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THE COLOR ENERGY BOUNDS FOR SPOKE-LACED ARCHITECTURAL GRAPHS

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Abstract The color energy of the graphs is achieved by first coloring the graphs and then building their coloring adjacency matrix based on the properties of colored vertices that are adjacent or not. In this study, the sunlet, helm, closed helm, and gear graphs are identified as spoke-structured graphs in line with their geometric pattern. For the graphs under consideration the color energy bounds, comprising lower, upper, and optimized lower as well as upper bounds, are established. Our computation allows us to develop an efficient algorithm to calculate the fluctuations of the color energies discussed.

1 Introduction

In 1978, Ivan Gutman introduced the concept of graph energy for an undirected network $G(V, E)$ inspired by the principles of Hückel Molecular Orbital theory. Extending the HMO model [1, 2, 3] to analyze the π-electron energy in molecular frameworks, Gutman redefined it to introduce a graph-based energy measure. This adaptation enabled fresh insights into energy within a purely mathematical context. As a result, researchers have systematically formulated and examined a broad spectrum of attributes and strict conditions surrounding graph energy [4, 5, 6]. In a molecular graph G, the adjacency matrix composed of binary entries 0 and 1 produces a spectrum of characteristic values $\lambda_1, \lambda_2, \ldots, \lambda_n$, providing information on the spectral properties of the graph. By adding the absolute values of its characteristic values, the graph energy highlights the structural properties of G [2].

The author Adiga et.al [7] introduced an idea in the color energy of a graph is the sum of the absolute values of the eigenvalues of its L-matrix $A_c(G)$. This matrix is defined based on vertex coloring, where adjacent vertices with different colors are assigned 1, non-adjacent vertices with the same color are assigned -1, and all other entries are 0. The eigenvalues of $A_c(G)$, known as color eigenvalues, collectively determine the graph's color energy. Concurrently, Cvetkovski et al. [8] address several inequalities that enhance the precision of graph energy calculations. Building on these techniques, we first derive the lower and upper bounds for the color energy in graphs that satisfy the necessary conditions. Subsequently, Jerlinkasmir et al. [9] propose optimized upper and lower bounds, further enhancing the precision of energy calculations in graph theory.

2 Motivation

The color energy of graphs provides insights into the structural interactions and allocation between vertices, particularly in network and optimization problems. By analyzing the color energy of spoke-structured graphs like sunlet, helm, closed Helm, and gear graphs, we can better understand their performance in applications such as network design, resource allocation, and traffic flow management. These bounds (lower, upper, and optimized) help to optimize energy consumption and reduce interference in systems like wireless networks. The developed algorithm offers an efficient way to compute these values, contributing to real-world problem solving in graph-based systems.

3 Preliminaries

The fundamental terminology and ideas pertaining to our work are covered in this part.

Definition 3.1. (**Color Energy**)[7] The color energy of a graph is defined as the sum of the absolute values of the latent metrics of its L-matrix $A_c(G)$, which encodes the adjacency and color relationships between vertices. The matrix entries are defined as follows:

$$A_c(G)_{ij} = \begin{cases} 1, & \text{if } v_i \text{ and } v_j \text{ are adjacent with } c(v_i) \neq c(v_j), \\ -1, & \text{if } v_i \text{ and } v_j \text{ are non-adjacent with } c(v_i) = c(v_j), \\ 0, & \text{otherwise.} \end{cases}$$

The latent metrics of $A_c(G)$, known as color latent metrics, collectively determine the color energy of the graph, computed as: $E_c(G) = \sum_{i=1}^{n} |\lambda_i|$, where λ_i are the eigenvalues of $A_c(G)$.

Definition 3.2. (**Sunlet**)[13] By adding n pendant edges to the cycle C_n, the n-sunlet graph, represented by S_n, is a graph with $2n$ vertices. Each pendant edge is connected to a unique cycle vertex, creating a structure in which the cycle serves as the core and pendant vertices radiate out from it.

Definition 3.3. (**Gear**) [10] The Gear graph G_n is created from the Wheel graph W_n by adding a new vertex to each edge of the cycle C_{n-1}. These new vertices are connected to the endpoints of their corresponding edges, forming a structure that resembles the teeth of a gear. The central vertex of the wheel graph remains connected to all vertices of the cycle.

Definition 3.4. Helm [12] A helm graph H_n is a graph constructed by linking a pendant edge to every vertex on the outer cycle C_n of a wheel graph W_n.

Definition 3.5. Closed helm[11] By joining the edges between each pendant node, the Helm graph H_n can be transformed into the Closed Helm graph CH_n.

Theorem 3.6. ***Cauchy-Schwarz inequality*** *[8]*
If $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ are real numbers, we have

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right). \tag{3.1}$$

Theorem 3.7. ***Arithmetic and Geometric mean inequality(AM-GM)****[8] Let n be a number of vertex and λ_i and λ_j be the characteristic values.*

$$\frac{1}{n(n-1)} \sum_{i \neq j} |\lambda_i|\,|\lambda_j| \geqslant \left(\prod_{i \neq j} |\lambda_i|\,|\lambda_j|\right)^{\frac{1}{n(n-1)}} \tag{3.2}$$

Theorem 3.8. ***Chebyshev's Inequality (monotonic Condition)*** *[8]*
If $\{x_i\}$ and $\{y_i\}$ are non-decreasing, then:

$$\sum_{\substack{i=1 \\ i \neq j}}^{n} x_i y_j \geq \frac{2}{n(n-1)} \left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right), \tag{3.3}$$

with the monotonicity condition from Chebyshev's Inequality.

