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Sigma coloring on Mycielski of some graphs

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Abstract For a non$-$trivial connected graph G, let $\tau : V\left(G\right) \rightarrow N$ be a vertex coloring of G, here adjacent vertices may be colored the same. Let $v \in G$, $N\left(v\right)$ be the set of vertices adjacent to v. The color sum $\sigma\left(v\right)$ of v is the sum of the colors of the vertices in $N\left(v\right)$. Let τ be a sigma coloring of G if $\sigma\left(u\right) \neq \sigma\left(v\right)$, for every two adjacent vertices u and v of G. The minimum number of colors needed in a sigma coloring of a graph G is called its sigma chromatic number $\sigma\left(G\right)$. The bounds of $\sigma\left(G\right)$ are sharp. For every graph G, $\sigma\left(G\right) \leq \chi\left(G\right)$. In this article, we obtain the results for sigma coloring on mycielski graph of star, double star, triple star, comb graph, sunlet graph and ladder graph.

1 Introduction

A graph G is the mathematical structure consisting of two sets $V(G)$ (vertices of G) and $E(G)$ (edges of G). Graph theory is a mathematical concept about graphs and its properties, plays a vital role in this modern era. Graph coloring is a important concept in the graph theory which contributes in network connections, data analysis, textile engineering areas, etc..,.There are many types of coloring occurs, sigma coloring is one of the type of neighbor$-$ distinguishing coloring. The $\sigma-$ coloring was introduced by Gary chartrand et.al in 2008 [11]. Gary chartrand et.al presented his first paper in 2010[5], obtaining sigma chromatic number for complete graphs, complete $r-$ partite graphs with $r \geq 2$ and cycles. He also stated that $\sigma\left(G\right) \leq \chi\left(G\right)$ where $\chi\left(G\right)$ is the minimum number of colors used in the proper vertex coloring G. Preethi K Pillai and J suresh kumar recently determined sigma coloring for Barbell graph, Twig graph, shell graph, tadpole graph, lollipop graph, fusing all the vertices of cycle and duplication of every edge by a vertex in C_n in 2023 [10]. Also they obtain sigma chromatic number for cycle related graphs [8]. J. Veninstine vivek. and P. Xavier investigated equitable total coloring of spiltting on double wheel and sunlet graphs in 2021[4]. T.V. Satees kumar and S. Meenakshi consider ladder graphs for their work under properly lucky labeling in 2022[3]. C. Yogalakshmi and B.J. Balamurugan investigated sigma coloring under mycielski transformation and conclude that $\sigma\left(G\right) \leq \chi\left(G\right)$ hold for path, complete bipartite graph, complete graph, cycle,wheel graph and helm graph in 2024[7] . P. Kowsalya and D. Vijayalakshmi discussed mycielski graphs under star and wheels in 2024 [2]. Also N.K. Sudev, C. Susanth and S.J. Kalayathankal investigated mycielski under rainbow neighbourhood number in 2018[6] . In this paper, our primary goal is to find sigma chromatic number of mycielski graph of some graphs.

Let G be simple connected graph and $\tau : V\left(G\right) \rightarrow N$, where N is set of positive integers, be a coloring of the vertices in G. We call $\tau\left(v\right)$ as the color of the vertex,v.For any $v \in V\left(G\right)$, let $\sigma\left(v\right)$ denotes the sum of colors of the vertices adjacent to v then τ is called a sigma coloring [10] of G if for any two adjacent vertices $u, v \in V\left(G\right)$, $\sigma\left(u\right) \neq \sigma\left(v\right)$. The least number of colors needed in a sigma coloring of a graph G is called its sigma chromatic number $\sigma\left(G\right)$[10]. The mycielski graph [6]of a given graph G with vertex set $V(G) = \{v_1, v_2, ..., v_n\}$ is the graph obtained by introducing a new vertex u_i and $U = \{u_i : 1 \leq i \leq n\}$ corresponding to each vertex v_i. Add edges from each vertex u_i of U to the vertex v_j if $v_iv_j \in E(G)$. Take another vertex u and add edges from u to all vertices in U. The new graph thus obtained is called the mycielski graph of G and is denoted by $\mu(G)$. The path graph P_n, [9]is a graph with n vertices that can be enumerated such that two vertices are adjacent if and only if they are consecutive in the enumeration. The cycle graph C_n,[9] is a graph with n vertices that can be enumerated such that two vertices are adjacent if and only if they are consecutive in the enumeration or are the first and last vertex in the enumeration. A tree containing exactly one vertex that is not a pendent vertex is called a star graph $K_{1,n}$[2]. The double star graph $K_{1,n,n}$ [2] is obtained from a star graph $K_{1,n}$ by joining a new pendent edge to each existing pendent node. The triple star graph $K_{1,n,n,n}$ [2] is obtained from a double star graph $K_{1,n,n}$ by joining a new pendent edge to each existing pendent node. The comb graph [1] is represented by $P_n \odot K_1$. The P_n is a path graph with $(n-1)$ edges and n vertices. The graph is constructed by connecting each vertex in the path with a pendent edge. The sunlet graph [4] composed of $2n$ vertices is acquired by adding n pendant edges to the cycle C_n and represented by S_n. The Cartesian product of P_n and K_2 is called ladder graph [3]. and is denoted as L_n, where $L_n = P_n \times K_2$.

