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On $r-$ Dynamic chromatic number of some brick product graphs $C(2n, 2, p)$

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Abstract *An r-dynamic coloring of a graph G is a proper coloring c of the vertices such that $|c(N(v))| \geq min\{r, d(v)\}$, for each $v \in V(G)$. The r-dynamic chromatic number of a graph G is the minimum k such that G has an r-dynamic coloring with k colors. In this paper, we obtain the $r-$ dynamic chromatic number of brick product graphs.*

1 Introduction

Graphs in this paper are simple and finite. For undefined terminologies and notations see [4, 18]. Thus for a graph $G, \delta(G), \Delta(G)$ and $\chi(G)$ denote the minimum degree, maximum degree and chromatic number of G respectively. When the context is clear we write, δ, Δ and χ for brevity. For $v \in V(G)$, let $N(v)$ denote the set of vertices adjacent to v in G and $d(v) = |N(v)|$. The r-dynamic chromatic number was first introduced by Montgomery [12].

An r-dynamic coloring of a graph G is a map c from $V(G)$ to the set of colors such that (i) if $uv \in E(G)$, then $c(u) \neq c(v)$ and (ii) for each vertex $v \in V(G)$,

$$|c(N(v))| \geq min\{r, d(v)\} \tag{1.1}$$

where $N(v)$ denotes the set of vertices adjacent to v, $d(v)$ its degree and r is a positive integer.

The $r-$ dynamic chromatic number has been studied by several authors, for instance in [3, 5, 6, 8, 1, 9, 10, 11, 13, 17].

In [2], Alspach et.al. have proved the brick product graphs associated with even cycles C_{2n} are Hamiltonian laceable, in the sence that any two vertices at an odd distance Hamiltonian path. Brick product graphs of even cycles, introduced by Alspach et al., are a class of three regular graphs that exhibit interesting path properties. Some results on the rainbow connection number of brick product graphs and modified brick product graphs have been determined by Srinivasa Rao and Murali in [14, 15, 16]. Deepa et al.[7] have proved the $r-$ dynamic chromatic number of some brick product graphs $C(2n, 1, p)$.

In this paper, we find the r-dynamic chromatic number of some brick product graphs associated with even cycles.

2 Preliminary

Definition 2.1. Let m, n and p be a positive integers. Let $C_{2n} = v_0, v_1, v_2, ..., v_{2n-1}, v_{2n}$ denote a cycle of order $2n$. The (m, p) brick product of C_{2n}, denoted by $C(2n, m, p)$ is defined in two cases as follows:

1. For $m = 1$, we reqire that p be odd and greater than 1. Then $C(2n, m, p)$ is obtained from C_{2n}

by adding chords $v_{2k}(v_{2k+p}), k = 1, 2, \cdots, n$, where the computation is performed modulo 2n.
2. For $m > 1$, we require that $m + p$ be even. Then $C(2n, m, p)$ is obtained by first taking the disjoint union of m copies of C_{2n}, namely $C_{2n}(1), C_{2n}(2), C_{2n}(3), \cdots, C_{2n}(m)$ where for each $i = 1, 2, \cdots, m, C_{2n}(i) = v_{i1}, v_{i2}, \cdots, v_{i2n}, v_{i0}$. Next for each odd $i = 1, 2, \cdots, m-1$ and each even $k = 0, 1, 2, \cdots, 2n-2$, an edge(called a brick edge) is drawn to join (v_i, v_k) to v_{i+1}, v_k whereas for each even $i = 1, 2, \cdots, m-1$ and each odd $k = 1, 2, \cdots, 2n-1$ an edge(also called a brick edge) is drawn to join (v_i, v_k) to v_{i+1}, v_k. Finally, for each odd $k = 1, 2, \cdots, 2n-1$, an edge(called a hooking edge) is drawn to join (v_1, v_k) to v_m, v_{k+p}. An edge in $C(2n, m, p)$ which is neither a brick edge nor a hooking edge is called a flat edge.

Lemma 2.2. *[12] Let G be a graph and let r be a positive integer. Then,*

$$\min\{r, \Delta(G)\} + 1 \leq \chi_r(G) \leq \chi_{r+1}(G).$$

Moreover, $\chi_r(G) \leq \chi_{\Delta(G)}(G)$.

3 Results

Lemma 3.1. *Let $G = C(2n, m, p)$. Then for $m = 2, n \geq 5$ and $p = 4$,the lower bound of r- dynamic chromatic number is*

$$\chi_r(G) \geq \begin{cases} 2, & r = 1 \\ 3, & r = 1 \\ 4, & r = 2 \\ 6, & r = 3 = \Delta \\ 7, & r = 3 = \Delta \end{cases}$$

Proof. Let us consider the two copies of C_{2n} namely $C_{2n}(1)$ and $C_{2n}(2)$. Let the vertices of $C_{2n}(1)$ is $v_{11}, v_{12}, v_{13}, \cdots, v_{1(2n-1)}, v_{1(2n)} = v_{10}$ and the vertices of $C_{2n}(2)$ is $v_{21}, v_{22}, v_{23}, \cdots,$ $v_{2(2n-1)}, v_{2(2n)} = v_{20}$ and the order $|V_{1i}| = 3$ and $|V_{2i}| = 3$.