Theorem 3.9. ***Weighted AM–GM inequality*** *[8] Let $a_i \in (0,\infty)$, $i = 1,2,\ldots,n$, and $\alpha_i \in [0,1]$, $i = 1,2,\ldots,n$, be such that $\alpha_1 + \alpha_2 + \cdots + \alpha_n = 1$. Then*

$$a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_n^{\alpha_n} \leq \alpha_1 a_1 + \alpha_2 a_2 + \cdots + \alpha_n a_n. \tag{3.4}$$

Theorem 3.10. ***Weighted Cauchy–Schwarz inequality*** *[8]*
Let $a_i, b_i \in \mathbb{R}$ and $m_i \in \mathbb{R}^+$ for $i = 1,2,\ldots,n$. Then:

$$\left(\sum_{i=1}^{n} a_i b_i m_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2 m_i\right)\left(\sum_{i=1}^{n} b_i^2 m_i\right), \tag{3.5}$$

with equality if and only if $a_1 b_1 = a_2 b_2 = \cdots = a_n b_n$.

4 Deriving the bounds of color energy in spokes structures

In this part, we examine the various color energy, including lower bound, upper bound, optimized lower bound and optimized upper bounds for the spoke families.

4.1 The color energy lower bounds for the sunlet graph

Theorem 4.1. *For a sunlet graph S_n, $n \geq 8$, with $p = 2n$ vertices and $2n$ edges.*
(i) The lower bound of color energy is given by:

$$E_c(S_n) \geq \sqrt{\left|\varrho(-1)\right| + \left|\varpi(1)\right| + \tfrac{2}{(p)(p-1)} \prod_{i=1}^{p} (\sigma_i^{w_i})^2}$$

here ϖ and ϱ represent the total number of 1's and -1's, respectively, in the color adjacency matrix. The weights denoted by w_i, subject to the normalization condition, $\sum_{i=1}^{p} w_i = 1$, which ensures that latent metrics weights sum to unity.
(ii) The optimized lower bound of the color energy is:

$$E_c(S_n) \geq \sqrt{\left|\varrho(-1)\right| + \left|\varpi(1)\right| + (p)(p-1)D^{\frac{2}{p}}} \text{ where } D = det(A_c(S_n))$$

Proof. The vertices on the circular portion of the sunlet S_n are nominated as $v_i, i = 1,2,\ldots,n$ and its pendant vertices named as $v_{n+i}, i = 1,2,\ldots,n$ which implies the whole vertex set of S_n as $V = \{v_i \cup v_{n+i}, i = 1,2,\ldots,n\}$.
The coloring of the graph follows the mapping $\Phi : V \longrightarrow \{1,2,3\}$.
Case (i) $n \equiv 1 \text{ mod } 2$ (odd)

$$\Phi(v_i) = \begin{cases} 1, & i \equiv 1 \text{ mod } 2 \\ 2, & i \equiv 0 \text{ mod } 2,\ 1 \leq i \leq n-1 \end{cases}$$

$$\Phi(v_n) = 3$$

$$\Phi(v_{n+i}) = \begin{cases} 2, & i \equiv 1 \text{ mod } 2 \\ 1, & i \equiv 0 \text{ mod } 2,\ 1 \leq i \leq n. \end{cases}$$

Case (ii) $n \equiv 0 \text{ mod } 2$ (even)

$$\Phi(v_i) = \begin{cases} 1, & i \equiv 1 \text{ mod } 2 \\ 2, & i \equiv 0 \text{ mod } 2,\ 1 \leq i \leq n \end{cases}$$

$$\Phi(v_{n+i}) = \begin{cases} 2, & i \equiv 1 \text{ mod } 2 \\ 1, & i \equiv 0 \text{ mod } 2,\ 1 \leq i \leq n. \end{cases}$$

From this coloring the color adjacency matrix C_{ij} is constructed as follows:

$$C_{ij} = \begin{cases} 1, & \text{if } (i,j) \in E, \quad \chi(i) \neq \chi(j) \\ -1, & \text{if } (i,j) \notin E, \quad \chi(i) = \chi(j) \\ 0, & \text{otherwise} \end{cases} \tag{4.1}$$

As per equation 4.1 we get the $p \times p$ equidimensional matrix's generalized latent polynomial equation

$$c_0\sigma^p + c_1\sigma^{p-1} + c_2\sigma^{p-2} + \ldots + c_n = 0$$

The latent metrics $\sigma_1, \sigma_2, \ldots, \sigma_p$ are acquired by solving $|A_c(S_n) - \sigma I| = 0$. The color energy $E_c(S_n)$ is the sum of the magnitude of the latent metrics of the vertex-colored color matrix. For the purpose of the proof, we obtain the total of the squared latent metrics as

$$\begin{aligned}\sum_{i=1}^{p} \sigma_i^2 &= \sum_{\substack{(i,j)\in E\\ \chi(i)\neq\chi(j)}} (1) + \sum_{\substack{(i,j)\notin E\\ \chi(i)=\chi(j)}} (-1) + \sum_{\substack{(i,j)\notin E\\ \chi(i)\neq\chi(j)}} 0 \\ &= \big|\varrho(-1)\big| + \big|\varpi(1)\big|\end{aligned} \tag{4.2}$$

Here E represents the set of edges and $\chi(i)$ denotes the color assigned to vertex i. In this case, ϖ and ϱ is the total count of 1's and -1's present in the color matrix respectively.

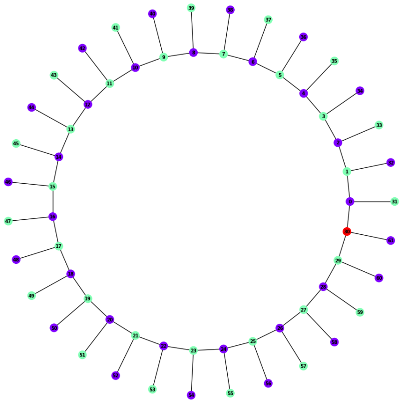

Fig. 1.a. Sunlet Graph S_{31} (n is odd).