The neighborhood of v, $N(v)$, is the set of all vertices adjacent to v. the closed neighborhood of v, $N[v]$, is the set $N(v) \cup v$. If for two vertices v and u have $N[u] = N[v]$ they are called strong twins[5].

Lemma 1.1 [5]

Let G be a nontrivial connected graph. Then $\sigma(G) = 1$, if and only if, every two adjacent vertices of G have different degrees.

Lemma 1.2 [5]

If u and v are two adjacent vertices in a graph G such that $N[u] = N[v]$, then $c(u) \neq c(v)$ for every sigma coloring c of G.

Lemma 1.3 [5]

If H is a complete subgraph of order k in a graph G such that $N[u] = N[v]$ for every two vertices u and v of H, then $\sigma(G) \geq k$

Lemma 1.4 [5]

Let G be a nontrivial connected graph of order n. Then $\sigma(G) = n$ if and only if $G = K_n$

2 Main results

In this section, discussion is made under sigma coloring and sigma chromatic number of mycielski graph of some new graphs.

Theorem 2.1

For $n \in \mathbb{N}$, where $(n \geq 1)$, the sigma chromatic number of mycielski graph of star graph $\mu(K_{1,n})$ is determined to be $\sigma(\mu(K_{1,n}) = \begin{cases} 3 \text{ if n=1} \\ 2 \text{ if } n \geq 2 \end{cases}$

Proof:

Let the vertex set of star graph $K_{1,n}$ be $V(K_{1,n}) = \{v\} \cup \{v_i : 1 \leq i \leq n\}$. For construction of Mycielski graph of $(K_{1,n})$, we introduce vertices $w, u, u_1, u_2, ..., u_n$. Therefore $V(\mu(K_{1,n})) = \{u, v, w\} \cup \{v_i : 1 \leq i \leq n\} \cup \{u_i : 1 \leq i \leq n\}$ and the edge set $E(\mu(K_{1,n})) = \{vv_i : 1 \leq i \leq n\} \cup \{vu_i : 1 \leq i \leq n\} \cup \{uw\} \cup \{u_iw : 1 \leq i \leq n\} \cup \{uv_i : 1 \leq i \leq n\}$.

Case(i): Assume n=1, then

Define the vertex color $\tau : V(\mu(K_{1,n})) \longrightarrow \{1, 2, 3\}$ as follows

$\tau(v) = 2$; $\tau(v_1) = 2$; $\tau(u) = 1$; $\tau(u_1) = 2$; $\tau(w) = 3$.

From the above procedure, it is evident that no pairwise adjacent vertices recieves same neighbourhood sum. Assume τ is a $\sigma-$ coloring with $\sigma(\mu(K_{1,n})) \leq 3$. Then consider $\sigma(\mu(K_{1,n})) = 1$, then $\sigma(v) = \sigma(v_1)$ where v and v_1 are adjacent which violates the condition of sigma coloring. Since by lemma 2.1, So $\sigma(\mu(K_{1,n}) \neq 1$. Next,$\sigma(\mu(K_{1,n})) = 2$, then atleast two vertices which are adjacent receives same neighbourhood sum. So, $\sigma(\mu(K_{1,n}) \neq 2$. Therefore $\sigma(\mu(K_{1,n})) = 3$.

Case(ii): Assume $n \geq 2$, then

Define the vertex color $\tau : V(\mu(K_{1,n})) \longrightarrow \{1, 2\}$ as follows

$\tau(v) = 1$; $\tau(v_i) = \begin{cases} 2 \text{ if i is odd} \\ 1 \text{ if i is even} \end{cases}$ for $1 \leq i \leq n$

$\tau(u) = 2$; $\tau(u_i) = 2$; for $1 \leq i \leq n$

$\tau(w) = 1$.

From the above, it is true that all the pairwise adjacent vertices receives distinct neighbourhood sum. Assume τ is a $\sigma-$ coloring with $\sigma(\mu(K_{1,n})) \leq 2$. Then consider $\sigma(\mu(K_{1,n})) = 1$, then u and w are adjacent and are of same degree receives same neighbourhood sum. Since by lemma 2.1, So, $\sigma(\mu(K_{1,n}) \neq 1$. Therefore $\sigma(\mu(K_{1,n})) = 2$.