For case $r = 1$, choose $r = 1$, based on definition $\chi_r(G) \geq min\{r, \Delta(G)\} + 1$.
i.e. $\chi_1(G) \geq min\{1, 3\} + 1 = 1 + 1 = 2$, then $\chi_1(G) \geq 2$.

For case $r = 2$, choose $r = 2$, based on definition $\chi_2(G) \geq min\{2, 3\} + 1 = 2 + 1 = 3$, then $\chi_2(C(2n, m, p)) \geq 3$.

Since by definition of dynamic coloring, every vertex set of G with degree has at least two neighbours that are colored differently. But $C(V_{1(i-1)}) \neq C(V_{1i}) \neq C(V_{1(i+1)}) \neq C(V_{2i})$ and also $C(V_{1i}) \neq C(V_{1(i+1)}) \neq C(V_{2(2i+3)})$. so we need to add one more color. so in this case $\chi_2(C(2n, m, p)) \geq 4$.

For case $r = 3 = \Delta$, choose $r = 3$, based on definition $\chi_3(G) \geq min\{3, \Delta\} + 1 = \Delta + 1$, then $\chi_3(C(2n, m, p)) \geq 6$ for $2n \equiv 0(mod\ \ 3)$. Also we need 7 colors for $2n \equiv 1(mod\ \ 3)$ *or* $2n \equiv 2(mod\ \ 3)$

This completes the proof. □

Theorem 3.2. *Let $G = C(2n, m, p)$. Then for $m = 2, n \geq 5$ and $p = 4$,*

$$\chi_r(G) = \begin{cases} 2, & if\, r = 1, \\ 3, & if\, r = 2, \\ 4, & if\, r = 2, \\ 6, & if\, r = 3 = \Delta, \\ 7, & if\, r = 3 = \Delta, \end{cases}$$

Proof. Let us consider the two copies of C_{2n} namely $C_{2n}(1)$ and $C_{2n}(2)$. Let the vertices of $C_{2n}(1)$ is $v_{11}, v_{12}, v_{13}, \cdots, v_{1(2n-1)}, v_{1(2n)} = v_{10}$ and the vertices of $C_{2n}(2)$ is $v_{21}, v_{22}, v_{23}, \cdots,$ $v_{2(2n-1)}, v_{2(2n)} = v_{20}$ and the edge set of $C(2n, m, p)$ as
$E(C(2n, m, p)) = \{v_{1i}, v_{1(i+1)} : 0 \leq i \leq 2n\}$ under modulo 2n $\cup \{v_{2i}, v_{2(i+1)} : 0 \leq i \leq 2n\}$under modulo 2n $\cup \{v_{1i}, v_{2i} : 0 \leq i \leq 2n-2, i\ \ even\} \cup \{v_{1(2i+1)}, v_{2(2i+5)} : 0 \leq i \leq n-3\}$

$\cup \left\{v_{2(2i+1)}, v_{1(2n+2i-3)} : 0 \leq i \leq 1\right\}$

Here $|N(v_{1i})| = deg(v_{1i}) = 3$ and $|N(v_{2i})| = deg(v_{2i}) = 3$.
We divide the proof into some cases.
Assign the $r-$dynamic coloring to $C(2n,m,p)$ as follows:

Case 1: For $r = 1$
The $r-$dynamic proper coloring $C : V(G) \to \{1, 2, \cdots k\}$ satisfying condition 1.1.

Since lemma 2.2 implies that $\chi_r(G) \geq 2$.
The r-dynamic proper 2-coloring is as follows:

$$C_1(v_{1i}) = \begin{cases} 1, \text{for i is odd}, 1 \leq i \leq 2n \\ 2, \text{for i is even}, 1 \leq i \leq 2n \end{cases} \quad C_1(v_{2i}) = \begin{cases} 2, \text{for i is odd}, 1 \leq i \leq 2n \\ 1, \text{for i is even}, 1 \leq i \leq 2n \end{cases}$$

An easy verification shows that $C : V(G) \to \{1, 2\}$.
Hence, $\chi_r(G) = 2$, for $r = 1$.

Case 2: For $r = 2$, $2n = 3t, t \geq 4$

By lemma 3.1, the lower bound for $r = 2$ is $\chi_r(C(G_n)) \geq 3$.
Consider the mapping $C : V(G) \to \{1, 2, \cdots k\}$ such that

$$C_2(v_{1i}) = \begin{cases} 1, & i = 3t+1, \quad t \geq 0, 1 \leq i \leq 2n \\ 2, & i = 3t+2, \quad t \geq 0, 1 \leq i \leq 2n \\ 3, & i = 3t, \quad t \geq 1, 1 \leq i \leq 2n \end{cases} \quad C_2(v_{2i}) = \begin{cases} 2, & i = 3t+1, \quad t \geq 0, 1 \leq i \leq 2n \\ 3, & i = 3t+2, \quad t \geq 0, 1 \leq i \leq 2n \\ 1, & i = 3t, \quad t \geq 1, 1 \leq i \leq 2n \end{cases}$$

An easy verification shows that $C : V(G) \to \{1, 2, 3\}$.
Hence, $\chi_r(G) = 3$, for $r = 2$.

