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ON THE GENUS OF GAMMA GRAPH OF ZERO-DIVISOR GRAPH FROM COMMUTATIVE RING

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Abstract Let R be a commutative ring with non-zero identity. The gamma graph of γ-sets in the zero-divisor graph, $\Gamma(R)$ is the graph, $\gamma.(\Gamma(R))$ with vertex set $\mathfrak{D}$ as the collection of all γ-sets of the zero-divisor graph, $\Gamma(R)$ and two distinct vertices $\mathfrak{D}_1$ and $\mathfrak{D}_2$ are adjacent if and only if $|\mathfrak{D}_1 \cap \mathfrak{D}_2| = \gamma(\Gamma(R)) - 1$, where $\gamma(\Gamma(R))$ denotes the cardinality of γ-set. In this work, we explore the gamma graph of the zero-divisor graph of $\mathbb{Z}_{p^n} \times \mathbb{Z}_{p^n} \times \mathbb{Z}_{p^n} \times ... \times \mathbb{Z}_{p^n}$, where the number p is prime and n is a non-zero whole number. Additionally, we classify the graphs that are planar and toroidal.

1 Introduction

To explore the relationship between algebraic form of the object supplied and the graph theoretic aspects of the graph associated with it, various graphs were allocated to rings. Beck (1998) defined a graph where the elements of the ring R are represented by vertices, and an edge connects two different vertices if and only if their product is zero, i.e., xy=0. [1] By focusing solely on the non-zero zero-divisors of a ring as the graphs' vertices, Anderson and Livingston (1999) slightly altered this idea. After defining the same adjacency criterion, they called this graph the zero-divisor graph which is represented as $\Gamma(R)$. [2] A dominating set S is a set that is a subset of V of vertices and regarding each vertex v in V, either v exists in S or there exists a vertex u in S such that u and v are adjacent. A dominating set S is considered minimal when there is no proper subset of S that can also be a dominating set. The least number of vertices needed to create a dominating set in a graph G is its domination number $\gamma(G)$. In a graph G, a dominating set of cardinality $\gamma(G)$ is known as γ-set. Let $\mathfrak{D}$ indicate the collection of all γ-sets within the graph G. [2] The gamma graph of a graph G, denoted as $\gamma.G$, is a graph whose vertex set is the set $\mathfrak{D}$ of all dominating sets of G and two vertices $\mathfrak{D}_1$ and $\mathfrak{D}_2$ in $\gamma.G$ are adjacent if and only if the intersection of $\mathfrak{D}_1$ and $\mathfrak{D}_2$ contains exactly $\gamma(G) - 1$vertices. [7] Let S_k represent an oriented surface of genus k, where k indicates a non-negative integer. This surface is commonly referred to as a sphere with k handles. [7, 8]The graph's genus, denoted by $g(G)$, is the smallest integer n that allows the graph to be embedded on a genus n surface, denoted by S_n. In other words, it is the smallest number of "handles" or "holes" needed in a surface such that the graph should be embedded without any edge crossings. Basically, a graph G is said to be embedded in a surface if it can be drawn so that edges intersect only at those vertices which are in common. A toroidal graph is a graph with genus 1, whereas a planar graph is a graph with genus 0. [7] If H is a subgraph of G, then the genus of H, $g(H)$, is at most the genus of G, $g(G)$. Throughout this paper, G denotes the zero-divisor graph of R and $\gamma.G$ denotes the corresponding gamma graph.

2 Prelimnaries

Lemma 2.1. *[7]* $g(K_n) = \lceil \frac{(n-3)(n-4)}{12} \rceil$ *if* $n \geq 3$. *In particular,* $g(K_n) = 1$ *if* $n = 5, 6, 7$.

Lemma 2.2. *[7]* $g(K_{m,n}) = \lceil \frac{(m-2)(n-2)}{4} \rceil$ *if* $m, n \geq 2$. *In particular,* $g(K_{4,4}) = g(K_{3,n}) = 1$ *if*

$n = 3, 4, 5, 6$. *Also* $g(K_{5,4}) = g(K_{6,4}) = g(K_{m,3}) = 2$ *if* $m = 7, 8, 9, 10$.

Lemma 2.3. *[3] If G is a finite connected graph with* n *vertices and* m *edges; then,* $n - m + f = 2 - 2g$, *where the graph is embedded upon a surface* S_k *with genus* k *and* f *is the number of faces created when G is embedded on* S_k.

Lemma 2.4. *[3] If G is a triangle-free graph with* n *vertices and* m *edges, then* $g(G) \geq \lceil \frac{m}{4} - \frac{n}{2} + 1 \rceil$.

