45:   for  $i = 0$  to 4 do
46:      $e[i] \leftarrow pattern[i]$ 
47:   end for
48: else if  $y > 6 \wedge y \bmod 2 = 0$  then
49:    $e[0] \leftarrow y/2; val \leftarrow 3$ 
50:   for  $i = 1$  to  $y/2 - 1$  do
51:      $e[2i] \leftarrow i$ 
52:   end for
53:   for  $i = 1$  to  $y/2 - 2$  do
54:      $e[2i - 1] \leftarrow val; val \leftarrow val + 1$ 
55:   end for
56:    $e[y - 5] \leftarrow y/2; e[y - 3] \leftarrow 1$ 
57: else if  $y = 7$  then
58:   for  $i = 0$  to 4 do
59:      $e[i] \leftarrow 4$ 
60:   end for
61:    $e[5] \leftarrow 1$ 
62: else if  $y = 5$  then
63:    $e[0..1] \leftarrow 3; e[2..3] \leftarrow 1$ 
64: else if  $y = 9 \vee y = 11$  then
65:   for  $i = 0$  to  $y - 2$  step 2 do
66:      $e[i] \leftarrow (y + 1)/2$ 
67:   end for
68:    $e[1] \leftarrow (y + 1)/2 - 1; val \leftarrow 1$ 
69:   for  $i = 3$  to  $y - 2$  step 2 do
70:      $e[i] \leftarrow val; val \leftarrow val + 1$ 
71:   end for
72: else
73:    $e[0] \leftarrow (y - 1)/2; val \leftarrow 1$ 
74:   for  $i = 2$  to  $y - 2$  step 2 do
75:      $e[i] \leftarrow val; val \leftarrow val + 1$ 
76:   end for
77:    $k \leftarrow (y + 1)/2$ 
78:   for  $i = 1$  to 9 step 2,  $i < y - 1$  do
79:      $e[i] \leftarrow k$ 
80:   end for
81:    $val_1 \leftarrow 1$ 
82:   for  $i = 11$  to  $y - 2$  step 2 do
83:      $e[i] \leftarrow val_1; val_1 \leftarrow val_1 + 1$ 
84:   end for
85: end if
86: end if
87:  $l \leftarrow (y = 3) ? 3 : \lfloor y/2 \rfloor + (y \bmod 2)$ 
88: Output: Arrays  $u, e, w, e'$  and equitable chromatic number  $l$ 

```

A counterexample from the code is provided below:

**Example of  $\chi_{=}(C[ATS_9])$**

**Input:** Enter the size of Alternate Triangular Snake Graph ( $y \geq 3$ ): 9

**Output:** The Equitable Chromatic Number of  $C[ATS_9]$  is = 5

| $A_u$      | $B_w$      | $C_e$      | $D_{e'}$    |
|------------|------------|------------|-------------|
| $u[1] = 1$ | $w[1] = 1$ | $e[1] = 5$ | $e'[1] = 4$ |
| $u[2] = 1$ | $w[2] = 2$ | $e[2] = 4$ | $e'[2] = 4$ |
| $u[3] = 2$ | $w[3] = 3$ | $e[3] = 5$ | $e'[3] = 1$ |
| $u[4] = 2$ | $w[4] = 4$ | $e[4] = 1$ | $e'[4] = 1$ |
| $u[5] = 3$ | —          | $e[5] = 5$ | $e'[5] = 2$ |
| $u[6] = 3$ | —          | $e[6] = 2$ | $e'[6] = 2$ |
| $u[7] = 4$ | —          | $e[7] = 5$ | $e'[7] = 3$ |
| $u[8] = 4$ | —          | $e[8] = 3$ | $e'[8] = 3$ |
| $u[9] = 5$ | —          | —          | —           |

By the above algorithm, we derive the upper bound:

$$\chi_{=}(C[ATS_y]) \leq \begin{cases} 3; & y = 3 \\ \lfloor \frac{y}{2} \rfloor + y \pmod{2}; & y \geq 4 \end{cases}$$

Upon examining the structural properties of  $C[ATS_y]$ , we identify that the vertex set

- If  $y$  is **even**:  $V_r = \{w_{m'} : 1 \leq m' \leq \lfloor \frac{y}{2} \rfloor\}$ .
- If  $y$  is **odd**:  $V_{r'} = \{w_{m'}, u_y : 1 \leq m' \leq \lfloor \frac{y}{2} \rfloor\}$ .

forms maximum induced complete sub-graph of order  $\lfloor \frac{y}{2} \rfloor + y \pmod{2} \forall y \geq 4$ . For  $y = 3$ , assume only 2 colors are used. Let all path vertices  $u_m$  ( $1 \leq m \leq 3$ ) be colored with color 1. Since  $u_3$  is adjacent to  $w_1$ , assign color 2 to  $w_1$ . Next, color all  $e_{m''}$  ( $1 \leq m'' \leq 2$ ) with color 2. Now, each vertex  $e'_{m''}$  is adjacent to both  $w_1$  (color 2) and  $u_m$  (color 1), so neither color 1 nor 2 is valid. Therefore, a third color is required, contradicting the assumption. Thus, the lower bound for  $\chi_{=}(C[ATS_y])$  is at least 3 when  $y = 3$ . Since  $\chi_{=}(H) \geq \omega(H)$  for any graph  $H$ , the result follows

$$\chi_{=}[C(ATS_y)] \geq \omega[C(ATS_y)] = \lfloor \frac{y}{2} \rfloor + y \pmod{2}, \forall y \geq 4$$

Therefore, we obtained the lower bound

$$\chi_{=}(C[ATS_y]) \geq \begin{cases} 3; & y = 3 \\ \lfloor \frac{y}{2} \rfloor + y \pmod{2}; & y \geq 4 \end{cases}$$

Hence, combing both the bounds,

$$\chi_{=}[C(ATS_y)] = \begin{cases} \lfloor \frac{y^2-1}{y} \rfloor; & y = 3 \\ \lfloor \frac{y}{2} \rfloor + y \pmod{2}; & y \geq 4 \end{cases}$$

□

**Theorem 3.3.** *The equitable chromatic number of the central graph associated with diamond triangular snake graph, denoted as  $\chi_{=}[C(DTS_y)]$ , for  $y \geq 3$ , is determined for all cases and is given by:*

$$\chi_{=}[C(DTS_y)] = 2 \lfloor \frac{y^2+1}{y} \rfloor, \forall y \geq 3$$

*Proof.* By the definition of  $DTS_y$ , we have  $V(DTS_y) = \{u_m \mid 1 \leq m \leq y+1\} \cup \{u'_{m'}, u''_{m'} \mid 1 \leq m' \leq y\}$ , each  $C_4$  forms the structure:  $u_m - u'_{m'} - u_{m+1} - u''_{m'} - u_m$ , for  $1 \leq m \leq y$ . Applying the central graph operation, the vertex set of  $V[C(DTS_y)]$  becomes

$$V[C(DTS_y)] = V(DTS_y) \cup \{w_{m''}, w'_{m''} \mid 1 \leq m'' \leq 2y\}$$

where  $w_{m''}$  be the sub divided vertices of edges  $u_m u'_{m'}$  and  $w'_{m''}$  be the sub divided vertices of edges  $u_m u''_{m'}$ , respectively.

