Palestine Journal of Mathematics

Vol 15(Special Issue II)(2026) , 266–278

Generalized Steiner Degree Distance index of Graphs

B. Sarveshkumar and B. Chaluvaraju

MSC 2020 Classifications: Primary 05C07; 05C09; 05C12; Secondary 05C05; 05C38.

Keywords and phrases: Wiener index, Steiner index, Steiner distance-distance index.

The authors would like to thank the reviewers and editor for their constructive comments and valuable suggestions that improved the quality of our paper.

Bheemanna Sarveshkumar was very thankful to the CSIR-UGC (the financial support provided by the University Grants Commission, New Delhi, in the form of the Senior Research Fellowship (UGC-Ref-No:959/CSIR-UGC NET JUNE 2018)).

Corresponding Author: B. Chaluvaraju

Abstract. The Steiner distance $d_G(S)$ of a graph $G = (V, E)$ is the minimum number of edges needed to connect all the vertices in $S \subseteq V(G)$ where $|S| = k$. In this paper, we introduced a generalized version of the Steiner degree distance index of a graph G, obtained the exact value for some specific families of graphs and established some sharp bounds based on the diameter of G. Additionally, comparisons are made among the Steiner-related indices.

1 Introduction

In this paper, we consider graphs that are undirected, simple, finite, and connected. The graph $G = (V(G), E(G))$ has p-vertices and q-edges. Here, $V = V(G)$ and $E = E(G)$ represent the collections of vertices and edges, respectively. The degree of vertex v_i, denoted $d_G(v_i)$, is the number of vertices adjacent to it. We use $\delta(G)$ and $\Delta(G)$ to denote the minimum and maximum vertex degree. A vertex with a degree of 1 is called a pendant vertex. The distance between any two vertices is the number of edges in the shortest path connecting them. This is denoted as $d_G(v_i, v_j)$. The diameter of a graph $diam(G) = max\{d(u, v) : u, v \in V(G)\}$, is the largest distance between any two vertices of a graph G. Please refer to [4, 18] if any notations used in this paper are undefined.

The Wiener index is a graph theoretical invariant introduced by Harry Wiener [32]. It is a well-established measure of the distance between all pairs of vertices in a connected graph G.

$$W(G) = \sum_{v_i, v_j \subseteq V(G)} d_G(v_i, v_j).$$

The degree distance index ($DD(G)$) introduced by Dobrynin [14] is a graph invariant that has gained significant attention in recent years and is defined as :

$$DD(G) = \sum_{\{v_i, v_j\} \in V(G)} (d_G(v_i) + d_G(v_j))d_G(v_i, v_j).$$

The Steiner distance of a graph introduced by Chartrand [7] is an important concept in graph theory. It is denoted by $d_G(S)$, where $S \subseteq V(G)$ and $2 \leq |S| \leq p$. The Steiner degree distance index of a graph introduced by Gutman [15] and is defined as

$$SDD^k(G) = \sum_{S \subseteq V(G), |S|=k} \left(\sum_{v_i \in S} d_G(v_i) \right) d_G(S).$$

Here, we define the Generalized Steiner degree distance index of G as

$$SDD^k_{(a,b)}(G) = \sum_{S \subseteq V(G), |S|=k} \left(\sum_{v_i \in S} d_G(v_i) \right)^a d_G(S)^b,$$

where a and b are any real numbers.

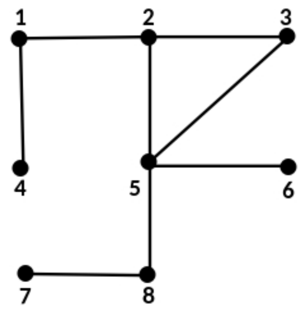

Figure 1. Graph G

For example, let's consider the graph G constructed with 8-vertices. If $S = \{1, 2, 3\}$, then $\left(\sum_{v_i \in S} d_G(v_i)\right)^a d_G(S)^b = 7^a 2^b$. Also, for $|S| = 3$,

$$\begin{aligned} SDD^k_{(a,b)}(G) &= (6^a + 3 \times 7^a + 3 \times 8^a + 3 \times 9^a)2^b \\ &+ (4^a + 3 \times 5^a + 6^{(a+1)} + 5 \times 7^a + 4 \times 8^a)3^b \\ &+ (3 \times 4^a + 5^{(a+1)} + 5 \times 6^a + 3 \times 7^a)4^b \\ &+ (5 \times 4^a + 3 \times 5^a + 6^a)5^b + (3^a + 4^a)6^b. \end{aligned}$$

If $|S| = 4$, then

$$\begin{aligned} SDD^k_{(a,b)}(G) &= (2 \times 8^a + 2 \times 9^a + 4 \times 10^a + 3 \times 11^a + 12^a)3^b \\ &+ (6^a + 2 \times 7^a + 8 \times 8^a + 8 \times 9^a + 4 \times 10^a + 11^a)4^b \\ &+ (5 \times 6^a + 10 \times 7^a + 5 \times 8^a + 3 \times 9^a + 10^a)5^b \\ &+ (2 \times 5^a + 4 \times 6^a + 2 \times 7^a + 9^a)6^b + 5^a 7^b. \end{aligned}$$