It is straight forwardly verified that condition 1.1 holds and hence $\chi_r(G) = 3$.

Case 3: For $r = 2$, $2n = 3t+1$ and $2n = 3t+2$,$t \geq 4$

By lemma 3.1, the lower bound for $r = 2$ is $\chi_r(C(G_n)) \geq 3$.
Consider the mapping $C : V(G) \to \{1, 2, \cdots k\}$ such that

$$C_3(v_{1i}) = \begin{cases} 1, \text{for i is odd}, 1 \leq i \leq 2n \\ 2, \text{for i is even}, 1 \leq i \leq 2n \end{cases} \quad C_3(v_{2i}) = \begin{cases} 4, \text{for i is odd}, 1 \leq i \leq 2n \\ 3, \text{for i is even}, 1 \leq i \leq 2n \end{cases}$$

An easy verification shows that $C : V(G) \to \{1, 2, 3, 4\}$.
Hence, $\chi_r(G) = 4$, for $r = 2$.

It is straight forwardly verified that condition 1.1 holds and hence $\chi_r(G) = 4$.

Case 4: For $r = 3 = \Delta$, $2n = 3t$, $t \geq 4$
By lemma 3.1, the lower bound for $r = 3$ is $\chi_r(C(G_n)) \geq 6$.
In this case, the 3-dynamic 6-proper coloring $C : V(G) \to \{1, 2, 3, \cdots k\}$ satisfying 1.1 is given.

$$C_4(v_{1i}) = \begin{cases} 1, & i = 3t+1, t \geq 0, 1 \leq i \leq 2n \\ 2, & i = 3t+2, t \geq 0, 1 \leq i \leq 2n \\ 3, & i = 3t, t \geq 0, 1 \leq i \leq 2n \end{cases} \quad C_4(v_{2i}) = \begin{cases} 6, & i = 3t+1, t \geq 0, 1 \leq i \leq 2n \\ 5, & i = 3t+2, t \geq 0, 1 \leq i \leq 2n \\ 4, & i = 3t, t \geq 0, 1 \leq i \leq 2n \end{cases}$$

It is verified that the map C is a proper coloring satisfying condition 1.1 and hence $\chi_r(G) = 6$, for $r = 3$.

Case 5: For $r = 3 = \Delta$, $2n = 3t+1$ and $2n = 3t+2$,$t \geq 4$

By lemma 3.1, the lower bound for $r = 3$ is $\chi_r(C(G_n)) \geq 6$.

In this case, the 3-dynamic 7-proper coloring $C : V(G) \to \{1, 2, 3, \cdots k\}$ satisfying 1.1 is given.

Subcase $2n = 3t + 1$, for some positive integer n.

$$C_5(v_{1i}) = \begin{cases} 1, & i = 3t+1, t \geq 0, 1 \leq i \leq 2n-1 \\ 2, & i = 3t+2, t \geq 0, 1 \leq i \leq 2n-1 \\ 3, & i = 3t, t \geq 1, 1 \leq i \leq 2n-1 \\ 4, & i = 2n \end{cases} \quad C_5(v_{2i}) = \begin{cases} 7, & i = 1 \\ 4, & i = 3t+2, t \geq 0, 1 \leq i \leq 2n \\ 6, & i = 3t, t \geq 1, 1 \leq i \leq 2n \\ 5, & i = 3t+1, t \geq 1, 1 \leq i \leq 2n \end{cases}$$

It is verified that the map C is a proper coloring satisfying condition 1.1 and hence $\chi_r(G) = 7$, for $r = 3$.

Subcase $2n = 3t + 2$ for some positive integer n.

$$C_6(v_{1i}) = \begin{cases} 1, & i = 3t+1, i = 2n-3, t \geq 0, 1 \leq i \leq 2n-5 \\ 2, & i = 3t+2, i = 2n-2, t \geq 0, 1 \leq i \leq 2n-5 \\ 3, & i = 3t, i = 2n-1, t \geq 1, 1 \leq i \leq 2n-5 \\ 4, & i = 2n-4 and i = 2n \end{cases}$$

$$C_6(v_{2i}) = \begin{cases} 7, & i = 4t+1, t \geq 0, 1 \leq i \leq 2n-2 \\ 6, & i = 4t+2, t \geq 0, 1 \leq i \leq 2n-2 \\ 5, & i = 4t+3, t \geq 0, 1 \leq i \leq 2n-2 \\ 4, & i = 4t, t \geq 1, 1 \leq i \leq 2n-2 \\ 1, & i = 2n-1 \\ 2, & i = 2n \end{cases}$$

It is verified that the map C is a proper coloring satisfying condition 1.1 and hence $\chi_r(G) = 7$, for $r = 3$.