Furthermore, let the vertex cardinality of  $C(DTS_y)$  be defined as  $|V(C(DTS_y))| = 7y + 1$ ,  $\forall y \geq 3$ . Since the order of the adjacency matrix of  $C(DTS_y)$ , which corresponds to the number of vertices in the graph, is given by:  $(7y + 1) \times (7y + 1) \forall y \geq 3$ . Then, the adjacency matrix representation for the first vertex  $u_1$  can be expressed as:

$$u_1 \begin{bmatrix} u_1 & u_2 & \cdots & u_{y+1} & u'_1 & u'_2 & \cdots & u'_y \\ 0 & 1 & \cdots & 1 & 0 & 1 & \cdots & 1 \\ u''_1 & u''_2 & \cdots & u''_y & w_1 & w_2 & \cdots & w_{2y} \\ 0 & 1 & \cdots & 1 & 1 & 0 & \cdots & 0 \\ w'_1 & w'_2 & \cdots & w'_{2y} & & & & \\ 1 & 0 & \cdots & 0 & & & & \end{bmatrix}$$

From the adjacency matrix, we systematically derive the **neighbourhood set**. The neighbourhood sets are listed as follows:

$$\begin{aligned}
N(u_1) &= \{w_1, w'_1, \{u_m\}_{m=2}^y, \{u'_{m'+1}, u''_{m'+1}\}_{m'=1}^{y-1}\} \\
N(u_2) &= \{w_2, w'_2, w_3, w'_3, u_1, \{u_m\}_{m=3}^{y+1}, \{u'_{m'}, u''_{m'}\}_{m'=3}^y\} \\
&\vdots \\
N(u_{y+1}) &= \{w_{2y}, w'_{2y}, \{u_m\}_{m=1}^y, \{u'_{m'}, u''_{m'}\}_{m'=1}^{y-1}\} \\
N(u'_1) &= \{w_1, w_2, \{u'_{m'}\}_{m'=2}^y, \{u''_{m'}\}_{m'=1}^y, \{u_m\}_{m=3}^{y+1}\} \\
N(u'_2) &= \{w_3, w_4, u_1, u'_1, \{u_m\}_{m=4}^{y+1}, \{u'_{m'}\}_{m'=3}^y, \{u''_{m'}\}_{m'=1}^y\} \\
&\vdots \\
N(u'_y) &= \{w_{2y-1}, w_{2y}, \{u_m\}_{m=1}^{y-1}, \{u'_{m'}\}_{m'=1}^{y-1}, \{u''_{m'}\}_{m'=1}^y\} \\
N(u''_1) &= \{w'_1, w'_2, \{u''_{m'}\}_{m'=2}^y, \{u'_{m'}\}_{m'=1}^y, \{u_m\}_{m=3}^{y+1}\} \\
N(u''_2) &= \{w'_3, w'_4, u_1, u''_1, \{u_m\}_{m=4}^{y+1}, \{u''_{m'}\}_{m'=3}^y, \{u'_{m'}\}_{m'=1}^y\} \\
&\vdots \\
N(u''_y) &= \{w'_{2y-1}, w'_{2y}, \{u_m\}_{m=1}^{y-1}, \{u''_{m'}\}_{m'=1}^{y-1}, \{u'_{m'}\}_{m'=1}^y\}
\end{aligned}$$

$$\begin{array}{c|c}
N(w_1) = \{u_1, u'_1\} & N(w'_1) = \{u_1, u''_1\} \\
N(w_2) = \{u_2, u'_1\} & N(w'_2) = \{u_2, u''_1\} \\
\vdots & \vdots \\
N(w_{2y}) = \{u_{y+1}, u'_y\} & N(w'_{2y}) = \{u_{y+1}, u''_y\}
\end{array}$$

We implement an algorithmic approach using a C++ procedure to assign colors optimally while satisfying equitable coloring constrains.

#### Algorithm: Equitable Chromatic Number of Diamond Triangular Snake Graph $DTS_y$

**Require:** Integer  $y \geq 3$

**Ensure:** Equitable color assignments for vertex sets  $A_u, B'_u, C''_u, D_w, E'_u$  and chromatic number  $\chi = (C[DTS_y])$

```

1: repeat
2:   Prompt user: "Enter the size of Diamond Triangular Snake Graph ( $y \geq 3$ ):"
3:   Read  $y$ 
4: until  $y \geq 3$ 
5: Define arrays:  $u[0 \dots y], u'[0 \dots y - 1], u''[0 \dots y - 1], w[0 \dots 2y - 1], w'[0 \dots 2y - 1]$ 
6: for  $i = 0$  to  $y$  do
7:    $u[i] \leftarrow i + 1$ 
8: end for
9: for  $i = 0$  to  $y - 1$  do
10:   $u'[i] \leftarrow i + 1$ 
11: end for
12:  $val \leftarrow \lfloor \frac{(y^2+1)}{y} \rfloor \times 2$ 
13: for  $i = 0$  to  $y - 1$  do
14:   $u''[i] \leftarrow val - -$ 
15: end for
16: for  $i = 0$  to  $y - 1$  do
17:   $w[2i] \leftarrow y + i + 1; w[2i + 1] \leftarrow y + i + 1$ 
18: end for
19:  $w'[0] \leftarrow y; w'[1] \leftarrow y$ 
20: for  $i = 0$  to  $y - 2$  do
21:   $w'[2i + 2] \leftarrow i + 1; w'[2i + 3] \leftarrow i + 1$ 
22: end for
23: Compute  $\ell \leftarrow \lfloor \frac{(y^2+1)}{y} \rfloor \times 2$ 
24: Output: Print arrays  $u[0 \dots y], u'[0 \dots y - 1], u''[0 \dots y - 1], w[0 \dots 2y - 1], w'[0 \dots 2y - 1]$ 

```

**Equitable Chromatic Number of C[DTS<sub>12</sub>]**

**Input:** Enter the size of Diamond Triangular Snake Graph ( $y \geq 3$ ): 12

**Output:** The Equitable Chromatic Number of  $C[DTS_{12}]$  is = 24

| $A_u$        | $B'_u$        | $C''_u$        | $D_w$        | $E'_w$        |
|--------------|---------------|----------------|--------------|---------------|
| $u[1] = 1$   | $u'[1] = 1$   | $u''[1] = 24$  | $w[1] = 13$  | $w'[1] = 12$  |
| $u[2] = 2$   | $u'[2] = 2$   | $u''[2] = 23$  | $w[2] = 13$  | $w'[2] = 12$  |
| $u[3] = 3$   | $u'[3] = 3$   | $u''[3] = 22$  | $w[3] = 14$  | $w'[3] = 1$   |
| $u[4] = 4$   | $u'[4] = 4$   | $u''[4] = 21$  | $w[4] = 14$  | $w'[4] = 1$   |
| $u[5] = 5$   | $u'[5] = 5$   | $u''[5] = 20$  | $w[5] = 15$  | $w'[5] = 2$   |
| $u[6] = 6$   | $u'[6] = 6$   | $u''[6] = 19$  | $w[6] = 15$  | $w'[6] = 2$   |
| $u[7] = 7$   | $u'[7] = 7$   | $u''[7] = 18$  | $w[7] = 16$  | $w'[7] = 3$   |
| $u[8] = 8$   | $u'[8] = 8$   | $u''[8] = 17$  | $w[8] = 16$  | $w'[8] = 3$   |
| $u[9] = 9$   | $u'[9] = 9$   | $u''[9] = 16$  | $w[9] = 17$  | $w'[9] = 4$   |
| $u[10] = 10$ | $u'[10] = 10$ | $u''[10] = 15$ | $w[10] = 17$ | $w'[10] = 4$  |
| $u[11] = 11$ | $u'[11] = 11$ | $u''[11] = 14$ | $w[11] = 18$ | $w'[11] = 5$  |
| $u[12] = 12$ | $u'[12] = 12$ | $u''[12] = 13$ | $w[12] = 18$ | $w'[12] = 5$  |
| $u[13] = 13$ | —             | —              | $w[13] = 19$ | $w'[13] = 6$  |
| —            | —             | —              | $w[14] = 19$ | $w'[14] = 6$  |
| —            | —             | —              | $w[15] = 20$ | $w'[15] = 7$  |
| —            | —             | —              | $w[16] = 20$ | $w'[16] = 7$  |
| —            | —             | —              | $w[17] = 21$ | $w'[17] = 8$  |
| —            | —             | —              | $w[18] = 21$ | $w'[18] = 8$  |
| —            | —             | —              | $w[19] = 22$ | $w'[19] = 9$  |
| —            | —             | —              | $w[20] = 22$ | $w'[20] = 9$  |
| —            | —             | —              | $w[21] = 23$ | $w'[21] = 10$ |
| —            | —             | —              | $w[22] = 23$ | $w'[22] = 10$ |
| —            | —             | —              | $w[23] = 24$ | $w'[23] = 11$ |
| —            | —             | —              | $w[24] = 24$ | $w'[24] = 11$ |

By the above programming algorithm, we obtain the upper bound

$$\chi_{=}[C(DTS_y)] \leq 2 \left\lfloor \frac{y^2 + 1}{y} \right\rfloor, \forall y \geq 3$$

Upon examining the structural properties of  $C(DTS_y)$ , we identify the vertex set  $V_r = \{u'_{m'}, u''_{m'} : 1 \leq m' \leq y\}$  forms a clique of order  $2 \left\lfloor \frac{y^2+1}{y} \right\rfloor, \forall y \geq 3$ . Since,  $\chi_{=}(H) \geq \omega(H)$  for any graph H.