Lemma 2.5. *[6] Let G be a connected graph with* $n \geq 3$ *vertices,* q *edges and genus* g. *Then* $g(G) \geq \lceil \frac{q}{6} - \frac{n}{2} + 1 \rceil$.

3 Genus of $\gamma.(\Gamma(\mathbb{Z}_{p^n} \times \mathbb{Z}_{p^n} \times \dots \times \mathbb{Z}_{p^n}$ (k times)))

Theorem 3.1. *Let* $R = \mathbb{Z}_{p^n} \times \mathbb{Z}_{p^n} \times \dots \times \mathbb{Z}_{p^n}$ *(k times), where the number* p *is prime,* $k > 1$ *and* n *are positive integers. Then* $\gamma.G$ *is a regular graph with* $(\phi(p))^k$ *vertices of degree* $(\phi(p) - 1)k$, *if* $p = 2, n \geq 2, k > 1$*;* $p = 3, n = 1, k > 2$*;* $p = 3, n \geq 2, k > 1$*;* $p > 3, n \geq 1, k > 1$

Proof. Let $p = 2, n \geq 2, k > 1$. Then $R = \mathbb{Z}_{2^n} \times \mathbb{Z}_{2^n} \times \dots \times \mathbb{Z}_{2^n}$. The zero-divisors of $\mathbb{Z}_{2^n}$ are $2, 2(2), 3(2), \dots, 2^{n-2}(2), \dots, (2^{n-1} - 1)2$. Consider the vertices, $a_{1i} = (u_1, u_1, \dots, 2, u_1, \dots, u_1)$, $a_{2i} = (u_2, u_2, \dots, 2, u_2, \dots, u_2)$, where u_1 and u_2 are units in $\mathbb{Z}_{2^n}$, and $v_i = (0, 0, \dots, 0, 2^{n-1}, 0, \dots, 0)$ in the zero-divisor graph G of R. Note that, $N(a_{1i}) = \{v_i\}$, $N(a_{2i}) = \{v_i\}, N[v_i] = \{(m_1, m_2, \dots, m_{i-1}, 0, m_{i+1}, \dots, m_k)$, $(n_{11}, n_{21}, \dots, n_{(i-1)1}, 2, n_{(i+1)1}, \dots, n_{k1}), (n_{12}, n_{22}, \dots, n_{(i-1)2}, 2^2, n_{(i+1)2}, \dots, n_{k2}), \dots$, $(n_{12^{n-2}}, n_{2(2^{n-2})}, \dots, n_{(i-1)2^{n-2}}, 2^{n-1}, n_{(i+1)(2^{n-2})}, \dots, n_{k(2^{n-2})})$, $(n_{1(2^{n-1}-1)}, n_{2(2^{n-1}-1)}, \dots, n_{(i-1)(2^{n-1}-1)}, 2(2^{n-1} - 1), n_{(i+1)(2^{n-1}-1)}, \dots, n_{k(2^{n-1}-1)}) |$ $m_i \in \mathbb{Z}_{2^n}$, m_i cannot be zero simultaneously, $n_{ij} \in \mathbb{Z}_{2^n}, i = 1, 2, \dots, k, j = 1, 2, \dots, 2^{n-1} - 1\}$. Therefore $\{v_1, v_2, \dots, v_k\}$ is the γ-set which is unique. Hence $\gamma.G$ is a regular graph of degree $(\phi(2) - 1)k$ with $\phi(2)^k$ vertices.