Subcase $2n = 12t + 8$ for some positive integer n.

$$C_7(v_{1i}) = \begin{cases} 1, & i = 3t+1, i = 2n-3, t \geq 0, 1 \leq i \leq 2n-5 \\ 2, & i = 3t+2, i = 2n-2, t \geq 0, 1 \leq i \leq 2n-5 \\ 3, & i = 3t, i = 2n-1, t \geq 1, 1 \leq i \leq 2n-5 \\ 4, & i = 2n-4 and i = 2n \end{cases}$$

$$C_7(v_{2i}) = \begin{cases} 5, & i = 4t+1, t \geq 0, 1 \leq i \leq 2n-2 \\ 4, & i = 4t+2, t \geq 0, 1 \leq i \leq 2n-2 \\ 7, & i = 4t+3, t \geq 0, 1 \leq i \leq 2n-2 \\ 6, & i = 4t, t \geq 1, 1 \leq i \leq 2n-2 \\ 1, & i = 2n-1 \\ 2, & i = 2n \end{cases}$$

It is verified that the map C is a proper coloring satisfying condition 1.1 and hence $\chi_r(G) = 7$, for $r = 3$.

□

Lemma 3.3. *Let $G = C(2n,m,p)$. Then for $m = 2, n \geq 5$ and $p = 6$,*

$$\chi_r(G) \geq \begin{cases} 2, & r = 1 \\ 3, & r = 2 \\ 4, & r = 2 \\ 6, & r = 3 = \Delta \\ 7, & r = 3 = \Delta \end{cases}$$

Proof. We consider the vertex set of G as $V(G)$ in lemma1 and the order $|V(G)| = 3$.

For case $r = 1$, choose $r = 1$, based on definition $\chi_r(G) \geq min\{r, \Delta(G)\} + 1$.
i.e. $\chi_1(G) \geq min\{1,3\} + 1 = 1 + 1 = 2$, then $\chi_1(C(2n,m,p)) \geq 2$.

For case $r = 2$, choose $r = 2$, based on definition $\chi_2(G) \geq min\{2,3\} + 1 = 2 + 1 = 3$, then $\chi_2(C(2n,m,p)) \geq 3$.

Since by definition of dynamic coloring, every vertex set of G with degree has at least two neighbours that are colored differently. But $C(V_{1(i-1)}) \neq C(V_{1i}) \neq C(V_{1(i+1)}) \neq C(V_{2i})$ and also $C(V_{1i}) \neq C(V_{1(i+1)}) \neq C(V_{2(2i+5)})$. so we need to add one more color. so in this case $\chi_2(C(2n,m,p)) \geq 4$.

For case $r = 3 = \Delta$, choose $r = 3$, based on definition $\chi_3(G) \geq min\{3,\Delta\} + 1 = \Delta + 1$, then $\chi_3(C(2n,m,p)) \geq 6$ for $2n \equiv 0(mod\ \ 3)$. Also we need 7 colors for $2n \equiv 1(mod\ \ 3)$ *or* $2n \equiv 2(mod\ \ 3)$

This completes the proof. □

Theorem 3.4. *Let $G = C(2n,m,p)$. Then for $m = 2, n \geq 5$ and $p = 6$,*

$$\chi_r(G) = \begin{cases} 2, & \textit{if } r = 1 \\ 3, & \textit{if } r = 2 \\ 4, & \textit{if } r = 2 \\ 6, & \textit{if } r = 3 = \Delta \\ 7, & \textit{if } r = 3 = \Delta \end{cases}$$

Proof. We consider the vertex set of G as $V(G)$ and the edge set of $E(G)$ defined in theorem 3.2.

Here $|N(v_i)| = deg(v_i) = 3$.

We divide the proof into some cases.

Assign the $r-$dynamic coloring to $C(2n,m,p)$ as follows:

Case 1: For $r = 1$

The $r-$dynamic proper coloring $C : V(G) \to \{1,2,\cdots k\}$ satisfying condition 1.1.

Since lemma 2.2 implies that $\chi_r(G) \geq 2$.

The r-dynamic proper 2-coloring is as follows:

$$c(v_i) = C_1(v_{1i}) = \begin{cases} 1, \text{for i is odd}, 1 \leq i \leq 2n \\ 2, \text{for i is even}, 1 \leq i \leq 2n \end{cases} \quad C_1(v_{2i}) = \begin{cases} 2, \text{for i is odd}, 1 \leq i \leq 2n \\ 1, \text{for i is even}, 1 \leq i \leq 2n \end{cases}$$

An easy verification shows that $C : V(G) \to \{1,2\}$.
Hence, $\chi_r(G) = 2$, for $r = 1$.