Hence,  $\chi_{=}[C(DTS_y)] \geq \omega[C(DTS_y)] = 2 \left\lfloor \frac{y^2+1}{y} \right\rfloor, \forall y \geq 3$ . Therefore, lower bound becomes

$$\chi_{=}[C(DTS_y)] \geq 2 \left\lfloor \frac{y^2+1}{y} \right\rfloor, \forall y \geq 3$$

Hence, by combining both bounds,

$$\chi_{=}[C(DTS_y)] = 2 \left\lfloor \frac{y^2+1}{y} \right\rfloor, \forall y \geq 4$$

□

**Theorem 3.4.** *The equitable chromatic number of the central graph associated with alternate quadrilateral snake graph, denoted as*

$\chi_{=}[C(AQS_y)], y \geq 4$  *is determined for all cases and is given by*

$$\chi_{=}[C(AQS_y)] = y, \forall y \geq 4$$

*Proof.* By the definition of alternate quadrilateral snake graph, we have

$$V(AQS_y) = \{u_m \mid 1 \leq m \leq y\} \cup \left\{ u'_{m'} \mid \begin{cases} 1 \leq m' \leq y - 1, & \text{if } y \text{ is odd,} \\ 1 \leq m' \leq y, & \text{if } y \text{ is even.} \end{cases} \right\}.$$

Applying the central graph operation, the vertex set of  $V[C(AQS_y)]$  becomes

$$V[C(AQS_y)] = V(AQS_y) \cup \{e_{m''} \mid 1 \leq m'' \leq y - 1\} \cup \{e'''_{m'''} \mid 1 \leq m''' \leq \lfloor y/2 \rfloor\} \cup \left\{ e'_{m'} \mid \begin{cases} 1 \leq m' \leq y - 1, & \text{if } y \text{ is odd,} \\ 1 \leq m' \leq y, & \text{if } y \text{ is even.} \end{cases} \right\}$$

where  $e_{m''}$  represents the sub divided vertices of base path graph  $u_m$ ;  $e_{m''}$  represents the sub divided vertices of upper vertices  $u'_{m'}u'_{m'+1}$  of cycle  $C_4$  and  $e'_{m'}$  represents the sub divided vertices of vertical edges  $u_m u'_{m'}$  respectively. Since the vertices are defined with two cases for  $e'_{m'}$ ,  $u'_{m''}$  as odd and even, this leads to two different sizes of vertex sets:

- If  $y$  is **odd**:  $|V(C(AQS_y))| = \frac{9y-7}{2}$ ,  $y \geq 4$
- If  $y$  is **even**:  $|V(C(AQS_y))| = \frac{9y-2}{2}$ ,  $y \geq 4$

The order of the adjacency matrix of  $C(AQS_y)$ , which corresponds to the number of vertices in the graph, is given by:

$$|V(C(AQS_y))| = \begin{cases} \frac{9y-7}{2} \times \frac{9y-7}{2}, & \text{if } y \text{ is odd,} \\ \frac{9y-2}{2} \times \frac{9y-2}{2}, & \text{if } y \text{ is even.} \end{cases} \quad (3.2)$$

Then, the adjacency matrix representation for the first vertex  $u_1$  for an odd case of  $y$  can be expressed as:

$$u_1 \begin{bmatrix} u_1 & u_2 & u_3 & \cdots & u_y & u'_1 & u'_2 & u'_3 & \cdots & u'_{y-1} \\ 0 & 0 & 1 & \cdots & 1 & 0 & 1 & 1 & \cdots & 1 \\ e_1 & e_2 & e_3 & \cdots & e_{y-1} & e'_1 & e'_2 & e'_3 & \cdots & e'_{y-1} \\ 1 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 \\ e''_1 & e''_2 & e''_3 & \cdots & e''_{\lfloor y/2 \rfloor} \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

From the adjacency matrix, we systematically derive the **neighbourhood set**. The neighbourhood sets are listed as follows:

$$N(u_1) = \begin{cases} e_1, e'_1, u_m, u'_{m'} : 3 \leq m \leq y, & 2 \leq m' \leq y-1, & 2 \leq m' \leq y \\ & \text{if } y \text{ is odd} & \text{if } y \text{ is even} \end{cases}$$

$$N(u_2) = \begin{cases} \{e_1, e_2, e'_2, u_m, u'_{m'}\} : 4 \leq m \leq y, \\ m' = 1, 3 \leq m' \leq y-1, \text{ if } y \text{ is odd,} \\ m' = 1, 3 \leq m' \leq y, \text{ if } y \text{ is even} \end{cases}$$

$$\vdots$$

$$N(u_y) = \{e_{y-1}, \{u_m\}_{m=1}^{y-2}, \{u'_{m'}\}_{m'=1}^{y-1}\} \text{ (if } y \text{ is odd)}$$

$$N(u_y) = \{e_{y-1}, e'_y, \{u_m\}_{m=1}^{y-2}, \{u'_{m'}\}_{m'=1}^{y-1}\} \text{ (if } y \text{ is even)}$$

$$N(u'_1) = \begin{cases} \{e'_1, e''_1, u_m, u'_{m'}\} : 2 \leq m \leq y, \\ 3 \leq m' \leq y-1, \text{ if } y \text{ is odd,} \\ 3 \leq m' \leq y, \text{ if } y \text{ is even} \end{cases}$$

$$N(u'_2) = \begin{cases} \{e''_1, e'_2, u_m, u'_{m'}\} : m = 1 \text{ and } 3 \leq m \leq y, \\ 3 \leq m' \leq y-1 \text{ (if } y \text{ is odd),} \\ 3 \leq m' \leq y \text{ (if } y \text{ is even)} \end{cases}$$

$$\vdots$$

$$N(u'_{y-1}) = \begin{cases} \{e'_{y-1}, e''_{\lfloor y/2 \rfloor}, u_m, u'_{m'}\} \\ m = y, 1 \leq m \leq y-3, 1 \leq m' \leq y-3, \text{ if } y \text{ is odd} \\ 1 \leq m, m' \leq y-2, \text{ if } y \text{ is even} \end{cases}$$

$$\begin{array}{l|l|l} N(e_1) = \{u_1, u_1\} & N(e'_1) = \{u_1, u'_1\} & N(e''_1) = \{u'_1, u'_2\} \\ N(e_2) = \{u_2, u_2\} & N(e'_2) = \{u_2, u'_2\} & N(e''_2) = \{u'_3, u'_4\} \\ \vdots & \vdots & \vdots \\ N(e_{y-1}) = \{u_{y-1}, u_y\} & N(e'_{y-1}) = \{u_{y-1}, u'_{y-1}\} & N(e''_{\lfloor y/2 \rfloor}) = \{u'_{y-2}, u'_y\} \\ & \text{if } y \text{ is odd} & \text{if } y \text{ is odd} \\ & N(e'_y) = \{u_y, u'_y\} & N(e''_{\lfloor y/2 \rfloor}) = \{u'_{y-1}, u'_y\} \\ & \text{if } y \text{ is even} & \text{if } y \text{ is even} \end{array}$$

We implement an algorithmic approach using a C++ procedure to assign colors optimally while satisfying equitable coloring constrains.