If $|S| = 5$, then

$$\begin{aligned} SDD^k_{(a,b)}(G) &= (2 \times 11^a + 3 \times 12^a + 13^a + 14^a)4^b \\ &+ (8^a + 3 \times 9^a + 5^10^a + 7 \times 11^a + 12^a + 13^a)5^a \\ &+ (7^a + 2 \times 8^a + 6 \times 9^a + 6 \times 10^a + 2 \times 11^a)6^b \\ &+ (7^a + 2 \times 8^a + 9^a + 10^a)7^b. \end{aligned}$$

If $|S| = 6$, then

$$\begin{aligned} SDD^k_{(a,b)}(G) &= (12^a + 4 \times 13^a + 2 \times 14^a + 15^a)5^b \\ &+ (10^a + 4 \times 11^a + 5 \times 12^a + 3 \times 13^a + 14^a)6^b \\ &+ (9^a + 2 \times 10^a + 2 \times 11^a + 12^a)7^b. \end{aligned}$$

If $|S| = 7$, then

$$SDD^k_{(a,b)}(G) = (2 \times 14^a + 2 \times 15^a)6^b + (12^a + 13^a + 2 \times 14^a)7^b.$$

Overall, these graph invariants have proven to be highly effective in various applications of graph theory. For more details on graphical indices, we refer to [5, 6, 8–13, 17, 24, 25, 27, 29–31].

2 The particular values of a, b and k in $SDD^k_{(a,b)}(G)$

Many such graphical indices are special cases of the Generalized Steiner degree distance index with specific a, b and k values.

(i) $SDD^2_{(0,1)}(G)$=$W(G)$, the Wiener index, [32].

(ii) $SDD^2_{(0,1)}(G) + SDD^2_{(0,2)}(G)$=$HW(G)$, the hyper-Wiener index, [22].

(iii) $SDD^2_{(0,\lambda)}(G)$=$W_\lambda(G)$, the general Wiener index, [21].

(iv) $SDD^2_{(0,-2)}(G)$=$H^2(G)$, the squared Harary index, [28].

(v) $SDD^2_{(0,-1)}(G)$=$H(G)$, the Harary index, [20].

(vi) $SDD^2_{(1,-1)}(G)$=$H_A(G)$, the additively weighted Harary index, [1].

(vii) $SDD^2_{(1,1)}(G)$=$DD(G)$, the degree distance index, [14].

(viii) $SDD^k_{(0,1)}(G)$=$SW^k(G)$, the Steiner Wiener index, [33].

(ix) $SDD^k_{(1,1)}(G)$=$SDD^k(G)$, the Steiner degree distance index, [15].

(x) $SDD^k_{(0,1)}(G) + SDD^k_{(0,2)}(G)$=$SWW^k(G)$, the Steiner hyper-Wiener index, [2].

(xi) $SDD^k_{(0,-1)}(G)$=$SH^k(G)$, the Steiner Harary index, [23].

(xii) $SDD^k_{(1,-1)}(G)$=$SRDD^k(G)$, the Steiner reciprocal degree, [34].

3 Specific families of graphs

Proposition 3.1. Let K_p be a complete graph with $p \geq 2$. Then

$$SDD^k_{(a,b)}(K_p) = \binom{p}{k}(k(p-1))^a(k-1)^b.$$

Proof. Let K_p be a complete graph with $p \geq 2$. The degree of an arbitrary vertex v_i is $p-1$, and every pair of vertices is adjacent. Hence, each set S corresponding a spanning tree of size $(k-1)$, which implies

$$SDD^k_{(a,b)}(K_p) = \sum_{S\subseteq V(K_p),|S|=k} \left[\sum_{v_i\in S}(p-1)\right]^a k-1^b.$$

$$SDD^k_{(a,b)}(K_p) = \binom{p}{k}\,(k(p-1))^a\,(k-1)^b.$$ □

Proposition 3.2. Let P_p be a path with $p \geq 4$. Then

$$\begin{aligned} SDD^k_{(a,b)}(P_p) = (2k)^a &\sum_{s=0}^{p-k-2}\binom{k+s-2}{k-2}(p-k-s-1)(k+s-1)^b \\ &+ 2\sum_{s=0}^{p-k-2}\binom{k+s-2}{k-2}(2k-1)^a(k+s-1)^b \\ &+ (2k-2)^a\binom{p-2}{k-2}(p-1)^b. \end{aligned}$$

Proof. Let P_p be a path with $p \geq 3$. We have
Case 1. If the vertices of a set S are of degree 2, then

$$\begin{aligned}\theta_1 &= \sum_{\substack{S\subseteq V(P_p),\\ d_{P_p}(v_i)=2}} \left[\sum_{v_i\in S} d_{P_p}(v_i)\right]^a d_{P_p}(S)^b.\\ &= (2k)^a \sum_{s=0}^{p-k-2} \binom{k+s-2}{k-2} (p-k-s-1)(k+s-1)^b.\end{aligned}$$