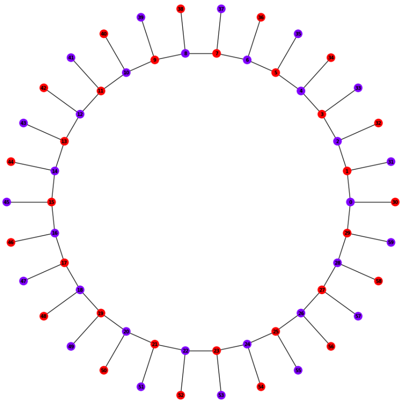

Fig. 1.b. Sunlet Graph S_{30} (n is even).

In the context of color energy, the following condition holds:

$$[E_c(S_n)]^2 = \left(\sum_{i=1}^{p} |\sigma_i|\right)^2$$

$$[E_c(S_n)]^2 = \left(\sum_{i=1}^{p} |\sigma_i|\right)\left(\sum_{j=1}^{r} |\sigma_j|\right).$$

$$[E_c(S_n)]^2 = \left(\sum_{i=1}^{p} |\sigma_i|^2\right) + \left(\sum_{i\neq j} |\sigma_i||\sigma_j|\right).$$

Case (i): Proof for the color energy lower bound of sunlet graph
From the Chebyshev's Inequality [3.3], we derive that,

$$[E_c(S_n)]^2 \geq \sum_{i=1}^{p} |\sigma_i|^2 + \frac{2}{p(p-1)}\left(\sum_{i=1}^{p} |\sigma_i|\right)\left(\sum_{i=1}^{p} |\sigma_i|\right)$$

Using Weighted Arithmetic–Geometric mean inequality [3.4], we conclude,

$$\begin{aligned}
[E_c(S_n)]^2 &\geq \sum_{i=1}^{p}|\sigma_i|^2 + \frac{2}{p(p-1)}\left(\sum_{i=1}^{p}\sigma_i w_i\right)\left(\sum_{i=1}^{p}\sigma_i w_i\right) \quad \text{where} \sum_{i=1}^{p} w_i = 1,\\
&\geq \sum_{i=1}^{p}|\sigma_i|^2 + \frac{2}{p(p-1)}\prod_{i=1}^{p}(\sigma_i^{w_i})\prod_{i=1}^{p}(\sigma_i^{w_i})\\
&\geq |\varrho(-1)| + |\varpi(1)| + \frac{2}{p(p-1)}\prod_{i=1}^{p}(\sigma_i^{w_i})^2 \ldots\ldots\ldots \text{[by (4.2)]}
\end{aligned}$$

Thus, the lower bound of the color energy for sunlet graph is,

$$E_c(S_n) \geq \sqrt{|\varrho(-1)| + |\varpi(1)| + \tfrac{2}{p(p-1)}\textstyle\prod_{i=1}^{p}(\sigma_i^{w_i})^2}.$$

Case (ii): Proof for the color energy optimized lower bound.
From the Arithmetic-Geometric mean inequalities in equation [3.2], we derive that

$$\frac{1}{p(p-1)}\sum_{i\neq j}|\sigma_i||\sigma_j| \geqslant \left(\prod_{i\neq j}|\sigma_i||\sigma_j|\right)^{\frac{1}{p(p-1)}}$$

$$\begin{aligned}
[E_c(S_n)]^2 &\geq \sum_{i=1}^{p}|\sigma_i|^2 + p(p-1)\left(\prod_{i\neq j}|\sigma_i|\,|\sigma_j|\right)^{\frac{1}{p(p-1)}}\\
&= \sum_{i=1}^{p}|\sigma_i|^2 + p(p-1)\left|\prod_{i=1}^{p}(\sigma_i)\right|^{\frac{2}{p}}\\
&= |\varrho(-1)| + |\varpi(1)| + p(p-1)D^{\frac{2}{p}} \quad \text{[by (4.2)]}
\end{aligned}$$

Thus, the optimized lower bound of the color energy for sunlet graph is,

$$E_c(S_n) \geq \sqrt{|\varrho(-1)| + |\varpi(1)| + p(p-1)D^{\frac{2}{p}}}.$$

□

4.2 The color energy upper bound for sunlet graph

Theorem 4.2. *For the sunlet graph S_n with $n \geq 8$ and rank r.*
(i) The upper bound of color energy is given by: $E_c(S_n) \leq \sqrt{r|\varrho(-1)| + |\varpi(1)|}$.
(ii) The optimized upper bound of color energy is: $E_c(S_n) \leq \sqrt{12|\varrho(-1)| + |\varpi(1)|}$.

Proof. The methodology used to determine the latent metrics echoes that is utilized in Theorem [4.1]. Let $\sigma_1 \geq \sigma_2 \geq \sigma_3 \geq \cdots \geq \sigma_{n-1} \geq \sigma_n \geq \cdots \geq \sigma_p$ denote the latent metrics of the $p \times p$ vertex-colored matrix of S_n.
Case (i): Proof for the color energy upper bound of sunlet graph
From the Cauchy-Schwarz inequality in the equation [3.1] we derive that,

$$\left(\sum_{i=1}^{p}\alpha_i\beta_i\right)^2 \leq \left(\sum_{i=1}^{p}\alpha_i^2\right)\left(\sum_{i=1}^{p}\beta_i^2\right).$$

To establish the upper bound, consider the expression $[E_c(S_n)]^2 \leq (\sum_{i=1}^{p}\alpha_i\beta_i)^2$, where the color energy is bounded on the right-hand side. Substituting $\alpha_i = 1$ and $\beta_i = |\sigma_i|$ into the inequality yields the desired result.

$$\begin{aligned}
[E_c(S_n)]^2 &\leq \left(\sum_{i=1}^{p} 1.|\sigma_i|\right)^2 \leq \left(\sum_{i=1}^{p} 1\right)^2 . \left(\sum_{i=1}^{p} |\sigma_i|^2\right) \ldots\ldots\ldots [\text{by } (3.1)] \\
&= r\big(|\varrho(-1)| + |\varpi(1)|\big) \ldots\ldots\ldots\ldots\ldots\ldots [\text{by } (4.2)] \\
[E_c(S_n)] &\leq \sqrt{r|\varrho(-1)| + |\varpi(1)|}.
\end{aligned}$$

Thus, the upper bound of the color energy, $E_c(S_n) \leq \sqrt{r|\varrho(-1)| + |\varpi(1)|)}$.