**Algorithm: Equitable Chromatic Number of Alternate Quadrilateral Snake Graph  $AQS_y$**

```

Require: Integer  $y \geq 4$ 
Ensure: Equitable coloring sets  $A_u, B_{u'}, C_e, D_{e'}, E_{e''}$  and chromatic number  $\chi = (C[AQS_y])$ 
1: repeat
2:   Prompt user: "Enter the size of Alternate Quadrilateral Triangular Snake Graph ( $y \geq 4$ ):"
3:   Read  $y$ 
4:   until  $y \geq 4$ 
5:    $k \leftarrow \lfloor y/2 \rfloor$ 
6:   Initialize arrays:  $u[0 \dots y-1], u'[0 \dots y-2], e[0 \dots y-2], e'[0 \dots y-1], e''[0 \dots k-1]$ 
7:   if  $y$  is even then
8:      $val \leftarrow 2$ 
9:     for  $i = 0$  to  $(y/2) - 1$  do
10:       $u[2i] \leftarrow i + val; u[2i + 1] \leftarrow i + val; val \leftarrow val + 1$ 
11:    end for
12:     $val_1 \leftarrow 2$ 
13:    for  $i = 0$  to  $(y/2) - 1$  do
14:       $u'[2i] \leftarrow val_1 - 1; u'[2i + 1] \leftarrow val_1 - 1; val_1 \leftarrow val_1 + 2$ 
15:    end for
16:  else
17:     $val \leftarrow 2$ 
18:    for  $i = 0$  to  $(y - 1)/2 - 1$  do
19:       $u[2i] \leftarrow i + val; u[2i + 1] \leftarrow i + val; val \leftarrow val + 1; u[y - 1] \leftarrow y$ 
20:    end for
21:     $val_1 \leftarrow 1$ 
22:    for  $i = 0$  to  $(y - 1)/2 - 1$  do
23:       $u'[2i] \leftarrow val_1 + 2i; u'[2i + 1] \leftarrow val_1 + 2i$ 
24:    end for
25:  end if
26:  if  $y$  is even then
27:    for  $i = 0$  to  $y - 2$  do
28:       $e[i] \leftarrow (i \bmod 2 = 0) ? (i + 4) : \lfloor \frac{i + 1}{2} \rfloor$ 
29:    end for
30:     $e[y - 2] \leftarrow 2$ 
31:  else if  $y = 5$  then
32:    Assign:  $e[0] \leftarrow 4, e[1] \leftarrow 5, e[2] \leftarrow 2, e[3] \leftarrow 1$ 
33:  else if  $y = 7$  or  $9$  then
34:     $e[y - 2] \leftarrow 2, e[y - 3] \leftarrow y$ 
35:    for  $i = 0$  to  $y - 4$  do
36:       $e[i] \leftarrow (i \bmod 2 = 0) ? i + 4 : y$ 
37:    end for
38:  else
39:    for  $i = 1$  to  $y - 6$  step 2 do
40:       $e[i - 1] \leftarrow i + 3$ 
41:    end for
42:     $e[y - 5] \leftarrow y - 1, e[y - 3] \leftarrow 2$ 
43:    for  $i = 2$  to 8 step 2 do
44:       $e[i - 1] \leftarrow y$ 
45:    end for
46:     $val \leftarrow 1$ 
47:    for  $i = 10$  to  $y - 1$  step 2 do
48:       $e[i - 1] \leftarrow val; val \leftarrow val + 1$ 
49:    end for
50:  end if
51:  if  $y$  is even then
52:     $val \leftarrow 3$ 
53:    for  $i = 0$  to  $(y/2) - 2$  do
54:       $e'[2i] \leftarrow i + val; e'[2i + 1] \leftarrow i + val; val \leftarrow val + 1$ 
55:    end for
56:     $e'[y - 2] \leftarrow 1, e'[y - 1] \leftarrow 1$ 
57:  else if  $y = 5$  then
58:     $val \leftarrow 3$ 
59:    for  $i = 0$  to 1 do
60:       $e'[2i] \leftarrow i + val; e'[2i + 1] \leftarrow i + val; val \leftarrow val + 1$ 
61:    end for
62:  else
63:     $e'[y - 2] \leftarrow 1, e'[y - 3] \leftarrow 1; val \leftarrow 3$ 
64:    for  $i = 0$  to  $(y - 1)/2 - 2$  do

```

```

65:     e'[2i] ← i + val; e'[2i + 1] ← i + val; val ← val + 1
66:   end for
67: end if
68: val ← 3, e''[k - 1] ← 2
69: for i = 0 to k - 2 do
70:   e''[i] ← i + val + 1; val ← val + 1
71: end for
72: Output: "The Equitable Chromatic Number of C[AQSy] = y"
73: Output: Print arrays u[0...y-1], u'[0...y-2], e[0...y-2], e'[0...y-1], e''[0...k-1]

```

**Example of**  $\chi_{=}(C[AQS_{11}])$

**Input:** Enter the size of Alternate Quadrilateral Triangular Snake Graph ( $y \geq 4$ ): 11

**Output:** The Equitable Chromatic Number of  $C[AQS_{11}]$  is = 11

| $A_u$        | $B_{u'}$     | $C_e$       | $D_{e'}$     | $E_{e''}$     |
|--------------|--------------|-------------|--------------|---------------|
| $u[1] = 2$   | $u'[1] = 1$  | $e[1] = 4$  | $e'[1] = 3$  | $e''[1] = 4$  |
| $u[2] = 2$   | $u'[2] = 1$  | $e[2] = 11$ | $e'[2] = 3$  | $e''[2] = 6$  |
| $u[3] = 4$   | $u'[3] = 3$  | $e[3] = 6$  | $e'[3] = 5$  | $e''[3] = 8$  |
| $u[4] = 4$   | $u'[4] = 3$  | $e[4] = 11$ | $e'[4] = 5$  | $e''[4] = 10$ |
| $u[5] = 6$   | $u'[5] = 5$  | $e[5] = 8$  | $e'[5] = 7$  | $e''[5] = 2$  |
| $u[6] = 6$   | $u'[6] = 5$  | $e[6] = 11$ | $e'[6] = 7$  | —             |
| $u[7] = 8$   | $u'[7] = 7$  | $e[7] = 10$ | $e'[7] = 9$  | —             |
| $u[8] = 8$   | $u'[8] = 7$  | $e[8] = 11$ | $e'[8] = 9$  | —             |
| $u[9] = 10$  | $u'[9] = 9$  | $e[9] = 2$  | $e'[9] = 1$  | —             |
| $u[10] = 10$ | $u'[10] = 9$ | $e[10] = 1$ | $e'[10] = 1$ | —             |
| $u[11] = 11$ | —            | —           | —            | —             |

By the above programming algorithm, we obtain the upper bound,

$$\chi_{=}[C(AQS_y)] \leq y, \forall y \geq 4$$

Upon examining the structural properties of the central graph of the alternate quadrilateral snake graph  $C(AQS_y)$ , suppose, for contradiction, that its equitable chromatic number is  $\chi_{=}(C(AQS_y)) = y - 1$ .