Case 2. If the set S contains at least one pendent vertex, then

$$\begin{aligned}\theta_2 &= \sum_{\substack{S\subseteq V(P_p),\\ \text{exists } d_{P_p}(v_i)\neq 2}} \left[\sum_{v_i\in S} d_{P_p}(v_i)\right]^a d_{P_p}(S)^b.\\ &= 2\sum_{s=0}^{p-k-2} \binom{k+s-2}{k-2} (2k-1)^a (k+s-1)^b\\ &+ (2k-2)^a \binom{p-2}{k-2} (p-1)^b.\end{aligned}$$

In view of the above cases, we have $SDD^k_{(a,b)}(P_p) = \theta_1 + \theta_2$.
On simplification, we have the desired result. □

Proposition 3.3. Let W_p be a wheel with $p \geq 5$. Then

$$\begin{aligned}SDD^k_{(a,b)}(W_p) &= \left(\binom{p-1}{k-1}(p+3k-4)^a + (p-k-1)(3k)^a\right)(k-1)^b\\ &+ \left(\binom{p-1}{k} - (p-k-1)\right)(3k)^a k^b.\end{aligned}$$

Proof. Let W_p be a wheel with $p \geq 5$. Then the set S has two possibilities: the central vertex is an element in the set S, or the central vertex is not in the set S, this implies that

$$\begin{aligned}SDD^k_{(a,b)}(W_p) &= \sum_{\substack{S\subseteq V(W_p),\\ \text{exists } d_{W_p}(v_i)\neq 3}} \left[\sum_{v_i\in S} d_{W_p}(v_i)\right]^a d_{W_p}S)^b\\ &+ \sum_{\substack{S\subseteq V(W_p),\\ d_{W_p}(v_i)=3}} \left[\sum_{v_i\in S} d_{W_p}(v_i)\right]^a d_{W_p}(S)^b.\end{aligned}$$

By the above facts, the desired result follows. □

Proposition 3.4. Let $K_{m,n}$ be a complete bipartite graph with $k \leq m \leq n$. Then

$$\begin{aligned}SDD^k_{(a,b)}(K_{m,n}) &= \left(\binom{m}{k}(mk)^a + \binom{n}{k}(nk)^a\right)k^b\\ &+ \sum_{s=1}^{m-1}\binom{m}{s}\binom{n}{k-s}(sn + (k-s)m)^a (k-1)^b.\end{aligned}$$

Proof. Let $K_{m,n}$ be a complete bipartite graph with $V(G) = V_1 \cup V_2$ and $k \leq m \leq n$. If S is selected from either V_1 or from V_2, then the size of a minimum spanning tree that spans S is k. However, if S is not a subset of either partition, then the size of a minimum spanning tree that spans S is $k-1$, with two real numbers a and b. We have

$$SDD^k_{(a,b)}(K_{m,n}) = \sum_{\substack{S\subseteq V_1 \text{ or } S\subseteq V_2,\\ |S|=k}} \left[\sum_{v_i\in S} d_{K_{m,n}}(v_i)\right]^a d_G(S)^b$$

$$+\sum_{\substack{S\nsubseteq V_1 \text{ and } S\nsubseteq V_2,\\ |S|=k}}\left[\sum_{v_i\in S} d_{K_{m,n}}(v_i)\right]^a d_G(S)^b.$$

By the above facts, the desired result follows. □

By Proposition 3.4, we state the next Proposition without proof.

Proposition 3.5. Let S_p be a star with $p\geq 2$. Then

$$SDD^k_{(a,b)}(S_p)=\binom{p-1}{k-1}(p+k-2)^a(k-1)^b+\binom{p-1}{k}k^{a+b}.$$

4 Bounds and Characterization

Theorem 4.1. *If G be a graph with $S=V(G)$. Then*

$$SDD^p_{(a,b)}(G)=(2q)^a(p-1)^b.$$

Proof. Let G be a graph with $S=V(G)$. Then $d_G(S)=p-1$ and $\sum_{v_i\in S} d_G(v_i)=2q$.

$$\begin{aligned} SDD^p_{(a,b)}(G) &= \sum_{S=V(G)}\left(\sum_{v_i\in S} d_G(v_i)\right)^a d_G(S)^b\\ &= (2q)^a(p-1)^b. \end{aligned}$$ □

Theorem 4.2. *Let G be a graph with $|S|=p-1$ and $a\geq 0$, $b\geq 0$. Then*

$$p(2q-\Delta(G))^a(p-2)^b\leq SDD^{(p-1)}_{(a,b)}(G)\leq p(2q-\delta(G))^a(p-1)^b.$$

Proof. Let G be a graph and $|S|=p-1$. Implies that

$$2q-\Delta(G)\leq\sum_{v_i\in S} d_G(v_i)\leq 2q-\delta(G).$$

This inequality remains the same if we raise a positive power a:

$$(2q-\Delta(G))^a\leq\left[\sum_{v_i\in S} d_G(v_i)\right]^a\leq(2q-\delta(G))^a. \tag{4.1}$$

The value of $d_G(S)$ is either $(p-2)$ or $(p-1)$ and $b\geq 0$, we have

$$(p-2)^b\leq d_G(S)^b\leq(p-1)^b. \tag{4.2}$$

Taking the product of equations (4.1) and (4.2) with respect to set S:

$$\begin{aligned} \sum_{\substack{S\subset V(G),\\ |S|=p-1}}(2q-\Delta(G))^a(p-2)^b &\leq SDD^{p-1}_{(a,b)}(G)\\ &\leq \sum_{\substack{S\subset V(G),\\ |S|=p-1}}(2q-\delta(G))^a(p-1)^b, \end{aligned}$$

$$p(2q-\Delta(G))^a(p-2)^b\leq SDD^{p-1}_{(a,b)}(G)\leq p(2q-\delta(G))^a(p-1)^b.$$

□

By Theorem 4.2, the following Theorem can be obtained, and we omit their proof.