Let $p = 3, n = 1, k > 2$ Then $R = \mathbb{Z}_3 \times \mathbb{Z}_3 \times \dots \times \mathbb{Z}_3$. Clearly 1 and 2 are the units in $\mathbb{Z}_3$. Consider the vertices $a_{1i} = (1, 1, \dots, 1, 0, 1, \dots, 1), a_{2i} = (2, 2, \dots, 2, 0, 2, \dots, 2)$ $v_i = (0, 0, \dots, 0, 1, 0, \dots, 0), w_i = (0, 0, \dots, 0, 2, 0, \dots, 0)$ in the zero-divisor graph G of R. Also $N(a_{1i}) = \{v_i, w_i\}, N(a_{2i}) = \{v_i, w_i\}, N(v_i) = N(w_i) = \{(n_1, n_2, \dots, n_{i-1}, 0, n_{i+1}, \dots, n_k) |$ $n_j \in \mathbb{Z}_3, j = 1, 2, \dots, k$ and n_j cannot be zero simultaneously$\}$. The γ-sets are
$\gamma_0 = \{v_1, v_2, \dots, v_k\}, \gamma_1 = \{w_1, v_2, \dots, v_k\}, \gamma_2 = \{v_1, w_2, \dots, v_k\}, \dots, \gamma_k = \{v_1, v_2, \dots, w_k\}$,
$\gamma_{12} = \{w_1, w_2, \dots, v_k\}, \gamma_{13} = \{w_1, v_2, w_3, v_4 \dots, v_k\}, \dots, \gamma_{1k} = \{w_1, v_2, \dots, w_k\}, \dots$,
$\gamma_{23} = \{v_1, w_2, w_3, v_4 \dots, v_k\}, \gamma_{24} = \{v_1, w_2, v_3, w_4, v_5 \dots, v_k\}, \dots, \gamma_{2k} = \{v_1, w_2, \dots, v_{k-1}, w_k\}, \dots$,
$\gamma_{(k-2)(k-1)} = \{v_1, v_2, \dots, w_{k-2}, w_{k-1}, v_k\}, \gamma_{(k-2)k} = \{v_1, v_2, \dots, w_{k-2}, v_{k-1}, w_k\}$,
$\gamma_{(k-1)k} = \{v_1, v_2, \dots, w_{k-1}, w_k\}, \dots, \gamma_{123\dots k} = \{w_1, w_2, \dots, w_k\}$
Total no. of γ-sets $= kC_0 + kC_1 + \dots + kC_k = 2^k$. Let us analyse the degree of each γ-sets in $\gamma.G$,$N(\gamma_0) = \{\gamma_1, \gamma_2, \dots, \gamma_k\}, N(\gamma_i) = \{\gamma_0, \gamma_{1i}, \gamma_{2i}, \dots, \gamma_{(i-1)i}, \gamma_{i(i+1)}, \dots, \gamma_{ik} |$ $\forall \gamma_{i_1 i_2}, i_1 < i_2\}, N(\gamma_{i_1 i_2}) = \{\gamma_{i_1}, \gamma_{i_2}, \gamma_{1 i_1 i_2}, \gamma_{2 i_1 i_2}, \dots, \gamma_{(i_1-1) i_1 i_2}, \gamma_{i_1 (i_1+1) i_2}, \dots, \gamma_{i_1 (i_2-1) i_2}$, $\gamma_{i_1 i_2 (i_2+1)}, \dots, \gamma_{i_1 i_2 k} | \forall \gamma_{i_1 i_2 i_3}, i_1 < i_2 < i_3$ and $\forall \gamma_{i_1 i_2}, i_1 < i_2\}$,$N(\gamma_{i_1 i_2 i_3}) = \{\gamma_{i_1 i_2}, \gamma_{i_1 i_3}, \gamma_{i_2 i_3}$, $\gamma_{1 i_1 i_2 i_3}, \dots, \gamma_{(i_1-1) i_1 i_2 i_3}, \gamma_{i_1 (i_1+1) i_2 i_3}, \dots, \gamma_{i_1 (i_2-1) i_2 i_3}, \gamma_{i_1 i_2 (i_2+1) i_3}, \dots, \gamma_{i_1 i_2 (i_3-1) i_3}, \gamma_{i_1 i_2 i_3 (i_3+1)}, \dots$, $\gamma_{i_1 i_2 i_3 k} | \forall \gamma_{i_1 i_2 i_3 i_4}, i_1 < i_2 < i_3 < i_4, \forall \gamma_{i_1 i_2 i_3}, i_1 < i_2 < i_3$ and $\forall \gamma_{i_1 i_2}, i_1 < i_2\}, \dots$,
$N(\gamma_{i_1 i_2 \dots i_{k-1}}) = \{\gamma_{i_1 i_2 \dots i_{k-2}}, \gamma_{i_1 i_2 \dots i_{k-3} i_{k-1}}, \dots, \gamma_{i_2 i_3 \dots i_{k-1}}, \gamma_{i_1 i_2 \dots i_{k-1} i_k} | \forall \gamma_{i_1 i_2 \dots i_{k-2}}$, $i_1 < i_2 < \dots < i_{k-2}$ and $\forall \gamma_{i_1 i_2 \dots i_k}, i_1 < i_2 < \dots < i_k\}$,
$N(\gamma_{i_1 i_2 \dots i_k}) = \{\gamma_{i_1 i_2 \dots i_{k-1}}, \gamma_{i_1 i_2 \dots i_{k-2} i_k}, \dots, \gamma_{i_2 i_3 \dots i_k} | \forall \gamma_{i_1 i_2 \dots i_{k-1}}, i_1 < i_2 < \dots < i_{k-1}\}$ Hence $\gamma.G$ is a regular graph of degree $(\phi(3) - 1)k$ with $\phi(3)^k$ vertices.