**Case 1: When  $y$  is odd**

Consider the vertex sets  $\{u_m, u'_{m'}\}_{m, m'=1}^{y-1}$ . In each quadrilateral  $C_4$  of  $AQS_y$ , the diagonal vertices are adjacent in the central graph  $C(AQS_y)$ . Odd-indexed vertices in  $u_m$  are adjacent to even-indexed vertices in  $u'_{m'}$ , necessitating distinct colors. However, even-indexed vertices in  $u_m$  inherit colors from preceding odd-indexed vertices in  $u_m$ , and similarly, odd-indexed vertices in  $u'_{m'}$  inherit colors from subsequent even-indexed vertices in  $u'_{m'}$ . Now, consider the vertex  $u_y$ . It is adjacent to all vertices in  $\{u_m : 1 \leq m \leq y - 2\} \cup \{u'_{m'} : 1 \leq m' \leq y - 1\}$ , each of which is assigned a unique color from the set  $\{1, 2, \dots, y - 1\}$ . As a result,  $u_y$  must be assigned a new color,  $y$ , which contradicts the assumption that  $C(AQS_y)$  is equitably  $(y - 1)$ -colorable.

**Case 2: If  $y$  is even:** For  $1 \leq m, m' \leq y - 2$ , in each quadrilateral  $C_4$ , the diagonal vertices are adjacent. Odd vertices of  $u_m$  and even vertices of  $u'_{m'}$  are assigned distinct colors, using  $y - 2$  colors. Even vertices of  $u_m$  inherit colors from preceding odd vertices in  $u_m$ , and odd vertices of  $u'_{m'}$  inherit colors from subsequent even vertices in  $u'_{m'}$ . The last two vertices of  $u_m$  are non-adjacent, so they receive color  $y - 1$ . For the last two vertices of  $u'_{m'}$ , color  $y - 1$  cannot be used, as they are adjacent to all vertices colored 1 through  $y - 1$ . Therefore, a new color must be introduced, contradicting the assumption of  $y - 1$  coloring.

From both the cases, we obtain the lower bound,

$$\chi_{=}[C(AQS_y)] \geq y, \forall y \geq 3$$

Therefore, combining both the bounds,

$$\chi_{=}[C(AQS_y)] = y, \forall y \geq 4$$

□

**Theorem 3.5.** *The equitable chromatic number of the central graph associated with double triangular snake graph, denoted as  $\chi=[C(D_y)]$ ,  $y \geq 3$  is determined for all cases and is given by*

$$\chi=[C(D_y)] = \left\lceil \frac{(y-1)^2 + 1}{y} \right\rceil + (y-1), \quad \forall y \geq 3$$

*Proof.* By the definition of  $D_y$ , we have  $V(D_y) = \{u_m \mid 1 \leq m \leq y\} \cup \{w_{m'}, w'_{m'} \mid 1 \leq m' \leq y-1\}$ . Applying the central graph operation, the vertex set of  $C(D_y)$  becomes:

$$V(C(D_y)) = V(D_y) \cup \{e_{m'} \mid 1 \leq m' \leq y-1\} \\ \cup \{e'_{m''}, e''_{m''} \mid 1 \leq m'' \leq 2y-2\},$$

where  $e_{m'}$  denotes the subdivided vertices along the path edges  $u_m u_{m+1}$ ,  $e'_{m''}$  are the subdivided vertices along edges  $u_m w_{m'}$ ,  $u_{m+1} w_{m'}$ , etc., from the upper triangular snake,  $e''_{m''}$  are the subdivided vertices along edges  $u_m w'_{m'}$ ,  $u_{m+1} w'_{m'}$ , etc., from the lower triangular snake.

Furthermore, let the vertex cardinality of  $C(D_y)$  be defined as  $|V(C(D_y))| = 8y-7$ ,  $\forall y \geq 3$ . Since the order of adjacency matrix of  $C(D_y)$ , which corresponds to the number of vertices in the graph, is given by:  $(8y-7) \times (8y-7)$ ,  $\forall y \geq 3$ . Then the adjacency matrix representation for the first vertex  $u_1$  can be expressed as:

$$u_1 \begin{bmatrix} u_1 & u_2 & u_3 & \cdots & u_y & w_1 & w_2 & w_3 & \cdots & w_{y-1} \\ 0 & 0 & 1 & \cdots & 1 & 0 & 1 & 1 & \cdots & 1 \\ e_1 & e_2 & e_3 & \cdots & e_{y-1} & e'_1 & e'_2 & e'_3 & \cdots & e'_{2y-2} \\ 1 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 \\ w'_1 & w'_2 & w'_3 & \cdots & w'_{y-1} & e''_1 & e''_2 & e''_3 & \cdots & e''_{2y-2} \\ 0 & 1 & 1 & \cdots & 1 & 1 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

From the adjacency matrix, we derive the neighbourhood set of each and every vertices such as,

$$N(u_1) = \{e_1, e'_1, e''_1, \{u_m\}_{m=3}^y, \{w_{m'}, w'_{m'}\}_{m'=2}^{y-1}\}$$

$$N(u_2) = \{e_1, e_2, e'_2, e'_3, e''_2, e''_3, \{u_m\}_{m=4}^y, \{w_{m'}, w'_{m'}\}_{m'=3}^{y-1}\}$$

⋮

$$N(u_y) = \{e_{y-1}, e'_{2y-2}, e''_{2y-2}, \{u_m\}_{m=1}^{y-2}, \{w_{m'}, w'_{m'}\}_{m'=1}^{y-2}\}$$

$$N(w_1) = \{e'_1, e'_2, \{u_m\}_3^y, \{w_{m'}\}_{m'=2}^{y-1}, \{w'_{m'}\}_{m'=1}^{y-1}\}$$

$$N(w_2) = \{e'_3, e'_4, u_1, w_1, \{u_m\}_{m=4}^y, \{w_{m'}\}_{m'=3}^{y-1}, \{w'_{m'}\}_{m'=1}^{y-1}\}$$

⋮

$$N(w_{y-1}) = \{e'_{2y-1}, e'_{2y-2}, \{u_m\}_{m=1}^{y-2}, \{w_{m'}\}_{m'=1}^{y-2}, \{w'_{m'}\}_{m'=1}^{y-1}\}$$

$$N(w'_1) = \{e''_1, e''_2, \{u_m\}_{m=3}^y, \{w'_{m'}\}_{m'=2}^{y-1}, \{w_{m'}\}_{m'=1}^{y-1}\}$$

$$N(w'_2) = \{e''_3, e''_4, u_1, w'_1, \{u_m\}_{m=4}^y, \{w'_{m'}\}_{m'=3}^{y-1}, \{w_{m'}\}_{m'=1}^{y-1}\}$$

⋮

$$N(w'_{y-1}) = \{e''_{2y-1}, e''_{2y-2}, \{u_m\}_{m=1}^{y-2}, \{w'_{m'}\}_{m'=1}^{y-2}, \{w_{m'}\}_{m'=1}^{y-1}\}$$

$$\begin{array}{l} N(e_1) = \{u_1, u_2\} \\ N(e_2) = \{u_2, u_3\} \\ \vdots \\ N(e_{y-1}) = \{u_{y-1}, u_y\} \end{array} \left| \begin{array}{l} N(e'_1) = \{w_1, u_1\} \\ N(e'_2) = \{w_1, u_2\} \\ \vdots \\ N(e'_{2y-2}) = \{w_{y-1}, u_y\} \end{array} \right. \left. \begin{array}{l} N(e''_1) = \{w'_1, u_1\} \\ N(e''_2) = \{w'_2, u_2\} \\ \vdots \\ N(e''_{2y-2}) = \{w'_{y-1}, u_y\} \end{array} \right.$$

We implement an algorithmic approach using a C++ procedure to assign colors optimally while satisfying equitable coloring constrains.