Case 2: For $r = 2$ and $2n = 3t, t \geq 4$

By lemma 3.3, the lower bound for $r = 2$ is $\chi_r(C(G_n)) \geq 3$.

Consider the mapping $C : V(G) \to \{1,2,\cdots k\}$ such that

$$C_2(v_{1i}) = \begin{cases} 1, & i = 3t+1, t \geq 0, 1 \leq i \leq 2n \\ 2, & i = 3t+2, t \geq 0, 1 \leq i \leq 2n \\ 3, & i = 3t, t \geq 0, 1 \leq i \leq 2n \end{cases} \quad C_2(v_{2i}) = \begin{cases} 2, & i = 3t+1, t \geq 0, 1 \leq i \leq 2n \\ 3, & i = 3t+2, t \geq 0, 1 \leq i \leq 2n \\ 1, & i = 3t, t \geq 1, 1 \leq i \leq 2n \end{cases}$$

It is straight forwardly verified that condition 1.1 holds and hence $\chi_r(G) = 3$.

Case 3: For $r = 2$, $2n = 3t + 1$ and $2n = 3t + 2$,$t \geq 4$, for some positive integer t.

By lemma 3.3, the lower bound for $r = 2$ is $\chi_r(C(G_n)) \geq 4$.
In this case, the 2-dynamic 4-proper coloring $C : V(G) \to \{1, 2, 3, \cdots k\}$ satisfying 1.1 is given.

$$C_3(v_{1i}) = \begin{cases} 1, \text{for i is odd}, 1 \leq i \leq 2n \\ 2, \text{for i is even}, 1 \leq i \leq 2n \end{cases} \quad C_3(v_{2i}) = \begin{cases} 4, \text{for i is odd}, 1 \leq i \leq 2n \\ 3, \text{for i is even}, 1 \leq i \leq 2n \end{cases}$$

It is verified that the map C is a proper coloring satisfying condition 1.1 and hence $\chi_r(G) = 4$, for $r = 2$.

Case 4: For $r = 3$, $2n = 3t$,$t \geq 4$ for some positive integer t.

By lemma 3.3, the lower bound for $r = 3$ is $\chi_r(C(G_n)) \geq 6$.
In this case, the 3-dynamic 6-proper coloring $C : V(G) \to \{1, 2, 3, \cdots k\}$ satisfying 1.1 is given.

$$C_4(v_{1i}) = \begin{cases} 1, & i = 3t + 1, t \geq 0, 1 \leq i \leq 2n \\ 2, & i = 3t + 2, t \geq 0, 1 \leq i \leq 2n \\ 3, & i = 3t, t \geq 1, 1 \leq i \leq 2n \end{cases} \quad C_4(v_{2i}) = \begin{cases} 6, & i = 3t + 1, t \geq 0, 1 \leq i \leq 2n \\ 5, & i = 3t + 2, t \geq 0, 1 \leq i \leq 2n \\ 4, & i = 3t, t \geq 1, 1 \leq i \leq 2n \end{cases}$$

It is verified that the map C is a proper coloring satisfying condition 1.1 and hence $\chi_r(G) = 6$, for $r = 3$.

Case 5: For $r = 3$, $2n = 12t + 8$ and $2n = 12t + 10$, $t \geq 4$

By lemma 3.3, the lower bound for $r = 3$ is $\chi_r(C(G_n)) \geq 7$.
In this case, the 3-dynamic 7-proper coloring $C : V(G) \to \{1, 2, 3, \cdots k\}$ satisfying 1.1 is given.
Subcase $2n = 12t + 8$, for some positive integer t.

$$C_5(v_{1i}) = \begin{cases} 1, & i = 3t + 1, i = 2n - 3, t \geq 0, 1 \leq i \leq 2n - 5 \\ 2, & i = 3t + 2, i = 2n - 2, t \geq 0, 1 \leq i \leq 2n - 5 \\ 3, & i = 3t, i = 2n - 1, t \geq 0, 1 \leq i \leq 2n - 5 \\ 4, & for \;\; i = 2n - 4 \;\; i = 2n \end{cases}$$

$$C_5(v_{2i}) = \begin{cases} 7, & i = 3t + 2, i = 2n - 3, t \geq 0, 1 \leq i \leq 2n - 4 \\ 6, & i = 3t, i = 2n - 2, t \geq 0, 1 \leq i \leq 2n - 4 \\ 5, & i = 3t + 1, i = 2n - 1, t \geq 0, 1 \leq i \leq 2n - 4 \\ 2, & for \;\; i = 2n \\ 1, & for \;\; i = 1 \end{cases}$$

It is verified that the map C is a proper coloring satisfying condition 1.1 and hence $\chi_r(G) = 7$, for $r = 3$.