Let $p = 3, n \geq 2, k > 1$. Then $R = \mathbb{Z}_{3^n} \times \mathbb{Z}_{3^n} \times \dots \times \mathbb{Z}_{3^n}$. The Zero-divisors of $\mathbb{Z}_{3^n}$ are $3, 2(3), 3(3), \dots, 3^{n-2}(3), (3^{n-2} + 1)3, \dots, (2(3^{n-2}))3, \dots, (3^{n-1} - 2)3, (3^{n-1} - 1)3$. Consider the vertices $a_{1i} = (u_1, u_1, \dots, u_1, 3, u_1, \dots, u_1), a_{2i} = (u_2, u_2, \dots, u_2, 3, u_2, \dots, u_2)$, $v_i = (0, 0, \dots, 0, 3^{n-1}, 0, \dots, 0), w_i = (0, 0, \dots, 0, 2(3^{n-1}), 0, \dots, 0)$ in the zero-divisor graph G of R, where u_1 and u_2 are units in $\mathbb{Z}_{3^n}$. Also $N(a_{1i}) = \{v_i, w_i\}, N(a_{2i}) = \{v_i, w_i\}$, $N[v_i] = N[w_i] = \{(m_1, m_2, \dots, m_{i-1}, 0, m_{i+1}, \dots, m_k), (n_{11}, n_{21}, \dots, n_{(i-1)1}, 3, n_{(i+1)1}, \dots, n_{k1})$, $(n_{12}, n_{22}, \dots, n_{(i-1)2}, 2(3), n_{(i+1)2}, \dots, n_{k2}), \dots$, $(n_{1(3^{n-1}-1)}, n_{2(3^{n-1}-1)}, \dots, n_{(i-1)(3^{n-1}-1)}, (3^{n-1} - 1)3, n_{(i+1)(3^{n-1}-1)}, \dots, n_{k(3^{n-1}-1)}) | m_i \in \mathbb{Z}_{3^n}$

and m_i cannot be zero simultaneously, $n_{ij} \in \mathbb{Z}_{3^n}, i = 1, 2, ..., k, j = 1, 2, ..., 3^{n-1} - 1\}$. As discussed above, $\gamma.G$ is a regular graph of degree $(\phi(3) - 1)k$ with $(\phi(3))^k$ vertices.

Let $p > 3, k > 1, n$ are natural numbers. Then $R = \mathbb{Z}_{p^n} \times \mathbb{Z}_{p^n} \times ... \times \mathbb{Z}_{p^n}$. The zero-divisors of $\mathbb{Z}_{p^n}$ are $p, 2(p), 3(p), ..., p^{n-2}(p), (p^{n-2} + 1)p, ..., (2(p^{n-2}))p, ..., ((p - 1)(p^{n-2}))p, ..., (p^{n-1} - 2)p, (p^{n-1} - 1)p$. Consider the vertices $a_{1i} = (1, 1, ..., 1, 0, 1, ..., 1)$,
$a_{2i} = (2, 2, ..., 2, 0, 2, ..., 2), a_{3i} = (3, 3, ..., 3, 0, 3, ..., 3), ...,$
$a_{(p-1)i} = (p - 1, p - 1, ..., p - 1, 0, p - 1, ..., p - 1)$, if $n = 1$ and $a_{1i} = (1, 1, ..., 1, p, 1, ..., 1)$,
$a_{2i} = (2, 2, ..., 2, p, 2, ..., 2), a_{3i} = (3, 3, ..., 3, p, 3, ..., 3), ...,$
$a_{(p-1)i} = (p - 1, p - 1, ..., p - 1, p, p - 1, ..., p - 1)$, if $n \geq 2$, $u_{i1} = (0, 0, ..., 1(p^{n-1}), 0, ..., 0)$,
$u_{i2} = (0, 0, ..., 2(p^{n-1}), 0, ..., 0), ..., u_{i(p-1)} = (0, 0, ..., (p-1)(p^{n-1}), 0, ..., 0)$ in the zero-divisor graph G of R. Also $N(a_{ji}) = \{u_{i1}, u_{i2}, ..., u_{i(p-1)}\}, j = 1, 2, ..., p - 1$, $N(u_{i1}) = N(u_{i2}) = ...$
$= N(u_{i(p-1)}) = \{(m_1, m_2, ..., m_{i-1}, 0, m_{i+1}, ..., m_k), (n_{11}, n_{21}, ..., n_{(i-1)1}, p, n_{(i+1)1}, ..., n_{k1}),$
$(n_{12}, n_{22}, ..., n_{(i-1)2}, 2(p), n_{(i+1)2}, ..., n_{k2}), ...,$
$(n_{1(p^{n-1}-1)}, n_{2(p^{n-1}-1)}, ..., n_{(i-1)(p^{n-1}-1)}, (p^{n-1} - 1)p, n_{(i+1)(p^{n-1}-1)}, ..., n_{k(p^{n-1}-1)})|$
$m_i \in \mathbb{Z}_{p^n}, i = 1, 2, ..., k$ and m_i cannot be zero simultaneously, $n_{ij} \in \mathbb{Z}_{p^n}, i = 1, 2, ..., k,$
$j = 1, 2, ..., p^{n-1} - 1\}$. The γ-sets are $\{u_{1j_1}, u_{2j_2}, ..., u_{kj_k}\}$, where $j_i = 1, 2, ..., p - 1$ and $i = 1, 2, ..., k$. The total no. of γ-sets$= (p - 1)^k = \phi(p)^k$ and $N(\{u_{1j_1}, u_{2j_2}, ..., u_{kj_k}\}) =$
$\{\{u_{1i_1}, u_{2j_2}, ..., u_{kj_k}\}, \{u_{1j_1}, u_{2i_2}, ..., u_{kj_k}\}, ..., \{u_{1j_1}, u_{2j_2}, ..., u_{(k-1)j_{(k-1)}}, u_{ki_k}\}|i_n \neq j_n,$
$\forall n = 1, 2, ..., k, i_n = 1, 2, ..., p-1, j_n = 1, 2, ..., p-1$ and $|N(\{u_{1j_1}, u_{2j_2}, ..., u_{kj_k}\})| = (p-2)k = (\phi(p) - 1)k$. Hence $\gamma.G$ is a regular graph of degree $(\phi(p) - 1)k$ with $\phi(p)^k$ vertices.