**Algorithm: Equitable Chromatic Number of Double Triangular Snake Graph  $D_y$**

**Require:** Integer  $y \geq 3$

**Ensure:** Equitable coloring sets  $A_u, B_w, C_{w'}, D_e, E_{e'}, F_{e''}$  and chromatic number

$$\chi = (C[D_y])$$

- 1: **repeat**
- 2:     Prompt user: "Enter the size of Double Triangular Snake Graph ( $y \geq 3$ ):"
- 3:     Read  $y$
- 4: **until**  $y \geq 3$
- 5: Initialize arrays:
- 6:  $u[0 \dots y-1], w[0 \dots y-2], w'[0 \dots y-2]$
- 7:  $e[0 \dots y-2], e'[0 \dots 2y-3], e''[0 \dots 2y-3]$
- 8:  $u[0] \leftarrow 1$
- 9: **for**  $i = 1$  to  $y-1$  **do**
- 10:      $u[i] \leftarrow i$
- 11: **end for**
- 12: **for**  $i = 0$  to  $y-2$  **do**
- 13:      $w[i] \leftarrow i + 1; w'[i], e[i] \leftarrow i + y; e'[2i], e'[2i + 1] \leftarrow i + 2$
- 14: **end for**
- 15: **if**  $y \geq 4$  **then**
- 16:     **for**  $i = 0$  to  $y-3$  **do**
- 17:          $e''[2i], e''[2i + 1] \leftarrow i + y + 1$
- 18:     **end for**
- 19:      $e''[2y-4], e''[2y-3] \leftarrow 1$
- 20: **else**
- 21:      $e''[y-3], e''[y-2] \leftarrow y + 1; e''[y-1] \leftarrow 2; e''[y] \leftarrow 1$
- 22: **end if**
- 23: Compute equitable chromatic number:

$$\chi = (C[D_y]) = \left\lceil \frac{(y-1)^2 + 1}{y} \right\rceil + (y-1)$$

24: **Output:**  $\chi = (C[D_y])$

25: **Output arrays:**  $u[0 \dots y-1], w[0 \dots y-2], w'[0 \dots y-2], e[0 \dots y-2], e'[0 \dots 2y-3], e''[0 \dots 2y-3]$

**Example of  $\chi = (C[D_{13}])$**

**Input:** Enter the size of Double Triangular Snake Graph ( $y \geq 3$ ): 13

**Output:** The Equitable Chromatic Number of  $C[D_{13}]$  is = 24

| $A_u$        | $B_w$        | $C_{w'}$      | $D_e$        | $E_{e'}$      | $E_{e''}$      |
|--------------|--------------|---------------|--------------|---------------|----------------|
| $u[1] = 1$   | $w[1] = 1$   | $w'[1] = 13$  | $e[1] = 13$  | $e'[1] = 2$   | $e''[1] = 14$  |
| $u[2] = 1$   | $w[2] = 2$   | $w'[2] = 14$  | $e[2] = 14$  | $e'[2] = 2$   | $e''[2] = 14$  |
| $u[3] = 2$   | $w[3] = 3$   | $w'[3] = 15$  | $e[3] = 15$  | $e'[3] = 3$   | $e''[3] = 15$  |
| $u[4] = 3$   | $w[4] = 4$   | $w'[4] = 16$  | $e[4] = 16$  | $e'[4] = 3$   | $e''[4] = 15$  |
| $u[5] = 4$   | $w[5] = 5$   | $w'[5] = 17$  | $e[5] = 17$  | $e'[5] = 4$   | $e''[5] = 16$  |
| $u[6] = 5$   | $w[6] = 6$   | $w'[6] = 18$  | $e[6] = 18$  | $e'[6] = 4$   | $e''[6] = 16$  |
| $u[7] = 6$   | $w[7] = 7$   | $w'[7] = 19$  | $e[7] = 19$  | $e'[7] = 5$   | $e''[7] = 17$  |
| $u[8] = 7$   | $w[8] = 8$   | $w'[8] = 20$  | $e[8] = 20$  | $e'[8] = 5$   | $e''[8] = 17$  |
| $u[9] = 8$   | $w[9] = 9$   | $w'[9] = 21$  | $e[9] = 21$  | $e'[9] = 6$   | $e''[9] = 18$  |
| $u[10] = 9$  | $w[10] = 10$ | $w'[10] = 22$ | $e[10] = 22$ | $e'[10] = 6$  | $e''[10] = 18$ |
| $u[11] = 10$ | $w[11] = 11$ | $w'[11] = 23$ | $e[11] = 23$ | $e'[11] = 7$  | $e''[11] = 19$ |
| $u[12] = 11$ | $w[12] = 12$ | $w'[12] = 24$ | $e[12] = 24$ | $e'[12] = 7$  | $e''[12] = 19$ |
| $u[13] = 12$ | —            | —             | —            | $e'[13] = 8$  | $e''[13] = 20$ |
|              |              |               |              | $e'[14] = 8$  | $e''[14] = 20$ |
|              |              |               |              | $e'[15] = 9$  | $e''[15] = 21$ |
|              |              |               |              | $e'[16] = 9$  | $e''[16] = 21$ |
|              |              |               |              | $e'[17] = 10$ | $e''[17] = 22$ |
|              |              |               |              | $e'[18] = 10$ | $e''[18] = 22$ |
|              |              |               |              | $e'[19] = 11$ | $e''[19] = 23$ |
|              |              |               |              | $e'[20] = 11$ | $e''[20] = 23$ |
|              |              |               |              | $e'[21] = 12$ | $e''[21] = 24$ |
|              |              |               |              | $e'[22] = 12$ | $e''[22] = 24$ |
|              |              |               |              | $e'[23] = 13$ | $e''[23] = 1$  |
|              |              |               |              | $e'[24] = 13$ | $e''[24] = 1$  |

By the above algorithm, we derive the upper bound:

$$\chi_{=} [C(D_y)] \leq \left\lceil \frac{(y-1)^2 + 1}{y} \right\rceil + (y-1), \forall y \geq 3$$

Upon examining the structural properties of  $C(D_y)$ , we observe that the vertex subset  $V_r = \{w_{m'}, w'_{m'} : 1 \leq m' \leq y-1\}$  forms a **clique** of order  $\left\lceil \frac{(y-1)^2 + 1}{y} \right\rceil + (y-1)$ . It is well-known that for any graph  $H$ , the equitable chromatic number satisfies  $\chi_{=} (H) \geq \omega(H)$ . Therefore, we obtain the lower bound as

$$\chi_{=} [C(D_y)] \geq \left\lceil \frac{(y-1)^2 + 1}{y} \right\rceil + (y-1), \quad \forall y \geq 3$$

Combining the upper and lower bounds, we conclude:

$$\chi_{=} [C(D_y)] = \left\lceil \frac{(y-1)^2 + 1}{y} \right\rceil + (y-1), \forall y \geq 3$$

□

**Theorem 3.6.** *The equitable chromatic number of the central graph associated with double alternate triangular snake graph, denoted as  $\chi_{=} [C(DA_y)]$ ,  $\forall y \geq 3$  is determined for all cases and is given by*

$$\chi_{=} [C(DA_y)] = 2 \lfloor \frac{y}{2} \rfloor + (y \bmod 2), \forall y \geq 3$$

*Proof.* By the definition of the double alternate triangular snake graph  $DA_y$ , the vertex set is given by  $V(DA_y) = \{u_m \mid 1 \leq m \leq y\} \cup \{w_{m'}, w'_{m'} \mid 1 \leq m' \leq \lfloor \frac{y}{2} \rfloor\}$ . Applying the central graph operation, the vertex set of the central graph  $C(DA_y)$  becomes

$$V[C(DA_y)] = V(DA_y) \cup \{e_{m''} \mid 1 \leq m'' \leq y-1\} \\ \cup \{e'_{m'''}, e''_{m'''} \mid 1 \leq m''' \leq 2 \lfloor \frac{y}{2} \rfloor\},$$

where  $u_m$  are the vertices of the path graph,  $e_{m''}$  are the subdivided vertices corresponding to edges  $u_m u_{m+1}$ ,  $w_{m'}$  are the vertices of the upper triangular snake graph and  $e'_{m'''}$  are the subdivided vertices of edges such as  $u_m w_{m'}$ ,  $u_{m+1} w_{m'}$ , etc.;  $w'_{m'}$  are the vertices of the lower triangular snake graph and  $e''_{m'''}$  are the subdivided vertices of edges such as  $u_m w'_{m'}$ ,  $u_{m+1} w'_{m'}$ , etc.