Subcase $2n = 12t + 10$, for some positive integer t.

$$C_6(v_{1i}) = \begin{cases} 1, & i = 3t + 1, t \geq 0, 1 \leq i \leq 2n - 1 \\ 2, & i = 3t + 2, t \geq 0, 1 \leq i \leq 2n - 1 \\ 3, & i = 3t, t \geq 1, 1 \leq i \leq 2n - 1 \\ 4, & i = 2n \end{cases}$$

$$C_6(v_{2i}) = \begin{cases} 7, & i = 3t+1, t \geq 0, 1 \leq i \leq 2n-3 \\ 6, & i = 3t+2, i = 2n-1, t \geq 0, 1 \leq i \leq 2n-4 \\ 5, & i = 3t, i = 2n, t \geq 0, 1 \leq i \leq 2n-4 \\ 4, & for \ \ i = 2n-2 \end{cases}$$

It is verified that the map C is a proper coloring satisfying condition 1.1 and hence $\chi_r(G) = 7$, for $r = 3$.

□

Lemma 3.5. *Let* $G = C(2n, m, p)$*. Then for* $m = 2, n \geq 5$ *and* $p = 8$*,*

$$\chi_r(G) = \begin{cases} 2, & r = 1 \\ 3, & r = 2 \\ 4, & r = 2 \\ 6, & r = 3 = \Delta \\ 7, & r = 3 = \Delta \end{cases}$$

Proof. We consider the vertex set of G as $V(G)$ in lemma1 and the order $|V(G)| = 3$.

For case $r = 1$, choose $r = 1$, based on definition $\chi_r(G) \geq min\{r, \Delta(G)\} + 1$.
i.e. $\chi_1(G) \geq min\{1, 3\} + 1 = 1 + 1 = 2$, then $\chi_1(C(2n, m, p)) \geq 2$.

For case $r = 2$, choose $r = 2$, based on definition $\chi_2(G) \geq min\{2, 3\} + 1 = 2 + 1 = 3$, then $\chi_2(C(2n, m, p)) \geq 3$.

Since by definition of dynamic soloring, every vertex set of G with degree has atleast two noighbours that are colorde differently. But $C(V_{1(i-1)}) \neq C(V_{1i}) \neq C(V_{1(i+1)}) \neq C(V_{2i})$ and also $C(V_{1i}) \neq C(V_{1(i+1)}) \neq C(V_{2(2i+7)})$. so we need to add one more color. so in this case $\chi_2(C(2n, m, p)) \geq 4$.

For case $r = 3 = \Delta$, choose $r = 3$, based on definition $\chi_3(G) \geq min\{3, \Delta\} + 1 = \Delta + 1$, then $\chi_3(C(2n, m, p)) \geq 6$ for $2n \equiv 0(mod \ \ 3)$ and $2n \equiv 1(mod \ \ 3)$. Also we need 7 colors for $2n \equiv 2(mod \ \ 3)$

This completes the proof. □

Theorem 3.6. *Let* $G = C(2n, m, p)$*. Then for* $m = 2, n \geq 5$ *and* $p = 8$*,*

$$\chi_r(G) = \begin{cases} 2, & r = 1 \\ 3, & r = 2 \\ 4, & r = 2 \\ 6, & r = 3 = \Delta \\ 7, & r = 3 = \Delta \end{cases}$$

Proof. We consider the vertex set of G as $V(G)$ and the edge set of $E(G)$ defined in theorem 3.2.

Here $|N(v_i)| = deg(v_i) = 3$.

We divide the proof into some cases.

Assign the $r-$dynamic coloring to $C(2n, m, p)$ as follows:

Case 1: For $r = 1$

The $r-$dynamic proper coloring $C : V(G) \to \{1, 2, \cdots k\}$ satisfying condition 1.1.

Since lemma 2.2 implies that $\chi_r(G) \geq 2$.

The r-dynamic proper 2-coloring is as follows:

$$c(v_i) = C_1(v_{1i}) = \begin{cases} 1, \text{for i is odd}, 1 \leq i \leq 2n \\ 2, \text{for i is even}, 1 \leq i \leq 2n \end{cases} \quad C_1(v_{2i}) = \begin{cases} 2, \text{for i is odd}, 1 \leq i \leq 2n \\ 1, \text{for i is even}, 1 \leq i \leq 2n \end{cases}$$

An easy verification shows that $C : V(G) \to \{1, 2\}$.
Hence, $\chi_r(G) = 2$, for $r = 1$.

Case 2: For $r = 2$ and $2n = 3t, t \geq 4$

By lemma 3.3, the lower bound for $r = 2$ is $\chi_r(C(G_n)) \geq 3$.
Consider the mapping $C : V(G) \to \{1, 2, \cdots k\}$ such that

$$C_2(v_{1i}) = \begin{cases} 1, & i = 3t+1, t \geq 0, 1 \leq i \leq 2n \\ 2, & i = 3t+2, t \geq 0, 1 \leq i \leq 2n \\ 3, & i = 3t, t \geq 1, 1 \leq i \leq 2n \end{cases} \quad C_2(v_{2i}) = \begin{cases} 3, & i = 3t+1, t \geq 0, 1 \leq i \leq 2n \\ 1, & i = 3t+2, t \geq 0, 1 \leq i \leq 2n \\ 2, & i = 3t, t \geq 1, 1 \leq i \leq 2n \end{cases}$$

It is straight forwardly verified that condition 1.1 holds and hence $\chi_r(G) = 3$.