Case (ii): Proof for the color energy optimized upper bound.

The Weighted Cauchy Schwarz inequality in [3.5] allows us to deduce that,

$$\left(\sum_{i=1}^{p} a_i b_i w_i\right)^2 \leq \left(\sum_{i=1}^{p} a_i^2 w_i\right)\left(\sum_{i=1}^{p} b_i^2 w_i\right)$$

For proving the optimized upper bound, consider $[E_c(S_n)]^2 \leq \left(\sum_{i=1}^{p} a_i.b_i.w_i\right)^2$, such that the color energy of the graph is less than RHS of the inequality, and substituting $a_i = \frac{1}{(p)^{\frac{3}{2}}}|\sigma_i|$ and $b_i = 1$ $w_i = \sqrt{12}$, in the above inequality,

$$\begin{aligned}
[E_c(S_n)]^2 &\leq \left(\sum_{i=1}^{p} \frac{\sigma_i^2}{(p)^3}.\sqrt{12}\right)\left(\sum_{i=1}^{p} 1.\sqrt{12}\right) \\
&\leq \frac{\sqrt{12}p}{p^3}\left(\sum_{i=1}^{p} \sigma_i^2\right).p^2\sqrt{12} \\
&= 12\big(|\varrho(-1)| + |\varpi(1)|\big) \ldots\ldots\ldots\ldots\ldots\ldots [\text{by } (4.2)] \\
E_c(S_n) &\leq \sqrt{12\big(|\varrho(-1)| + |\varpi(1)|\big)}.
\end{aligned}$$

Thus, the optimized upper bound of the color energy, $E_c(S_n) \leq \sqrt{12\big(|\varrho(-1)| + |\varpi(1)|\big)}$. □

Remark 4.3. The color energy lower, upper bounds, optimized lower and upper bounds for sunlet graphs with n vary from 8 to 100 are computed and plotted in Figure 2.

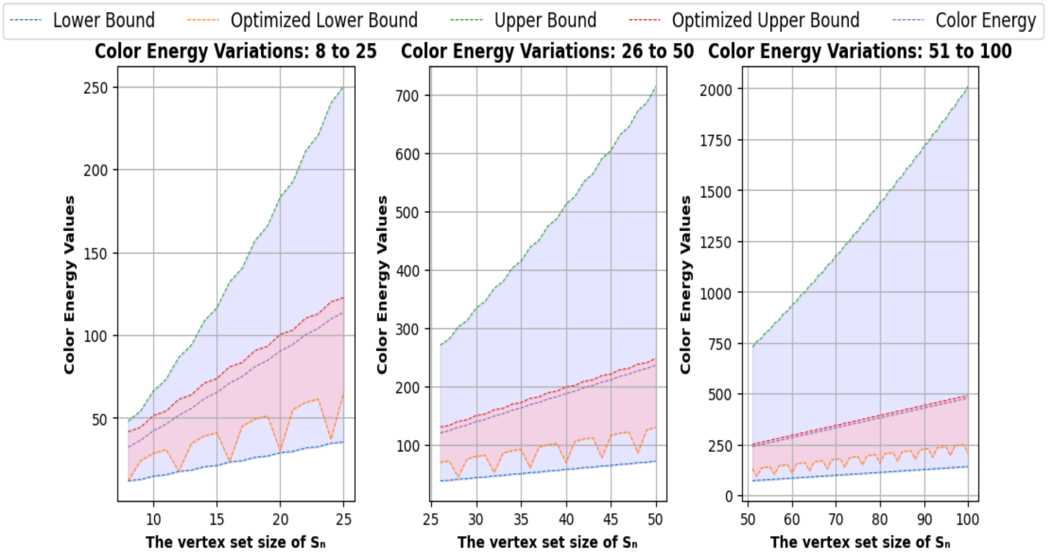

Fig. 2. The color energy bounds of sunlet graph

4.3 The color energy lower bounds for the Helm graph

Theorem 4.4. *For a helm graph H_n, $n \geq 8$, with $p = 2n + 1$ vertices and $3n$ edges.*
(i) The lower bound of color energy is given by:

$$E_c(H_n) \geq \sqrt{|\Theta(-1)| + |\Omega(1)| + \tfrac{2}{(p)(p-1)} \prod_{i=1}^{p} (\sigma_i^{w_i})^2}$$

where Θ and Ω correspond to the total occurrences of -1's and 1's color adjacency matrix respectively. The weights w_i adhere to the normalization condition$\sum_{i=1}^{p} w_i = 1$, which ensures that the latent metric weights collectively sum to unity.
(ii) The optimized lower bound of the color energy is:

$$E_c(H_n) \geq \sqrt{|\Theta(-1)| + |\Omega(1)| + (p)(p-1)D^{\frac{2}{p}}} \quad \text{where } D = |A_c(H_n)|.$$

Proof. The vertices of the helm graph H_n is categorized into three layers, the inner central point v_0, the middle annular part are marked as $v_i, i = 1, 2, \ldots, n$ and the outer leaf nodes as $v_{n+i}, i = 1, 2, \ldots, n$. Therefore the entire vertex set of H_n as $V = \{v_0 \cup v_i \cup v_{n+i}, i = 1, 2, \ldots, n\}$.