Theorem 4.3. *Let G be a graph and $S \subset V(G)$.*

(i) If $a \leq 0$ and $b \geq 0$, then

$$p(2q - \delta(G))^a (p-2)^b \leq SDD^{p-1}_{(a,b)}(G) \leq p(2q - \Delta(G))^a (p-1)^b.$$

(ii) If $a \leq 0$ and $b \leq 0$, then

$$p(2q - \delta(G))^a (p-1)^b \leq SDD^{p-1}_{(a,b)}(G) \leq p(2q - \Delta(G))^a (p-2)^b.$$

(iii) If $a \geq 0$ and $b \leq 0$, then

$$p(2q - \Delta(G))^a (p-1)^b \leq SDD^{p-1}_{(a,b)}(G) \leq p(2q - \delta(G))^a (p-2)^b.$$

Theorem 4.4. *Let G be a graph with $a \geq 0$ and $b \geq 0$. Then*

$$\binom{p}{k} [k\delta(G)]^a (k-1)^b \leq SDD^k_{(a,b)}(G) \leq \binom{p}{k} [k\Delta(G)]^a (p-1)^b.$$

Proof. Let G be a graph. Then the set S with $a,\ b \geq 0$, satisfies

$$[k\delta(G)]^a \leq \left[\sum_{v_i \in S} d_G(G)\right]^a \leq [k\Delta(G)]^a,$$

and

$$(k-1)^b \leq d_G(S)^b \leq (p-1)^b.$$

Therefore,

$$\binom{p}{k} [k\delta(G)]^a (k-1)^b \leq SDD^k_{(a,b)}(G) \leq \binom{p}{k} [k\Delta(G)]^a (p-1)^b.$$

Hence the proof. □

Corollary 4.1. Let G be a graph.

(i) If $a \geq 0$ and $b \leq 0$, then

$$\binom{p}{k} [k\delta(G)]^a (p-1)^b \leq SDD^k_{(a,b)}(G) \leq \binom{p}{k} [k\Delta(G)]^a (k-1)^b.$$

(ii) If $a \leq 0$ and $b \geq 0$, then

$$\binom{p}{k} [k\Delta(G)]^a (k-1)^b \leq SDD^k_{(a,b)}(G) \leq \binom{p}{k} [k\delta(G)]^a (p-1)^b.$$

(iii) If $a, b \leq 0$, then

$$\binom{p}{k} [k\Delta(G)]^a (p-1)^b \leq SDD^k_{(a,b)}(G) \leq \binom{p}{k} [k\delta(G)]^a (k-1)^b.$$

Theorem 4.5. *Let G be a graph with $diam(G) = 2$, $k \geq 3$, $p \geq 6$ and $a \geq 0,\ b \leq 0$. Then*

$$\binom{p-1}{k-1} (p+k-2)^a (k-1)^b + \binom{p-1}{k} k^{a+b}$$

$$\begin{aligned}&\leq SDD^k_{(a,b)}(G)\\ &\leq \binom{p-2}{k}(k(p-1))^a(k-1)^b\\ &+2\binom{p-2}{k-1}(k(p-1)-1)^a(k-1)^b\\ &+\binom{p-2}{k-2}(k(p-1)-2)^a(k-1)^b.\end{aligned}$$

Proof. Let G be a graph with $diam(G)=2$. If $a\geq 0$, then removing an edge from G will not increase the values of $d_G(v_i)$ and $(d_G(v_i))^a$.
The lower bound attains for a tree with $a\geq 0$ and $b\leq 0$. The star is a tree with a $diam(G)=2$, which leads to the lower bound.
By Proposition (3.5), we have

$$SDD^k_{(a,b)}(G)\geq \binom{p-1}{k-1}(p+k-2)^a(k-1)^b+\binom{p-1}{k}k^{a+b}. \tag{4.3}$$

On deleting one edge in a complete graph, we get the graph with the maximum possible number of edges with $diam(G)=2$, giving us the upper bound.

$$\begin{aligned}SDD^k_{(a,b)}(G)\leq &\sum_{\substack{S\subseteq V(G),\\ d_G(v_i)=p-1 \text{ for all } v_i}}\left[\sum_{v_i\in V(G)}(p-1)\right]^a d_G(S)^b\\ &+\sum_{\substack{S\subseteq V(G),\\ \text{exists } v_i: d_G(v_i)\neq p-1}}\left[\sum_{v_i\in V(G)}d_G(v_i)\right]^a d_G(S)^b.\end{aligned}$$

$$\begin{aligned}SDD^k_{(a,b)}(G)\leq &\left[\binom{p-2}{k}(k(p-1))^a+2\binom{p-2}{k-1}(k(p-1)-1)^a\right](k-1)^b\\ &+\binom{p-2}{k-2}(k(p-1)-2)^a(k-1)^b.\end{aligned} \tag{4.4}$$