Case 3: For $r = 2$, $2n = 3t + 1$ and $2n = 3t + 2$, for some positive integer t.

By lemma 3.3, the lower bound for $r = 2$ is $\chi_r(C(G_n)) \geq 4$.
In this case, the 2-dynamic 4-proper coloring $C : V(G) \to \{1, 2, 3, \cdots k\}$ satisfying 1.1 is given.
Subcase $n = 3t + 1$, for some positive integer t.

$$C_3(v_{1i}) = \begin{cases} 1, \text{for i is odd}, 1 \leq i \leq 2n \\ 2, \text{for i is even}, 1 \leq i \leq 2n \end{cases} \quad C_3(v_{2i}) = \begin{cases} 4, \text{for i is odd}, 1 \leq i \leq 2n \\ 3, \text{for i is even}, 1 \leq i \leq 2n \end{cases}$$

It is verified that the map C is a proper coloring satisfying condition 1.1 and hence $\chi_r(G) = 4$, for $r = 2$.

Case 4: For $r = 3$, $2n = 12t$ and $2n = 12t + 10, t \geq 4$

By lemma 3.3, the lower bound for $r = 3$ is $\chi_r(C(G_n)) \geq 6$.
In this case, the 3-dynamic 6-proper coloring $C : V(G) \to \{1, 2, 3, \cdots k\}$ satisfying 1.1 is given.
Subcase $2n = 12t$, for some positive integer t.

$$C_4(v_{1i}) = \begin{cases} 1, & i = 3t+1, t \geq 0, 1 \leq i \leq 2n \\ 2, & i = 3t+2, t \geq 0, 1 \leq i \leq 2n \\ 3, & i = 3t, t \geq 1, 1 \leq i \leq 2n \end{cases} \quad C_4(v_{2i}) = \begin{cases} 6, & i = 3t+1, t \geq 0, 1 \leq i \leq 2n \\ 5, & i = 3t+2, t \geq 0, 1 \leq i \leq 2n \\ 4, & i = 3t, t \geq 1, 1 \leq i \leq 2n \end{cases}$$

It is verified that the map C is a proper coloring satisfying condition 1.1 and hence $\chi_r(G) = 6$, for $r = 3$.

Subcase $2n = 12t + 10$, for some positive integer t.

$$C_5(v_{1i}) = \begin{cases} 1, & i = 3t+1, i = 2n-3, t \geq 0, 1 \leq i \leq 2n-5 \\ 2, & i = 3t+2, i = 2n-2, t \geq 0, 1 \leq i \leq 2n-5 \\ 3, & i = 3t, i = 2n-1, t \geq 0, 1 \leq i \leq 2n-5 \\ 4, & for \quad i = 2n-4 \quad i = 2n \end{cases}$$

$$C_5(v_{2i}) = \begin{cases} 6, & i = 3t+2, i = 2n-3, t \geq 0, 1 \leq i \leq 2n-4 \\ 4, & i = 3t, i = 2n-2, t \geq 0, 1 \leq i \leq 2n-4 \\ 5, & i = 3t+1, i = 2n-1, t \geq 0, 1 \leq i \leq 2n-4 \\ 2, & for \quad i = 2n \end{cases}$$

It is verified that the map C is a proper coloring satisfying condition 1.1 and hence $\chi_r(G) = 6$, for $r = 3$.

Case 5: For $r = 3$
By lemma 3.5, the lower bound for $r = 3$ is $\chi_r(C(G_n)) \geq 7$.

Subcase $2n = 12t + 8$, for some positive integer t.

$$C_6(v_{1i}) = \begin{cases} 1, & i = 3t+1, i = 2n-3, t \geq 0, 1 \leq i \leq 2n-5 \\ 2, & i = 3t+2, i = 2n-2, t \geq 0, 1 \leq i \leq 2n-5 \\ 3, & i = 3t, i = 2n-1, t \geq 1, 1 \leq i \leq 2n-5 \\ 4, & i = 2n-4 \ \ andi = 2n \end{cases}$$

$$C_6(v_{2i}) = \begin{cases} 7, & i = 3t+1, i = 2n-3, t \geq 0, 1 \leq i \leq 2n-4 \\ 6, & i = 3t+2, i = 2n-1, t \geq 0, 1 \leq i \leq 2n-4 \\ 5, & i = 3t, i = 2n, t \geq 1, 1 \leq i \leq 2n-4 \\ 4, & for \ \ i = 2n-2 \end{cases}$$

It is verified that the map C is a proper coloring satisfying condition 1.1 and hence $\chi_r(G) = 7$, for $r = 3$.