The coloring of this graph follows the mapping $\Phi : V \longrightarrow \{1, 2, 3, 4\}$.
Case (i) $n \equiv 1$ **mod** 2 **(odd)**

$$\Phi(v_0) = 1$$

$$\Phi(v_i) = \begin{cases} 2, & i \equiv 1 \text{ mod } 2 \\ 3, & i \equiv 0 \text{ mod } 2, \ 1 \leq i \leq n-1 \end{cases}$$

$$\Phi(v_n) = 4$$

$$\Phi(v_{n+i}) = 1, \ 1 \leq i \leq n.$$

Case (ii) $n \equiv 0$ **mod** 2 **(even)**

$$\Phi(v_0) = 1$$

$$\Phi(v_i) = \begin{cases} 2, & i \equiv 1 \text{ mod } 2 \\ 3, & i \equiv 0 \text{ mod } 2, \ 1 \leq i \leq n \end{cases}$$

$$\Phi(v_{n+i}) = 1, \ 1 \leq i \leq n.$$

Similar to the equation 4.1 form the color adjacency matrix of helm graph C_{ij} as followed in the sunlet graph. Obviously, it is derived from a wheel with additional outer vertices connected to each spoke, the color energy properties are preserved as per 4.1, hence the adjacency matrix is adjusted to reflect its unique vertex-edge interactions. Then we get the $p \times p$ dimensional matrix's latent polynomial equation as

$$c_0\sigma^p + c_1\sigma^{p-1} + c_2\sigma^{p-2} + \ldots + c_n = 0 \tag{4.3}$$

The matrix structure varies due to differences in connectivity, ensuring an accurate representation on each criteria of the color matrix. The total of each squared latent metrics for H_n is derived as; refer to 4.2.

$$\sum_{i=1}^{p} \sigma_i^2 = |\Theta(-1)| + |\Omega(1)| \tag{4.4}$$

In this case, Θ and Ω represent the total count of -1's and 1's respectively.

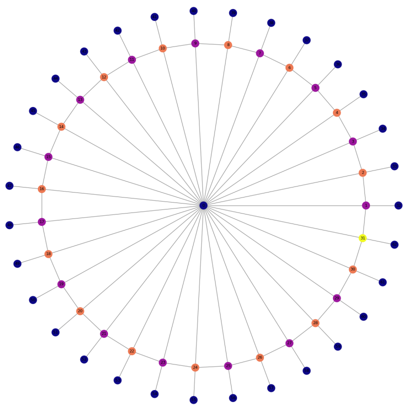

Fig. 3.a. Helm Graph H_{31} (n is odd).

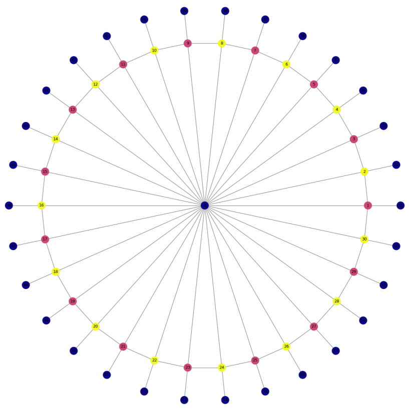

Fig. 3.b. Helm Graph H_{30} (n is even).

The remaining part of the proof is similar to Theorem [4.1] and is left to the interest of the reader. □

4.4 The color energy upper bounds for Helm graph

Theorem 4.5. *For helm graph H_n with $n \geq 8$ and rank r*
(i) The upper bound of color energy is given by: $E_c(H_n) \leq \sqrt{r\big|\Theta(-1)\big| + \big|\Omega(1)\big|}$
(ii) The optimized upper bound of color energy is: $E_c(H_n) \leq \sqrt{15\big(\big|\Theta(-1)\big| + \big|\Omega(1)\big|\big)}$.

Proof. The proof is similar to Theorem [4.2] by considering the weights $w_i = 15$ in the weighted Cauchy-Schwarz inequality while optimizing the upper bound of H_n, and its derivation is left to the reader's interest. □

Remark 4.6. The color energy of upper and lower bound, optimized upper and lower bound helm graphs with n varying from 8 to 100 are computed and plotted in Fig. 4.