The desired result can be obtained by simplifying equation (4.3) and equation (4.4). □

Lemma 4.6. *Let G be a graph with the maximum possible number of edges and $diam(G)=3$. Then*

$$\begin{aligned}SDD^k_{(a,b)}(G)=&\binom{p-2}{k}(k(p-3))^a(k-1)^b\\ &+\binom{d_G(v_1)}{k-1}(d_G(v_1)+(k-1)(p-3))^a(k-1)^b\\ &+\binom{d_G(v_2)}{k-1}(d_G(v_1)+(k-1)(p-3))^a k^b\\ &+\binom{d_G(v_2)}{k-1}(d_G(v_2)+(k-1)(p-3))^a(k-1)^b\\ &+\binom{d_G(v_1)}{k-1}(d_G(v_2)+(k-1)(p-3))^a k^b\\ &+\binom{d_G(v_1)}{k-2}((k-1)(p-3)+1)^a(k^b-(k-1)^b)\\ &+\binom{d_G(v_2)}{k-2}((k-1)(p-3)+1)^a(k^b-(k-1)^b)\end{aligned}$$

$$+\binom{p-2}{k-2}((k-1)(p-3)+1)^a(k-1)^b,$$

where $d_G(v_1)+d_G(v_2)=p-2$*.*

Proof. Let G be a graph with $diam(G)=3$ and the maximum possible number of edges. Since $diam(G)=3$, a pair of vertices (v_1, v_2) must exist such that the distance between them is 3. The remaining $(p-2)$-vertices are adjacent to each other, and these $(p-2)$-vertices are connected to either v_1 or v_2. Now $(p-2)$-vertices are partitions into two sets V_1 and V_2, such that the set V_1 contains the vertices which are adjacent to v_1 with $|V_1|=d_G(v_1)$ and the set V_2 contains the vertices which are adjacent to v_2 with $|V_2|=d_G(v_2)$. The degree of $(p-2)$-vertices is $(p-3)$ and $d_G(v_1)+d_G(v_4)=p-2$.
The value of $d_G(S)$ is $(k-1)$ for $S\subseteq V(G)$ with $|S|=k$, if any of the following conditions hold

(i) the vertices in S are chosen from $(p-2)$-vertices, excluding v_1 and v_2.

(ii) the vertices in S include v_1 and v_2, along with at least one vertex from the partitions V_1 and V_2.

(iii) the vertices in S include v_1 but not v_2, and at least one vertex from V_1, and vice versa.

In all other cases, $d_G(S)=k$.

If $d_G(v_1)\geq d_G(v_2)\geq k=|S|$, then we have

$$\begin{aligned}
SDD^k_{(a,b)}(G) &= \sum_{S\subseteq V_1\cup V_2}\left[\sum_{v_i\in S} d_G(v_i)\right]^a d_G(S)^b \\
&+ \sum_{\substack{S\subseteq V_1\cup\{v_1\},\\ v_1\in S}}\left[\sum_{v_i\in S} d_G(v_i)\right]^a d_G(S)^b \\
&+ \sum_{\substack{S\subseteq V_2\cup\{v_1\},\\ v_1\in S}}\left[\sum_{v_i\in S} d_G(v_i)\right]^a d_G(S)^b \\
&+ \sum_{\substack{S\subseteq V_1\cup\{v_2\},\\ v_1\in S}}\left[\sum_{v_i\in S} d_G(v_i)\right]^a d_G(S)^b \\
&+ \sum_{\substack{S\subseteq V_2\cup\{v_2\},\\ v_1\in S}}\left[\sum_{v_i\in S} d_G(v_i)\right]^a d_G(S)^b \\
&+ \sum_{\substack{S\subseteq\{v_1,v_2\}\cup V_1\cup V_2,\\ v_1,v_2\in S}}\left[\sum_{v_i\in S} d_G(v_i)\right]^a d_G(S)^b.
\end{aligned}$$

On simplification, we have the desired result. □

Lemma 4.7. *Let* $S_{m,n}$ *be a double star with* $k\leq m\leq n$*. Then*

$$\begin{aligned}
SDD^k_{(a,b)}(S_{m,n}) &= \binom{m-1}{k-1}\left((m+k-1)^a(k-1)^b+(n+k-1)^a k^b\right) \\
&+ \binom{n-1}{k-1}\left((n+k-1)^a(k-1)^b+(m+k-1)^a k^b\right) \\
&+ \sum_{h=1} k-1\left((m+k-1)^a+(n+k-1)^a\right)k^b
\end{aligned}$$

$$+\binom{p-2}{k-2}(m+n+k-2)^a(k-1)^b$$
$$+\left(\binom{m-1}{k}+\binom{n-1}{k}\right)\left(k^b+(k-1)^b\right)k^a$$
$$+\binom{m+n-2}{k}k^a(k+1)^b,$$

where double star $S_{m,n}$ is obtained by connecting the centers of two stars S_m and S_n with an edge v_1v_2 (i.e., $d_G(v_1)+d_G(v_2)=p$).