□

Corollary 3.2. *Let* $R = \mathbb{Z}_{3^n} \times \mathbb{Z}_{3^n} \times ... \times \mathbb{Z}_{3^n}$*(k times),* $k > 1, n \in \mathbb{N}$*, the graph* $\gamma.G$ *is triangle free, if* $k = 2, n > 1$*;* $k > 2, n \in \mathbb{N}$

Proof. Assume $k = 2, n > 1$. Then $R = \mathbb{Z}_{3^n} \times \mathbb{Z}_{3^n}$. By theorem 3.1, $\gamma.G$ is a regular graph of degree 2 with 4 vertices, which is nothing but C_4. Hence $\gamma.G$ is triangle free.

Assume $k > 2, n \in \mathbb{N}$. From theorem 3.1, Let us divide the vertices of $\gamma.G$ as
$X = \{\gamma_0, \gamma_{i_1i_2}, \gamma_{i_1i_2i_3i_4}, \gamma_{i_1i_2i_3i_4i_5i_6}, ..., \gamma_{i_1i_2i_3...i_k}\}$,
$Y = \{\gamma_{i_1}, \gamma_{i_1i_2i_3}, \gamma_{i_1i_2i_3i_4i_5}, ..., \gamma_{i_1i_2i_3...i_{k-1}}\}$, k is even and
$X = \{\gamma_0, \gamma_{i_1i_2}, \gamma_{i_1i_2i_3i_4}, \gamma_{i_1i_2i_3i_4i_5i_6}, ..., \gamma_{i_1i_2i_3...i_k-1}\}, Y = \{\gamma_{i_1}, \gamma_{i_1i_2i_3}, \gamma_{i_1i_2i_3i_4i_5}, ..., \gamma_{i_1i_2i_3...i_k}\}$,
k is odd. Since the indices of the vertices belongs to X differ by atleast two variables with each other, no two vertices in X are adjacent and hence for Y. Also, by theorem 3.1, $\gamma.G$ is a regular graph of degree $(\phi(3) - 1)k$ with $\phi(3)^k$ vertices, $\gamma.G$ is connected. Hence every edge of $\gamma.G$ has two ends: one in X and one in Y. Therefore, $\gamma.G$ is bipartite. Thus, $\gamma.G$ contains no odd cycle and hence it is triangle free. □

Corollary 3.3. *Let* $R = \mathbb{Z}_{p^n} \times \mathbb{Z}_{p^n} \times ... \times \mathbb{Z}_{p^n}$*(k times),* $n \geq 1, k \geq 2, p > 3$ *and p is prime, G and G′ be the zero-divisor graphs of R and* $\mathbb{Z}_p \times \mathbb{Z}_p$ *respectively. Then* $\gamma.G'$ *is always a subgraph of* $\gamma.G$