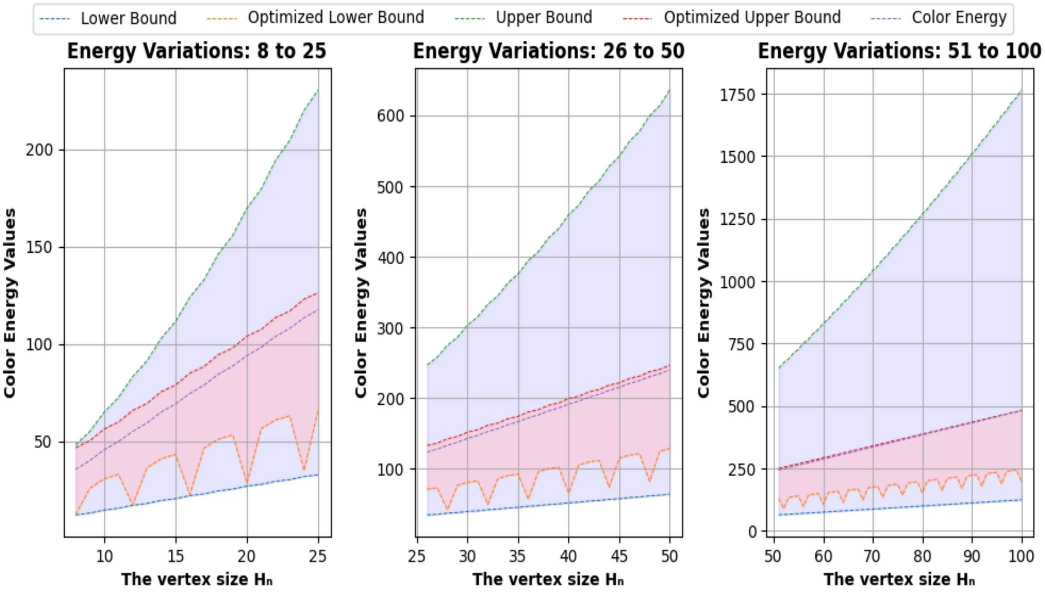

Fig. 4. The color energy bounds of Helm graph

4.5 The color energy lower bounds for the Closed Helm graph

Theorem 4.7. *For a closed helm CH_n, $n \geq 9$, with $p = 2n + 1$ vertices and $4n$ edges.*
(i) The lower bound of color energy is given by:
$E_c(CH_n) \geq \sqrt{|\Phi(-1)| + |\Psi(1)| + \frac{2}{(p)(p-1)} \prod_{i=1}^{p} (\sigma_i^{w_i})^2}$
here Φ and Ψ represent the total count of -1's and 1's respectively, in the color adjacency matrix, and w_i are the weights, $\sum_{i=1}^{p} w_i = 1$ gives the total sum of the latent metrics weights.
(ii) The optimized lower bound of the color energy is:
$E_c(CH_n) \geq \sqrt{|\Phi(-1)| + |\Psi(1)| + (p)(p-1)D^{\frac{2}{p}}}$ *where $D = |A_c(CH_n)|$.*

Proof. The vertices of the closed helm graph CH_n are structured into three layers, the inner central point v_0, the middle annular part is marked as $v_i, i = 1, 2, \ldots, n$ and the outer disc nodes as $v_{n+i}, i = 1, 2, \ldots, n$. Therefore, the entire vertex set of CH_n is $V = \{v_0 \cup v_i \cup v_{n+i}, i = 1, 2, \ldots, n\}$.

The coloring of this graph follows the mapping $\Phi : V \longrightarrow \{1, 2, 3, 4\}$.

Case (i) $n \equiv 1$ **mod** 2 **(odd)**

$$\Phi(v_0) = 1$$

$$\Phi(v_i) = \begin{cases} 2, & i \equiv 1 \bmod 2 \\ 3, & i \equiv 0 \bmod 2, \ 1 \leq i \leq n-1 \end{cases}$$

$$\Phi(v_n) = 4$$

$$\Phi(v_{n+i}) = \begin{cases} 1, & i \equiv 1 \bmod 2 \\ 2, & i \equiv 0 \bmod 2, \ 1 \leq i \leq n-1 \end{cases}$$

$$\Phi(v_{2n}) = 3$$

Case (ii) $n \equiv 0$ **mod** 2 **(even)**

$$\Phi(v_0) = 1$$

$$\Phi(v_i) = \begin{cases} 2, & i \equiv 1 \bmod 2 \\ 3, & i \equiv 0 \bmod 2, \ 1 \leq i \leq n \end{cases}$$

$$\Phi(v_{n+i}) = \begin{cases} 1, & i \equiv 1 \bmod 2 \\ 2, & i \equiv 0 \bmod 2, \ 1 \leq i \leq n \end{cases}$$

The color adjacency matrix of closed helm C_{ij} (refer 4.1) is constructed as per sunlet graph. It originates from a wheel graph, where additional outer vertices are linked to each spoke and further interconnected to form an enclosing cycle. Although the energy property remains consistent 4.1, the adjacency matrix adapts to accommodate its distinct vertex-edge relationships. We get the $p \times p$ equidimensional matrix's latent polynomial equation same as 4.3 .

The structural variation in connectivity influences the matrix formulation, ensuring the depiction of color adjacency matrix configuration (see 4.2). The sum of squares of latent values for CH_n is formulated as

$$\sum_{i=1}^{p} \sigma_i^2 = |\Phi(-1)| + |\Psi(1)| \qquad (4.5)$$

In this case, Φ and Ψ represent the all count of -1's and 1's respectively.

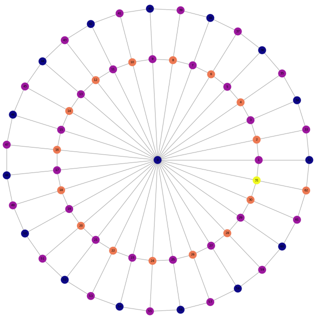

Fig. 5.a. Closed Helm Graph CH_{31} (n is odd).

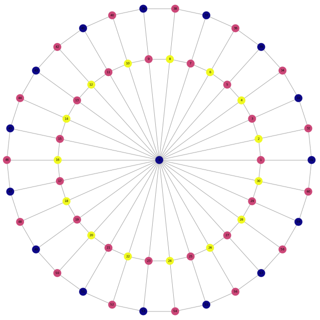

Fig. 5.b. Closed Helm Graph CH_{30} (n is even).

The remaining part of the proof adhere to the same procedure of Theorem [4.1] and is left to the interest of the reader. □

4.6 The color energy upper bounds for Closed Helm graph

Theorem 4.8. *For closed helm graph CH_n with $n \geq 9$ and rank r*
(i) The upper bound of color energy is given by: $E_c(CH_n) \leq \sqrt{r(\left|\Phi(-1)\right| + \left|\Psi(1)\right|)}$
(ii) The optimized upper bound of color energy is: $E_c(CH_n) \leq \sqrt{20(\left|\Phi(-1)\right| + \left|\Psi(1)\right|)}$.

Proof. The proof is similar to Theorem [4.2] by taking the weights $w_i = 20$ in weighted Cauchy-Schwarz inequality in the process of optimizing the upper bound of CH_n and is left to the interest of the reader. □

Remark 4.9. The color energy upper and lower bound, optimized upper and lower bounds of closed helm graphs with n varying from 9 to 100 are computed and plotted in Fig. 6.