Proof. Let $S_{m,n}$ be a double star with $V(S_m)=V_1\cup\{v_1\}$ and $V(S_n)=V_2\cup\{v_2\}$ for $k\leq m\leq n$, where v_1 and v_2 are central vertices of S_m and S_n, respectively. Then, for any $S\subseteq V(G)$, the value of $d_G(S)$ is lies between $k-1$ and $k+1$. We have

$$SDD^k_{(a,b)}(S_{m,n})=\sum_{S\subseteq V_1}\left[\sum_{v_i\in S}d_G(v_i)\right]^a d_G(S)^b$$
$$+\sum_{S\subseteq V_2}\left[\sum_{v_i\in S}d_G(v_i)\right]^a d_G(S)^b$$
$$+\sum_{\substack{S\subseteq V_1\cup V_2,\\ S\nsubseteq V_1,S\nsubseteq V_2}}\left[\sum_{v_i\in S}d_G(v_i)\right]^a d_G(S)^b$$
$$+\sum_{\substack{S\subseteq V_1\cup\{v_1\},\\ v_1\in S}}\left[\sum_{v_i\in S}d_G(v_i)\right]^a d_G(S)^b$$
$$+\sum_{\substack{S\subseteq V_2\cup\{v_1\},\\ v_1\in S}}\left[\sum_{v_i\in S}d_G(v_i)\right]^a d_G(S)^b$$
$$+\sum_{\substack{S\subseteq V(G)-\{v_2\},\\ v_1\in S,\\ S\nsubseteq V_1\cup\{v_1\},\\ S\nsubseteq V_2\cup\{v_1\}}}\left[\sum_{v_i\in S}d_G(v_i)\right]^a d_G(S)^b$$
$$+\sum_{\substack{S\subseteq V_1\cup\{v_1\},\\ v_2\in S}}\left[\sum_{v_i\in S}d_G(v_i)\right]^a d_G(S)^b$$
$$+\sum_{\substack{S\subseteq V_2\cup\{v_2\},\\ v_2\in S}}\left[\sum_{v_i\in S}d_G(v_i)\right]^a d_G(S)^b$$
$$+\sum_{\substack{S\subseteq V(G)-\{v_1\},\\ v_2\in S,\\ S\nsubseteq V_1\cup\{v_2\},\\ S\nsubseteq V_2\cup\{v_2\}}}\left[\sum_{v_i\in S}d_G(v_i)\right]^a d_G(S)^b$$
$$+\sum_{\substack{S\subseteq V(G)\\ \{v_1,v_2\in\}}}\left[\sum_{v_i\in S}d_G(v_i)\right]^a d_G(S)^b.$$

Therefore,

$$SDD^k_{(a,b)}(S_{m,n})=\binom{m-1}{k}k^ak^b+\binom{n-1}{k}k^ak^b$$

$$+ \left[\binom{m+n-2}{k} - \binom{m-1}{k} - \binom{n-1}{k}\right] (k)^a(k+1)^b$$
$$+ \binom{m-1}{k-1}(m+k-1)^a(k-1)^b$$
$$+ \binom{n-1}{k-1}(m+k-1)^a k^b$$
$$+ \sum_{h=1}^{k-1}\binom{m}{h}\binom{n}{k-h-1}(m+k-1)^a k^b$$
$$+ \binom{m-1}{k-1}(n+k-1)^a k^b$$
$$+ \binom{n-1}{k-1}(n+k-1)^a(k-1)^b$$
$$+ \sum_{h=1}^{k-1}\binom{m}{h}\binom{n}{k-h-1}(n+k-1)^a k^b$$
$$+ \binom{p-2}{k-2}(m+n+k-2)^a(k-1)^b.$$

On simplification, we have the required result. □

By Lemma 4.7, we have

Corollary 4.2. Let $S_{m,n}$ be a double star.

(i) If $m < k \leq n$, then

$$SDD^k_{(a,b)}(S_{m,n}) = \binom{n-1}{k-1}\left((n+k-1)^a(k-1)^b + (m+k-1)^a k^b\right)$$
$$+ \sum_{h=1}^{m-1}\left((m+k-1)^a + (n+k-1)^a\right)k^b$$
$$+ \binom{p-2}{k-2}(m+n+k-2)^a(k-1)^b$$
$$+ \binom{n-1}{k}\left(k^b + (k-1)^b\right)k^a + \binom{m+n-2}{k}k^a(k+1)^b.$$

(ii) If $m \leq n < k$, then

$$SDD^k_{(a,b)}(S_{m,n}) = \sum_{h=1}^{m-1}\left((m+k-1)^a + (n+k-1)^a\right)k^b$$
$$+ \binom{p-2}{k-2}(m+n+k-2)^a(k-1)^b$$
$$+ \binom{m+n-2}{k}k^a(k+1)^b.$$

Observation 4.1. Let G be a graph with $a \leq 0$, $b \geq 0$, $k \geq 4$, $p \geq 8$ and $diam(G) = 3$. Then, the lower bound is attained when there is only one pair of vertices with diameter 3, while the upper bound is attained for the double star.