□

4 Concluding Remarks

The study of r - dynamic coloring on the brick product graph $(C_{2n}, 2, p)$ highlights the intricate relationship between structural parameters of the graph and the chromatic constraints imposed by the r - dynamic condition. The results demonstrate that the additional neighborhood color diversity requirement significantly elevates the chromatic number compared to classical coloring. Moreover, the periodic and layered structure of the brick product ensures that the derived bounds are both tight and extendable to broader classes of product graphs. These findings not only enrich the theory of r - dynamic coloring but also provide a foundation for exploring its applications in network design, frequency assignment, and fault-tolerant systems where balanced color distribution across neighborhoods is essential. The study of r - dynamic coloring on the brick product graph $(C_{2n}, 2, p)$ highlights the intricate relationship between structural parameters of the graph and the chromatic constraints imposed by the r - dynamic condition. The results demonstrate that the additional neighborhood color diversity requirement significantly elevates the chromatic number compared to classical coloring. Moreover, the periodic and layered structure of the brick product ensures that the derived bounds are both tight and extendable to broader classes of product graphs. These findings not only enrich the theory of r - dynamic coloring but also provide a foundation for exploring its applications in network design, frequency assignment, and fault-tolerant systems where balanced color distribution across neighborhoods is essential.

References

[1] *I. H. Agustin, Dafik and A. K. Harsya, On $r-$Dynamic coloring of some graph operations,* Indonesian Journal of Combinatories 1*, (1) (2016), 22–30.*

[2] *B. Alspach, C. C. Chen and Kevin McAvaney, On a class of Hamiltonian leceable 3-regular graphs,* Discrete Mathematics*, 1511(1996), 19-38.*

[3] *Arika Indah Kristiana, Dafik, M. Imam Utoyo and Ika Hesti Agustin, On $r-$Dynamic chromatic number of the corronation of path and several graphs,* International Journal of Advanced Engineering Research and Science*, 4(4)(2017),96–101.*

[4] *J. A. Bondy, U. S. R. Murty, Graph Theory, Springer, 2008.*

[5] *T. Deepa, M. Venkatachalam, R. M. Falcón,* On the r-dynamic coloring of the direct product of a path with either a path or a cycle*, AIMS Mathematics,* **5** *(2020), 6496–6520.*

[6] *T. Deepa and M. Venkatachalam and R. M. Falcon, On the r-Dynamic Coloring of the Direct Product of a Path with either a Complete graph or a Wheel graph,* AIMS Mathematics*, 6(2020), 1470–1496.*

[7] *T. Deepa, M. Venkatachalam, Ismail Naci Cangul,* On r-dynamic chromatic number of some brick product graphs $C(2n, 1, p)$*, Asian-Europian Journal of Mathematics, 15(6)(2022), 2250102(1–11).*

[8] *H. Furmanczyk, J. Vernold Vivin and N. Mohanapriya, $r-$dynamic chromatic number of some line graphs,* Indian Journal of Pure and Applied Mathematics*, 49 (4) (2018), 591–600.*

[9] *S. Jahanbekam, Jaehoon Kim, Suil O, Douglas B. West, On $r-$Dynamic coloring of graph,* Discrete Applied Mathematics*, 206 (2016), 65–72.*

[10] *N. Mohanapriya, J. Vernold Vivin and M. Venkatachalam, δ- dynamic chromatic number of helm graph families,* Cogent Mathematics*, 3(2016), No. 1178411.*

[11] *N. Mohanapriya, J. Vernold Vivin, Johan Kok and M. Venkatachalam, On dynamic coloring of certain cycle-related graphs,* Arabian Journal of Mathematics*, (2018), pp 1–9.*

[12] *B. Montgomery, Dynamic coloring of graphs, ProQuest LLC, Ann Arbor, MI, (2001), Ph.D Thesis, West Virginia University.*

[13] *B. J Septory, Dafik, A. I. Kristiana, I. H. Agustin and D. A. R. Wardani, On $r-$Dynamic chromatic number of corronation of order two of any graphs with path graph,* IOP Conf. Series: Earth and Environmental Science*, 243(2019),012113.*

[14] *K.Srinivasa Rao and R. Murali, Rainbow connection in modified brick product graphs,* Far East J. Math. Sciences*, 101(2)(2017), 289–300.*

[15] *K.Srinivasa Rao, R. Murali and S. K. Rajendra, Rainbow connection in brick product graphs,* Bulletin of the International Mathematical Virtual Institute*, 8(2017), 55–66.*

[16] *K.Srinivasa Rao and R. Murali, Rainbow connection number in the brick product graphs $C(2n, m, r)$,* Int. J. of Math. Comb.*, 2(2017), 70–83.*

[17] *A. Taherkhani, r-Dynamic chromatic number of graphs,* Discrete Applied Mathematics*, 201(2016), 222–227.*

[18] *A. T. White, Graphs, Groups and Surfaces, American Elsevier, New York, 1973.*

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