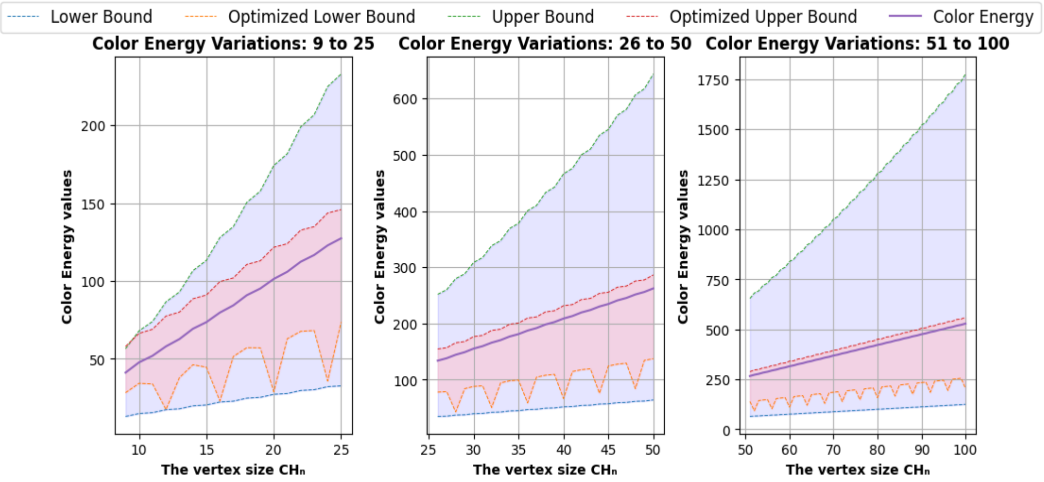

Fig. 6. The color energy bounds of Closed Helm graph

4.7 The color energy lower bounds for the Gear graph

Theorem 4.10. *For a gear graph G_n, $n \geq 9$, with $p = 2n + 1$ vertices and $3n$ edges.*
(i) The lower bound of color energy is given by:

$$E_c(G_n) \geq \sqrt{\left|\xi(-1)\right| + \left|\Lambda(1)\right| + \frac{2}{(p)(p-1)} \prod_{i=1}^{p} (\sigma_i^{w_i})^2}$$

where ξ and Λ denote the total number of -1's and 1's respectively, in the color adjacency matrix, while the weights w_i satisfy the condition $\sum_{i=1}^{p} w_i = 1$, thereby normalizing the eigen weights.

(ii) The optimized lower bound of the color energy is:
$E_c(G_n) \geq \sqrt{\left|\xi(-1)\right| + \left|\Lambda(1)\right| + (p)(p-1)D^{\frac{2}{p}}}$ *where* $D = |A_c(G_n)|$.

Proof. The vertices of the gear graph G_n is constructed with the inner hub node v_0, and the outer disc nodes adjacent with central point as $v_i, i = 1, 2, \ldots, n$ and non-adjacent to it as $v_{n+i}, i = 1, 2, \ldots, n$. The whole vertex set of G_n is combined as $V = \{v_0 \cup v_i \cup v_{n+i}, i = 1, 2, \ldots, n\}$.

The coloring of this graph follows the mapping $\Phi : V \longrightarrow \{1, 2\}$.
When n is odd or even.$\Phi(v_0) = 1$; $\Phi(v_i) = 2,\ 1 \leq i \leq n$ and $\Phi(v_{n+i}) = 1,\ 1 \leq i \leq n$.

The tracing of color adjacency matrix for gear graph is similar to C_{ij} 4.1 of the sunlet graph. The structure of gear evolves from the wheel graph and consists of an additional vertex inserted between each adjacent pair of outer cycle vertices, forming a spiked pattern, Though the color energy property remains unchanged, the color adjacency matrix 4.1 is restructured to accommodate the altered vertex-edge relationships and its polynomial equation is derived as in 4.3.

This structural variations in connectivity influence the color matrix composition, ensuring a precise representation of the graph's topology; refer to 4.2. The sum of squares of latent values for G_n is calculated as

$$\sum_{i=1}^{p} \sigma_i^2 = \left|\xi(-1)\right| + \left|\Lambda(1)\right| \tag{4.6}$$

In this case, ξ and Λ represent the total count of -1's and 1's respectively.

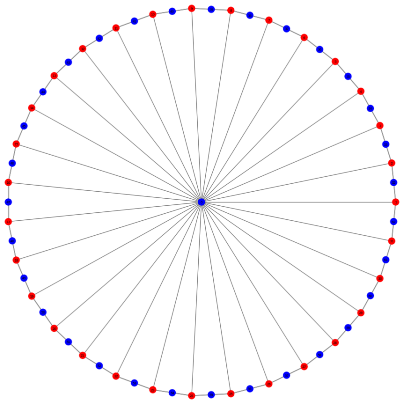

Fig. 7. a. Gear Graph G_{31} (n is odd).

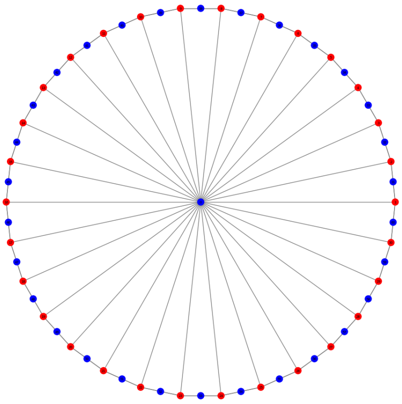

Fig. 7. b. Gear Graph G_{30} (n is even).