5 Comparison among degree distance indices

If $f(x, y) = \left(\sum_{i=1}^{k} a_i\right)^x b^y$ is a function in two variables, where $a_i \in \{1, 2, \ldots, p-1\}$, $b \in \{k-1, k, \ldots, p-1\}$ and x and y are real numbers, then $f(x, y)$ is a strictly increasing function.

We have the following inequalities among the existing indices for a fixed value of a, b, and k. For fixing $a = 0$ and $k = 2$,

$$SDD^2_{(0,-2)}(G) < SDD^2_{(0,-1)}(G) < SDD^2_{(0,1)}(G)$$

$$\sum_{\{v_i,v_j\}\subseteq V(G)} \frac{1}{d(v_i,v_j)^2} < \sum_{\{v_i,v_j\}\subseteq V(G)} \frac{1}{d(v_i,v_j)} < \sum_{\{v_i,v_j\}\subseteq V(G)} d(v_i,v_j)$$

$$H^2(G) < H(G) < W(G).$$

For fixing $k = 2$.

$$SDD^2_{(1,-1)}(G) < SDD^2_{(1,1)}(G)$$

$$\sum_{\{v_i,v_j\}\subseteq V(G)} \frac{(d_G(v_i)+d_G(v_j))}{d(v_i,v_j)^2} < \sum_{\{v_i,vJ\}\subseteq V(G)} (d_G(v_i)+d_G(v_j))d(v_i,v_j)^2$$

$$H_A(G) < DD(G).$$

For fixed values a and b.

$$SDD^k_{(0,-1)}(G) < SDD^k_{(0,1)}(G)$$

$$\sum_{S\subseteq V(G),|S|=k} \frac{1}{d_G(S)} < \sum_{S\subseteq V(G),|S|=k} d_G(S)$$

$$SH^k(G) < SW^k(G),$$

and

$$SDD^k_{(1,-1)}(G) < SDD^k_{(1,1)}(G)$$

$$\sum_{S\subseteq V(G),|S|=k} \left(\sum_{v_i\in S} d_G(v_i)\right) \frac{1}{d_G(S)} < \sum_{S\subseteq V(G),|S|=k} \left(\sum_{v_i\in S} d_G(v_i)\right) d_G(S)$$

$$SRDD^k(G) < SDD^k(G)$$

Theorem 5.1. *Let G be a connected graph and $S \subseteq V(G)$ with $2 \le k \le p-2$, $a \ge 1$ and $b \ge 1$. Then*

$$DD^k(G) \le SDD^k_{(a,b)}(G).$$

Proof. Let G be a connected graph and $S \subseteq V(G)$ with $2 \le k \le p-2$. If there exist each pair of vertices v_i and v_j in S (i.e., $\{v_i, v_j\} \subseteq S$) such that $d_G(v_i) + d_G(v_j) \le \sum_{v_i\in S} d_G(v_i)$, then $d_G(v_i, v_j) \le d_G(S)$. Therefore,

$$\sum_{v_i,v_j\in V(G)} (d_G(v_i)+d_G(v_j))\, d_G(v_i,v_j) \le \sum_{S\subseteq V(G)} \left(\sum_{v_i\in S} d_G(v_i)\right) d_G(S)$$

$$DD^k(G) \le SDD^k_{(a,b)}(G).$$

□

6 Conclusion

The Generalized Steiner degree distance index of a graph lies on the claim that their special cases, for pertinently chosen values of the parameters a, b, and k, with the vast majority of previously considered vertex degree distance-based topological indices. Here are some key observations: The complete graph achieves the upper bound for any fixed k and $a \ge b \ge 0$. However, the lower bound is reached by either path or star, depending on the value of a. The function corresponding to $SDD^k_{(a,b)}(G)$ strictly increases with the independent variables a and b. For a tree G with diameter d, the number of pendent vertices ranges from $\lceil \frac{p}{d} \rceil$ to $(p - d + 1)$.

7 Conflicts of Interest

The authors have reported that they have no conflicts of interest to declare.

References

[1] Y. Alizadeh, A. Iranmanesh and T. Doslic, Additively weighted Harary index of some composite graphs, Discrete Mathematics, **313**, 26–34, (2013).

[2] A. Babu and J. B. Babujee, Steiner Hyper Wiener Index, International Journal of Mathematics Trends and Technology (IJMTT) – Special Issue NCCFQET May 2018.

[3] S. A. Basha, T. N. Asha and B. Chaluvaraju, Generalized Schultz and Gutman Indices, Iranian Journal of Mathematical Chemistry, **13(4)**, 301–316, (2022).

[4] F. Buckley and F. Harary, Distance in graphs, Addison–Wesley, Redwood, (1990).

[5] B. Chaluvaraju, H. S. Boregowda and I. N. Cangul, Some Inequalities for the First General Zagreb Index of Graphs and Line Graphs,Proceedings of the National Academy of Sciences, India Section A: Physical Sciences, **91**, 79–88, (2021).

[6] B. Chaluvaraju and S. A. Basha, Different versions of atom-bond connectivity indices of some molecular structures: Applied for the treatment and prevention of COVID-19, Polycyclic Aromatic Compounds, **42(6)**, 3748–3761, (2022).

[7] G. Chartrand, Steiner distance in graphs, casopis pro pestovani matematiky, **144(4)**, 399–410, (1989).