Proof. From theorem 3.1, the $\gamma-$sets of $\gamma.G$ are $\{u_{1j_1}, u_{2j_2}, ..., u_{kj_k}\}$, where $j_i = 1, 2, ..., p - 1$ and $i = 1, 2, ..., k$. Consider the ring $\mathbb{Z}_p \times \mathbb{Z}_p$, by theorem 3.1, $\gamma.G'$ is a regular graph with $(p - 1)^2$ vertices of degree $2p - 4$. In order to prove $\gamma.G'$ is a subgraph of $\gamma.G$, it is enough to show that $\gamma.G$ contains a regular graph with $(p-1)^2$ vertices of degree $2p-4$. From theorem 3.1, consider the $(p - 1)^2$ vertices from $\gamma.G$, $v_{ij} = \{u_{1i}, u_{2j}, u_{31}, ..., u_{k1}\}$, where $i = 1, 2, ..., p - 1$ and $j = 1, 2, ..., p - 1$

Now, $N[v_{1j}] = \{v_{11}, v_{12}, ..., v_{1(p-1)}, v_{2j}, v_{3j}, ..., v_{(p-1)j}\}$,
$N[v_{2j}] = \{v_{21}, v_{22}, ..., v_{2(p-1)}, v_{1j}, v_{3j}, ..., v_{(p-1)j}\}, ...,$
$N[v_{(p-1)j}] = \{v_{(p-1)1}, v_{(p-1)2}, ..., v_{(p-1)(p-1)}, v_{1j}, v_{2j}, ..., v_{(p-2)j}\}, j = 1, 2, ..., p - 1$ and
$|N(v_{ij})| = 2(p - 2) = 2p - 4$. Hence $\gamma.G'$ is a subgraph of $\gamma.G$.

□

Theorem 3.4. *Let* $R = \mathbb{Z}_{p^n} \times \mathbb{Z}_{p^n} \times ... \times \mathbb{Z}_{p^n}$*(k times), where the number p is prime,* $k > 1, n$ *are positive integers. Then* $g(\gamma.G) = 0$ *if and only if R is isomorphic to one of the rings listed* *below,* $\mathbb{Z}_{2^n} \times \mathbb{Z}_{2^n} \times ... \times \mathbb{Z}_{2^n}$*(k times),*$\mathbb{Z}_{3^n} \times \mathbb{Z}_{3^n}, \mathbb{Z}_{3^n} \times \mathbb{Z}_{3^n} \times \mathbb{Z}_{3^n}$

Proof. Assume $g(\gamma.G) = 0$

Claim: $p < 5$

Suppose $p \geq 5$, by theorem 3.1, $|V(\gamma.G)| = \phi(p)^k, |E(\gamma.G)| = \frac{(\phi(p)-1)k\phi(p)^k}{2}$. By Lemma 2.5, $g \geq \lceil \frac{(\phi(p)-1)k\phi(p)^k}{12} - \frac{(\phi(p))^k}{2} + 1 \rceil \geq 1$, which refutes our supposition. Hence $p < 5$. Thus p=2,3

For $p = 3$, Claim: $k \leq 3$

Suppose $k > 3$, we have $R = \mathbb{Z}_{3^n} \times \mathbb{Z}_{3^n} \times ... \times \mathbb{Z}_{3^n}(k \text{ times})$, by corollary 3.2 and Lemma 2.4, $g \geq \lceil \frac{k2^k}{8} - \frac{2^k}{2} + 1 \rceil \geq 1$, which disagrees with our assumption. Hence $k \leq 3$. Therefore R is isomorphic to $\mathbb{Z}_{2^n} \times \mathbb{Z}_{2^n} \times ... \times \mathbb{Z}_{2^n}(k \text{ times})$,$\mathbb{Z}_{3^n} \times \mathbb{Z}_{3^n}$, $\mathbb{Z}_{3^n} \times \mathbb{Z}_{3^n} \times \mathbb{Z}_{3^n}$

Converse of the result is from the following graphs:

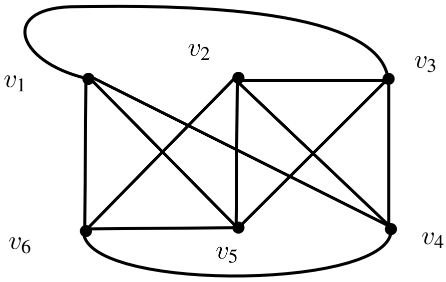

Figure 1. $\gamma.G$ of $\mathbb{Z}_3 \times \mathbb{Z}_3$

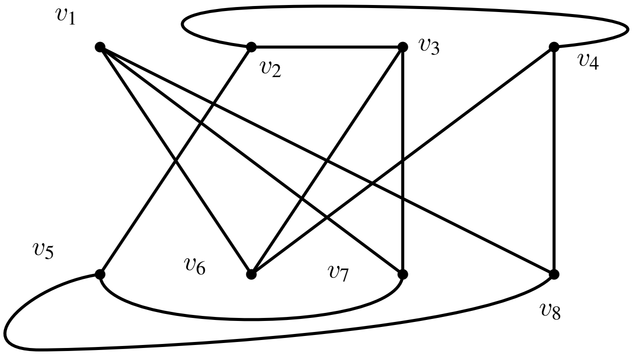

Figure 2. $\gamma.G$ of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

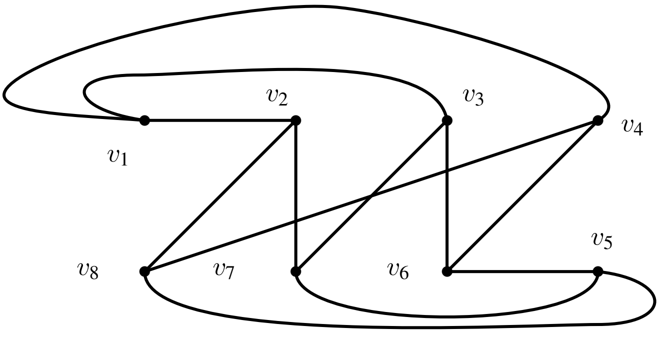

Figure 3. $\gamma.G$ of $\mathbb{Z}_{3^n} \times \mathbb{Z}_{3^n} \times \mathbb{Z}_{3^n}$

□

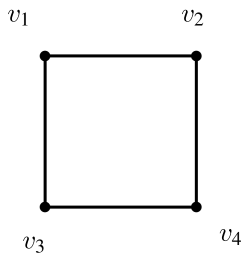

Figure 4. $\gamma.G$ of $\mathbb{Z}_{3^n} \times \mathbb{Z}_{3^n} (n > 1)$

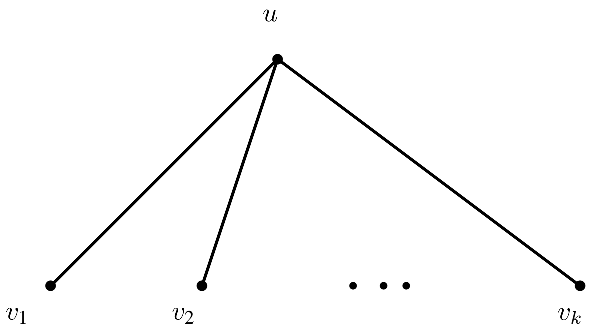

Figure 5. $\gamma.G$ of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times ... \times \mathbb{Z}_2 (k$ times, $k \geq 4)$

Figure 6. $\gamma.G$ of $\mathbb{Z}_{2^n} \times \mathbb{Z}_{2^n} \times ... \times \mathbb{Z}_{2^n} (k$ times, $k > 1, n \geq 2)$

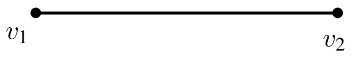

Figure 7. $\gamma.G$ of $\mathbb{Z}_2 \times \mathbb{Z}_2$

Theorem 3.5. *Let $R = \mathbb{Z}_{p^n} \times \mathbb{Z}_{p^n} \times ... \times \mathbb{Z}_{p^n}$(k times), where p is a prime number, $k > 1, n$ are positive integers. Then $g(\gamma.G) = 1$ if and only if R is isomorphic to one of the following rings, $\mathbb{Z}_{3^n} \times \mathbb{Z}_{3^n} \times \mathbb{Z}_{3^n} \times \mathbb{Z}_{3^n}$*

Proof. Assume $g(\gamma.G) = 1$

Claim: $p < 7$

Suppose $p \geq 7$, by theorem 3.1 and Lemma 2.5, $g \geq \left\lceil \frac{(\phi(p)-1)k\phi(p)^k}{12} - \frac{(\phi(p))^k}{2} + 1 \right\rceil \geq 13$, arrived a refutation. Hence $p < 7$. Thus $p = 2, 3, 5$

If $p = 2, R = \mathbb{Z}_{2^n} \times \mathbb{Z}_{2^n} \times ... \times \mathbb{Z}_{2^n}$ and by theorem 3.4, $g(\gamma.G) = 0$, arrived a refutation. Thus p cannot be 2.

For p=3, Claim: $k \leq 4$, Suppose $k > 4$, then by corollary 3.2 and lemma 2.4,
$g \geq \left\lceil \frac{k2^k}{8} - \frac{2^k}{2} + 1 \right\rceil \geq 5$, refutes our supposition. Hence $k \leq 4$ and so $k = 2, 3, 4$

If $p = 3, k = 3, R = \mathbb{Z}_{3^n} \times \mathbb{Z}_{3^n} \times \mathbb{Z}_{3^n}$ and by theorem 3.4, $g(\gamma.G) = 0$, which violates our supposition. Thus k cannot be 3. Therefore, $k = 2, 4$

For $p = 3, k = 2, R = \mathbb{Z}_{3^n} \times \mathbb{Z}_{3^n}$ and by theorem 3.4, $g(\gamma.G) = 0$, which violates our supposition. Thus k cannot be 2.