The portion of the proof is similar to Theorem [4.1] and is left to the interest of the reader. □

4.8 The color energy upper bounds for Gear graph

Theorem 4.11. *For Gear graph* G_n *with* $n \geq 9$ *and rank* r.
(i) The upper bound of color energy is given by: $E_c(G_n) \leq \sqrt{r(\left|\xi(-1)\right| + \left|\Lambda(1)\right|)}$
(ii) The optimized upper bound of color energy is: $E_c(G_n) \leq \sqrt{17(\left|\xi(-1)\right| + \left|\Lambda(1)\right|)}$.

Proof. The proof is similar to Theorem [4.2] by considering the weights $w_i = 17$ in weighted Cauchy-Schwarz inequality when refining the upper bound of G_n and is left to the interest of the reader. □

Remark 4.12. The color energies lower, upper and optimized lower as well as upper bounds for gear graphs with n varying from 9 to 100 are computed and plotted in Figure 7.

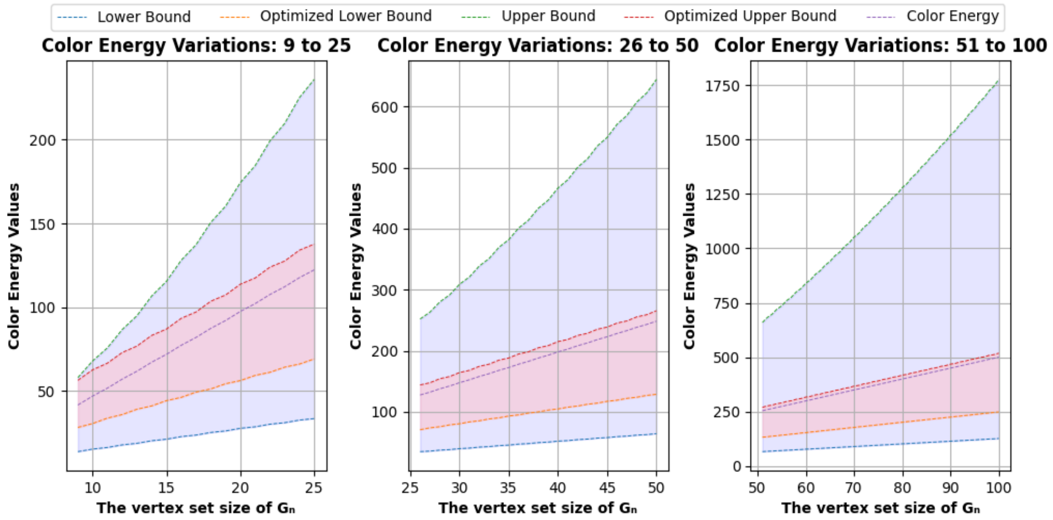

Figure 7 The color energy limits of Gear graph

Algorithm: Generalized Algorithm for Color Energy Bounds

```
Require: Graph type $G \in \{\text{gear}, \text{sunlet}, \text{helm}, \text{closed-helm}\}$, size $p \times p$
Ensure: Color energy bounds of the graph
  procedure GRAPH ANALYSIS(G, p)
      Step 1: Generate Graph Structure
      if $G = \text{gear}$ then
          Initialize graph $G = (V, E)$
          Add central hub node $v_0$
          Create cycle nodes $\{v_1, v_2, \ldots, v_p\}$ and connect them in a cycle
          Insert mid-edge nodes between consecutive cycle nodes
      else if $G = \text{sunlet}$ then
          Initialize graph $G = (V, E)$
          Create cycle nodes $\{v_1, v_2, \ldots, v_p\}$ and connect them in a cycle
          Attach a pendant node to each cycle node
      else if $G = \text{helm}$ then
          Initialize graph $G = (V, E)$
          Add central hub node $v_0$
          Create cycle nodes $\{v_1, v_2, \ldots, v_p\}$ and form a cycle
          Attach a pendant node to each cycle node
      else if $G = \text{closed-helm}$ then
          Initialize graph $G = (V, E)$
          Add central hub node $v_0$
          Create cycle nodes $\{v_1, v_2, \ldots, v_p\}$ and form a cycle
          Attach a pendant node to each cycle node
          Connect pendant nodes to form an outer cycle
      end if
      Step 2: Compute Adjacency Matrix with Color Rules
      Initialize matrix $C$ of size $|V| \times |V|$ with zeros
      for all edges $(u, v) \in E$ do
          Set $C[u, v] = C[v, u] = 1$ if adjacent vertices receive different colors
      end for
      for all non-adjacent vertex pairs $(i, j)$ do
          Set $C[i, j] = C[j, i] = -1$ if they receive the same color
      end for
  end procedure
```

```
procedure GRAPH SPECTRAL ANALYSIS(G, p)
    Step 3: Compute Eigenvalues, Energy, and Bounds
    Compute latent metrics $\sigma_1, \sigma_2, \ldots, \sigma_n$ of $C$
    Compute squared sum $R = \sum_{i=1}^{n} \sigma_i^2$
    Compute determinant $\Delta = \det(C)$
    Compute energy $E(G) = \sum_{i=1}^{n} |\sigma_i|$
    Lower bound $LB = \sqrt{R + \frac{2GM^2}{p(p-1)}}$ where $GM$ is the geometric mean
    Upper bound $UB = \sqrt{\text{rank}(C) \cdot R}$
    Improved upper bound $IUB = \sqrt{\kappa(G) \cdot R}$        ▷ $\kappa(G)$ depends on the graph structure
end procedure
```

5 Conclusion remarks

In this study introduces a comprehensive framework for analyzing the color energy of spoke-structured graphs, including Sunlet, Helm, Closed Helm, and Gear graphs. By deriving the lower, upper, and optimized bounds for color energy, we provide a deeper understanding of the energy dynamics in these graphs. The efficient algorithm developed in this study enables precise computation of these bounds, offering significant contributions to practical applications in network design, resource allocation, and traffic flow management. Our findings pave the way for more efficient optimization techniques in graph-based systems, ultimately enhancing the performance and sustainability of modern networks.

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