Furthermore, let the vertex cardinality of  $C(DA_y)$  be defined as  $|V(C(DA_y))| = 5y - 4 + 3((y-3) \bmod 2)$ ,  $y \geq 3$ . Since the order of adjacency matrix of  $C(DA_y)$ , which corresponds to the number of vertices in the graph, is given by:  $5y - 4 + 3((y-3) \bmod 2) \times (5y - 4 + 3((y-3) \bmod 2))$ ,  $y \geq 3$ . Then the adjacency matrix representation for the first vertex  $u_1$  can be expressed as:

$$u_1 \begin{bmatrix} u_1 & u_2 & u_3 & \cdots & u_y & w_1 & w_2 & w_3 & \cdots & w_{y-1} \\ 0 & 0 & 1 & \cdots & 1 & 0 & 1 & 1 & \cdots & 1 \\ e_1 & e_2 & e_3 & \cdots & e_{y-1} & e'_1 & e'_2 & e'_3 & \cdots & e'_{2y-2} \\ 1 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 \\ w'_1 & w'_2 & w'_3 & \cdots & w'_{y-1} & e''_1 & e''_2 & e''_3 & \cdots & e''_{2y-2} \\ 0 & 1 & 1 & \cdots & 1 & 1 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

From the adjacency matrix, let us derive the neighbourhood set of each and every vertices as,

$$N(u_1) = \{e_1, e'_1, e''_1, \{u_m\}_{m=3}^y, \{w_{m'}, w'_{m'}\}_{m'=2}^{\lfloor \frac{y}{2} \rfloor}\}$$

$$N(u_2) = \{e_1, e_2, e'_2, e''_2, \{u_m\}_{m=4}^y, \{w_{m'}, w'_{m'}\}_{m'=2}^{\lfloor \frac{y}{2} \rfloor}\}$$

$$\vdots$$

$$N(u_y) = \{e_{y-1}, \{u_m\}_{m=1}^{y-2}, \{w_{m'}, w'_{m'}\}_{m'=1}^{\lfloor \frac{y}{2} \rfloor}\} \text{ (if } y \text{ is odd)}$$

$$N(u_y) = \{e_{y-1}, e'_y, e''_y, \{u_m\}_{m=1}^{y-2}, \{w_{m'}, w'_{m'}\}_{m'=1}^{\lfloor \frac{y}{2} \rfloor - 1}\} \text{ (if } y \text{ is even)}$$

$$\begin{aligned}
N(w_1) &= \{e'_1, e'_2, \{u_m\}_{m=3}^y, \{w_{m'}\}_{m'=2}^{\lfloor \frac{y}{2} \rfloor}, \{w'_{m'}\}_{m'=1}^{\lfloor \frac{y}{2} \rfloor}\} \\
N(w_2) &= \{e'_3, e'_4, u_1, u_2, w_1, \{u_m\}_{m=5}^y, \{w_{m'}\}_{m'=3}^{\lfloor \frac{y}{2} \rfloor}, \{w'_{m'}\}_{m'=1}^{\lfloor \frac{y}{2} \rfloor}\} \\
&\vdots \\
N(w_{\lfloor \frac{y}{2} \rfloor}) &= \{e'_{y-2}, e'_{y-1}, u_y, \{u_m\}_{m=1}^{y-3}, \{w_{m'}\}_{m'=1}^{\lfloor \frac{y}{2} \rfloor - 1}, \{w'_{m'}\}_{m'=1}^{\lfloor \frac{y}{2} \rfloor}\} \text{ (if } y \text{ is odd)} \\
N(w_{\lfloor \frac{y}{2} \rfloor}) &= \{e'_{y-1}, e'_y, \{u_m\}_{m=1}^{y-2}, \{w_{m'}\}_{m'=1}^{\lfloor \frac{y}{2} \rfloor - 1}, \{w'_{m'}\}_{m'=1}^{\lfloor \frac{y}{2} \rfloor}\} \text{ (if } y \text{ is even)} \\
N(w'_1) &= \{e''_1, e''_2, \{u_m\}_{m=3}^y, \{w'_{m'}\}_{m'=2}^{\lfloor \frac{y}{2} \rfloor}, \{w_{m'}\}_{m'=1}^{\lfloor \frac{y}{2} \rfloor}\} \\
N(w'_2) &= \{e''_3, e''_4, u_1, u_2, w'_1, \{u_m\}_{m=5}^y, \{w'_{m'}\}_{m'=3}^{\lfloor \frac{y}{2} \rfloor}, \{w_{m'}\}_{m'=1}^{\lfloor \frac{y}{2} \rfloor}\} \\
&\vdots \\
N(w'_{\lfloor \frac{y}{2} \rfloor}) &= \{e''_{y-2}, e''_{y-1}, u_y, \{u_m\}_{m=1}^{y-3}, \{w'_{m'}\}_{m'=1}^{\lfloor \frac{y}{2} \rfloor - 1}, \{w_{m'}\}_{m'=1}^{\lfloor \frac{y}{2} \rfloor}\} \text{ (if } y \text{ is odd)} \\
N(w'_{\lfloor \frac{y}{2} \rfloor}) &= \{e''_{y-1}, e''_y, \{u_m\}_{m=1}^{y-2}, \{w'_{m'}\}_{m'=1}^{\lfloor \frac{y}{2} \rfloor - 1}, \{w_{m'}\}_{m'=1}^{\lfloor \frac{y}{2} \rfloor}\} \text{ (if } y \text{ is even)} \\
\end{aligned}$$

|                                 |                                                                               |                                                                                 |
|---------------------------------|-------------------------------------------------------------------------------|---------------------------------------------------------------------------------|
| $N(e_1) = \{u_1, u_2\}$         | $N(e'_1) = \{u_1, w_1\}$                                                      | $N(e''_1) = \{w'_1, u_1\}$                                                      |
| $N(e_2) = \{u_2, u_3\}$         | $N(e'_2) = \{u_2, w_1\}$                                                      | $N(e''_2) = \{w'_2, u_2\}$                                                      |
| $\vdots$                        | $\vdots$                                                                      | $\vdots$                                                                        |
| $N(e_{y-1}) = \{u_{y-1}, u_y\}$ | $N(e'_{y-1}) = \{u_{y-1}, w_{\lfloor \frac{y}{2} \rfloor}\}$<br>if $y$ is odd | $N(e''_{y-1}) = \{w'_{\lfloor \frac{y}{2} \rfloor}, u_{y-1}\}$<br>if $y$ is odd |
|                                 | $N(e'_y) = \{u_y, w_{\lfloor \frac{y}{2} \rfloor}\}$<br>if $y$ is even        | $N(e''_y) = \{w'_{\lfloor \frac{y}{2} \rfloor}, u_y\}$<br>if $y$ is even        |

We implement an algorithmic approach using a C++ procedure to assign colors optimally while satisfying equitable coloring constrains.