Claim: $p \neq 5$, Suppose $p = 5, R = \mathbb{Z}_{5^n} \times \mathbb{Z}_{5^n} \times ... \times \mathbb{Z}_{5^n}$($k$ times). Let G and G' be the zero-divisor graphs of R and $\mathbb{Z}_5 \times \mathbb{Z}_5$ respectively. By Corollary 3.3, $\gamma.G'$ is a subgraph of $\gamma.G$ and by theorem 3.1, $\gamma.G'$ is a regular graph with 16 vertices of degree 6. Hence $|V(\gamma.G')| = 16$ and $|E(\gamma.G')| = 48$. Suppose $g(\gamma.G') = 1$, By Lemma 2.3, $f = 32$. Hence while embedding $\gamma.G'$ on the torus there must be 32 faces and all such faces are triangular. Thus, for any graph H with $n = 16, m = 48, f = 32$, we have the following such embedding on Torus.

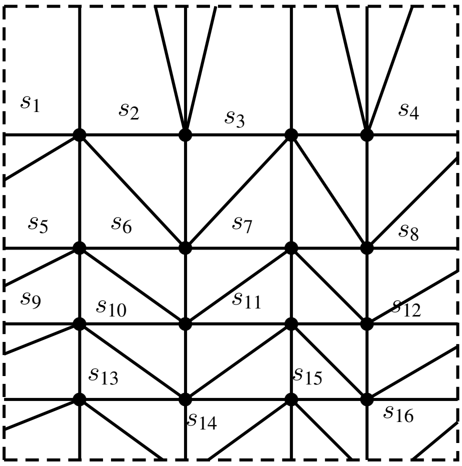

Figure 8. Planar embedding of H in torus

In $\gamma.G'$, we will have four vertices say r_1, r_2, r_3, r_4 such that $N(r_1) \supseteq \{r_2, r_3, r_4\}$, $N(r_2) \supseteq \{r_1, r_3, r_4\}, N(r_3) \supseteq \{r_1, r_2, r_4\}, N(r_4) \supseteq \{r_1, r_2, r_3\}$. The following structure will be there in $\gamma.G'$

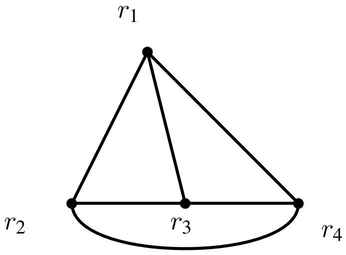

Figure 9. Structure contained in$\gamma.G'$

Similarly, there exists three more vertex sets $\{r_5, r_6, r_7, r_8\}, \{r_9, r_{10}, r_{11}, r_{12}\}$, $\{r_{13}, r_{14}, r_{15}, r_{16}\}$ consequently three more such structures, which is not in figure 8. Hence $\gamma.G'$ cannot be embedded on Torus, which implies $g(\gamma.G') > 1$. Hence $g(\gamma.G) \geq g(\gamma.G') > 1$, which is a contradiction. Thus $p \neq 5$. Consequently, R is isomorphic to one of the rings listed below, $\mathbb{Z}_{3^n} \times \mathbb{Z}_{3^n} \times \mathbb{Z}_{3^n} \times \mathbb{Z}_{3^n}$.

Converse of the result is from the following figures:

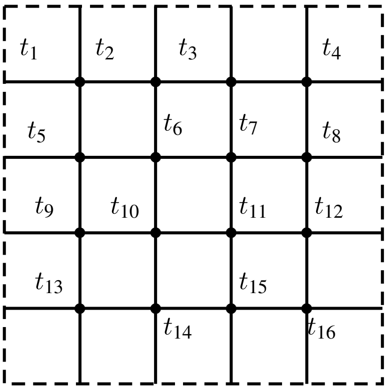

Figure 10. Planar embedding of $\gamma.G$ of $\mathbb{Z}_{3^n} \times \mathbb{Z}_{3^n} \times \mathbb{Z}_{3^n} \times \mathbb{Z}_{3^n}$ on torus

□

4 Conclusion remarks

Through this paper, we have analysed gamma graph of the zero-divisor graph of the ring $\mathbb{Z}_{p^n} \times \mathbb{Z}_{p^n} \times ... \times \mathbb{Z}_{p^n}(k$ times). Also, we have dealt with its embedding nature and found for those rings the gamma graph of zero-divisor graph is planar and toroidal.

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