[8] P. Dankelmann, The Steiner k-Wiener index of graphs with given minimum degree, Discrete Applied Mathematics, **268**, 35–43, (2019).

[9] P. Dankelmann, R. O. Oellermann and H. C. Swart, The average Steiner distance of a graph, Journal of Graph Theory, **22(1)**, 15–22, (1996).

[10] P. Dankelmann and A. Alochukwu, Wiener index in graphs with given minimum degree and maximum degree, Discrete Mathematics and Theoretical Computer Science, Graph Theory, **23**, (2021).

[11] P. Dankelmann, I. Gutman, S. Mukwembi, and C. S. Henda, On the degree distance of a graph, Discrete Applied Mathematics **157(13)**, 2773–2777, (2009).

[12] P. Dankalmann and R. Entringer, Average distance, minimum degree, and spanning trees, Journal of Graph Theory, **33(1)**, 1–13, (2000).

[13] K. V. Das, G. Su and L. Xiong, Relation between degree distance and Gutman index of graphs, MATCH Communications in Mathematical and in Computer Chemistry, **76(1)**, 221–232, (2016).

[14] A. A. Dobrynin and A. A. Kochetova, Degree distance of a graph: A degree analogue of the Wiener index, Journal of Chemical Information and Computer Sciences, **34** , 1082–1086, (1994).

[15] I. Gutman, On Steiner degree distance of trees, Applied Mathematics and Computation, **283**, 163–167, (2016).

[16] I. Gutman, Selected properties of the Schultz molecular topological index, Journal of chemical information and computer sciences, **34**, 1087–1089, (1994).

[17] I. Gutman, B. Furtula and K. C. Das, On some degree-and-distance-based graph invariants of trees, Applied Mathematics and Computation, **289**, 1–6, (2016).

[18] F. Harary, Graph Theory, Addison Wesley, Reading Reading Massachusets USA, (1969).

[19] H. Hua and S. Zhang, On the reciprocal degree distance of graphs, Discrete Applied Mathematics, **160(7-8)**, 1152–1163, (2012).

[20] O. Ivanciuc, T.S. Balaban and A.T. Balaban, Design of topological indices. Part 4. Reciprocal distance matrix, related local vertex invariants and topological indices, Journal of Mathematical Chemistry, **12**, 309–318, (1993).

[21] D. J. Klein and I. Gutman, Wiener-number-related sequences, Journal of chemical information and computer sciences, **39**, 534–536, (1999).

[22] D. J. Klein, I. Lukovits and I. Gutman, On the definition of the hyper-Wiener index for cycle-containing structures, Journal of chemical information and computer sciences, **35(1)**, 50–52, (1995).

[23] Y. Mao, Steiner Harary index, Kragujevac Journal of Mathematics, **42(1)**, 29–39, (2018).

[24] Y. Mao, Z. Wang, I. Gutman and A. Klobucar, Steiner degree distance. MATCH Communications in Mathematical and in Computer Chemistry, **78(1)**, 221–230, (2017).

[25] Y. Mao, W. Zhao and I. Gutman, Steiner Wiener index of graph products, Transactions on Combinatorics,**5(3)**, 39–50, (2016).

[26] Y. Mao and K. C. Das, Steiner Gutman index,MATCH Communications in Mathematical and in Computer Chemistry, **79(3)**, 779–94, (2018).

[27] Y. Mao, P. Dankelmann and Z. Wang, Steiner diameter, maximum degree and size of a graph, Discrete Mathematics, **344(8)**, 112468, (2021).

[28] Z. Mihalic and N. Trinajsti, A graph-theoretical approach to structure-property relationships, Journal of Chemical Education, **69**, 701–712, (1992).

[29] I. Redzepovic, Y. Mao, Z. Wang and B. Furtula, Steiner degree distance indices: Chemical applicability and bounds. International Journal of Quantum Chemistry, **120(12)**, e26209, (2020).

[30] B. Sarveshkumar and B. Chaluvaraju, Sharp Inequalities for the Generalized Steiner-Gutman Index, Journal of International Academy of Physical Sciences, **29(1)**, 1–12, (2025).

[31] B. Sarveshkumar, B. Chaluvaraju and M. C. Maheshkumar, Some Bounds for the Steiner-Harary index of a graph, Journal of Nonlinear Modeling and Analysis, **7(4)**, 1446-1460, (2025).

[32] H. Wiener, Structural determination of paraffin boiling points, Journal of the American chemical society, **69(1)**, 17–20, (1947).

[33] L. Xueliang, M. Yaping and I. Gutman, The Steiner Wiener index of a graph, Discussiones Mathematicae: Graph Theory, **36(2)**, 455–465, (2016).

[34] X. Zhang, Reciprocal sterner degree distance, Utilitas Mathematica, **113**, 1–16, (2019).

Author information

B. Sarveshkumar, Department of Mathematics, Bangalore University, Jnana Bharathi Campus
Bangalore -560 056, India.
E-mail: sarveshbub@gmail.com

B. Chaluvaraju, Department of Mathematics, Bangalore University, Jnana Bharathi Campus
Bangalore -560 056, India.
E-mail: bchaluvaraju@gmail.com