#### Algorithm: Equitable Chromatic Number of Double Alternate Triangular Snake Graph $DA_y$

```

Require: Integer  $y \geq 4$ 
Ensure: Equitable coloring sets  $A_u, B_w, C_{w'}, D_e, E_{e'}, F_{e''}$  and chromatic number  $\chi_{=}(C[DA_y])$ 
1: repeat
2:   Prompt user: "Enter the size of Double Alternate Triangular Snake Graph ( $y \geq 4$ ):"
3:   Read  $y$ 
4: until  $y \geq 4$ 
5:  $k \leftarrow \lfloor y/2 \rfloor$ 
6: Initialize arrays:
7:  $u[0 \dots y-1], w[0 \dots k-1], w'[0 \dots k-1]$ 
8:  $e[0 \dots y-2], e'[0 \dots 2k-1], e''[0 \dots 2k-1]$ 
9: if  $y$  is even then
10:    $val \leftarrow 1, val_1 \leftarrow 1$ 
11:   for  $i = 0$  to  $y - 1$  do
12:     if  $i \bmod 2 = 0$  then
13:        $u[i] \leftarrow val; val \leftarrow val + 1$ 
14:     else
15:        $u[i] \leftarrow val_1 + \frac{y}{2}; val_1 \leftarrow val_1 + 1$ 
16:     end if
17:   end for
18: else
19:    $val \leftarrow 1; val_1 \leftarrow 1$ 
20:   for  $i = 0$  to  $y - 2$  do
21:     if  $i \bmod 2 = 0$  then
22:        $u[i] \leftarrow val; val \leftarrow val + 1$ 
23:     else
24:        $u[i] \leftarrow val_1 + k; val_1 \leftarrow val_1 + 1$ 
25:     end if
26:   end for
27:    $u[y-1] \leftarrow y$ 
28: end if
29: for  $i = 0$  to  $k - 1$  do
30:    $w[i] \leftarrow i + 1; w'[i] \leftarrow i + k + 1$ 
31: end for

```

```

32: if y is odd then
33:   e'[0] ← y - 1; e'[1] ← y - 1
34: else
35:   e'[0] ← y; e'[1] ← y
36: end if
37: val ← 1
38: for i = 0 to k - 2 do
39:   e'[2i + 2] ← val; e'[2i + 3] ← val; val ← val + 1
40: end for
41: m ← y/2
42: for i = 0 to k - 1 do
43:   e''[2i] ← m; e''[2i + 1] ← m; m ← m + 1
44: end for
45: if y is even then
46:   val ← y/2
47:   for i = 0 to y - 2 do
48:     if i mod 2 = 1 then
49:       e[i] ←  $\frac{i+1}{2}$ 
50:     else
51:       e[i] ← val; val ← val + 1
52:     end if
53:   end for
54: else if y = 5 then
55:   e[0] ← y, e[1] ← y, e[2] ← y, e[3] ← 1
56: else
57:   for i = 1 to y - 2 step 2 do
58:     e[i] ←  $\frac{i+1}{2}$ 
59:   end for
60:   val ← e[y - 2]
61:   for i = 0 to y - 2 step 2 do
62:     if i ≤ 7 then
63:       e[i] ← y
64:     else
65:       val ← val + 1, e[i] ← val
66:     end if
67:   end for
68: end if
69: Compute chromatic number:  $\chi_{=}(C[DA_y]) \leftarrow y$ 
70: Output: Equitable Chromatic Number  $\chi_{=}(C[DA_y])$ 
71: Output: Print arrays  $u[0 \dots y-1], w[0 \dots k-1], w'[0 \dots k-1], e[0 \dots y-2], e'[0 \dots 2k-1], e''[0 \dots 2k-1]$ 

```

**Example of  $\chi_{=}(C[DA_{15}])$**  Input: Enter the size of Double Alternate Triangular Snake Graph ( $y \geq 4$ ): 15  
Output: The Equitable Chromatic Number of  $C[DA_{15}]$  is = 15

| $A_u$        | $B_w$      | $C_{w'}$     | $D_e$        | $E_{e'}$     | $F_{e''}$      |
|--------------|------------|--------------|--------------|--------------|----------------|
| $u[1] = 1$   | $w[1] = 1$ | $w'[1] = 8$  | $e[1] = 15$  | $e'[1] = 14$ | $e''[1] = 7$   |
| $u[2] = 8$   | $w[2] = 2$ | $w'[2] = 9$  | $e[2] = 1$   | $e'[2] = 14$ | $e''[2] = 7$   |
| $u[3] = 2$   | $w[3] = 3$ | $w'[3] = 10$ | $e[3] = 15$  | $e'[3] = 1$  | $e''[3] = 8$   |
| $u[4] = 9$   | $w[4] = 4$ | $w'[4] = 11$ | $e[4] = 2$   | $e'[4] = 1$  | $e''[4] = 8$   |
| $u[5] = 3$   | $w[5] = 5$ | $w'[5] = 12$ | $e[5] = 15$  | $e'[5] = 2$  | $e''[5] = 9$   |
| $u[6] = 10$  | $w[6] = 6$ | $w'[6] = 13$ | $e[6] = 3$   | $e'[6] = 2$  | $e''[6] = 9$   |
| $u[7] = 4$   | $w[7] = 7$ | $w'[7] = 14$ | $e[7] = 15$  | $e'[7] = 3$  | $e''[7] = 10$  |
| $u[8] = 11$  | —          | —            | $e[8] = 4$   | $e'[8] = 3$  | $e''[8] = 10$  |
| $u[9] = 5$   | —          | —            | $e[9] = 8$   | $e'[9] = 4$  | $e''[9] = 11$  |
| $u[10] = 12$ | —          | —            | $e[10] = 5$  | $e'[10] = 4$ | $e''[10] = 11$ |
| $u[11] = 6$  | —          | —            | $e[11] = 9$  | $e'[11] = 5$ | $e''[11] = 12$ |
| $u[12] = 13$ | —          | —            | $e[12] = 6$  | $e'[12] = 5$ | $e''[12] = 12$ |
| $u[13] = 7$  | —          | —            | $e[13] = 10$ | $e'[13] = 6$ | $e''[13] = 13$ |
| $u[14] = 14$ | —          | —            | $e[14] = 7$  | $e'[14] = 6$ | $e''[14] = 13$ |
| $u[15] = 15$ | —          | —            | —            | —            | —              |

By the above programming algorithm, we obtain the upper bound,

$$\chi_{=}[C(DA_y)] \leq 2 \left\lfloor \frac{y}{2} \right\rfloor + (y \bmod 2), \forall y \geq 3$$

Upon examining the structural properties of  $C(DA_y)$ , we observe that the vertex subset  $V_r = \{w_{m'}, w'_{m'} : 1 \leq m' \leq \lfloor \frac{y}{2} \rfloor\}$  forms a **clique** of order  $2 \lfloor \frac{y}{2} \rfloor + (y \bmod 2)$ . It is well-known that for any graph H, the equitable chromatic number satisfies  $\chi_{=}(H) \geq \omega(H)$ . Therefore, we obtain

the lower bound as

$$\chi = [C(DA_y)] \geq 2 \left\lfloor \frac{y}{2} \right\rfloor + (y \bmod 2), \forall y \geq 3$$

Combining the upper and lower bounds, we conclude:

$$\chi = [C(DA_y)] = 2 \left\lfloor \frac{y}{2} \right\rfloor + (y \bmod 2), \forall y \geq 3$$

□

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