Theorem 2.2
For $n \in \mathbb{N}$, where $(n \geq 1)$, the sigma chromatic number of mycielski graph of double star graph $\mu(K_{1,n,n})$ is determined to be $\sigma(\mu(K_{1,n,n}) = \begin{cases} 2 \text{ if } n \leq 3 \\ 1 \text{ if } n \geq 4 \end{cases}$

Proof:
Consider the vertex set of double star graph $K_{1,n,n}$ be $V(K_{1,n,n}) = \{v\} \cup \{v_i : 1 \leq i \leq n\} \cup \{u_i : 1 \leq i \leq n\}$. By adding, the vertices $w, v^{'}, v_1^{'}, v_2^{'}, ..., v_n^{'}, u_1^{'}, u_2^{'}, ..., u_n^{'}$, we can construct the mycielski graph with vertex set $V(\mu(K_{1,n,n})) = \{v\} \cup \{v^{'}\} \cup \{w\} \cup \{v_i : 1 \leq i \leq n\} \cup \{u_i : 1 \leq i \leq n\} \cup \{v_i^{'} : 1 \leq i \leq n\} \cup \{u_i^{'} : 1 \leq i \leq n\}$ and the edge set $E(\mu(K_{1,n})) = \{vv_i : 1 \leq i \leq n\} \cup \{v_iu_i : 1 \leq i \leq n\} \cup \{vv_i^{'}; 1 \leq i \leq n\} \cup \{v_iu_i^{'} : 1 \leq i \leq n\} \cup \{v^{'}v_i : 1 \leq i \leq n\} \cup \{wv^{'}\} \cup \{wv_i^{'} : 1 \leq i \leq n\} \cup \{wu_i^{'}; 1 \leq i \leq n\} \cup \{v_i^{'}u_i; 1 \leq i \leq n\}$.
Case(i): when $n \leq 3$, then
Define the coloring function $\tau : V(\mu(K_{1,n,n})) \longrightarrow \{1, 2\}$ as follows
$\tau(v) = 1; \tau(v^{'}) = 2; \tau(w) = 1$
$\tau(v_i) = 2$; for $1 \leq i \leq n$
$\tau(u_i) = 1$; for $1 \leq i \leq n$
$\tau(v_i^{'}) = 2$; for $1 \leq i \leq n$
$\tau(u_i^{'}) = 2$; for $1 \leq i \leq n$
From the above procedure, it is true that all the pairwise adjacent vertices gets distinct neighbourhood sum. Assume τ is a $\sigma-$ coloring with $\sigma(\mu(K_{1,n,n})) \leq 2$.
(i): If $n = 1$, then the vertices w and $u^{'}$ are adjacent and of same degree gets equal neighbourhood sum.
(ii): If $n = 2$, then the vertices v and v_i $(1 \leq i \leq n)$ are adjacent and are of same degree receives equal neighbourhood sum.
(iii): If $n = 3$, then the vertices $v^{'}$ and v_i $(1 \leq i \leq n)$ are adjacent and are of equal degree receives equal neighbourhood sum. Since by lemma 2.1, So, $\sigma(\mu(K_{1,n,n})) \neq 1$ for $n \leq 3$, therefore $\sigma(\mu(K_{1,n,n})) = 2$.
Case(ii): when $n \geq 4$, then
Define the coloring function $\tau : V(\mu(K_{1,n,n})) \longrightarrow \{1\}$ as follows
From the structure of mycielski graph of double star graph $\mu(K_{1,n,n})$, it is clear that all the pairwise adjacent vertices gets distinct neighbourhood sum. Since $\mu(K_{1,n,n}))$ is a non-empty graph atleast one color is needed to color the graph to satisfy sigma coloring, so $\sigma(\mu(K_{1,n,n})) = 1$.

Theorem 2.3
For $n \in \mathbb{N}$, where $(n \geq 1)$, the sigma chromatic number of mycielski graph of triple star graph $\mu(K_{1,n,n,n})$ is determined to be $\sigma(\mu(K_{1,n,n,n})) = 2$

Proof:
Let the vertex set of triple star graph $K_{1,n,n,n}$ be $V(K_{1,n,n,n}) = \{v\} \cup \{v_i : 1 \leq i \leq n\} \cup \{u_i : 1 \leq i \leq n\} \cup \{w_i : 1 \leq i \leq n\}$. By adding, the new vertices $w, v^{'}, v_1^{'}, v_2^{'}, ..., v_n^{'}, u_1^{'}, u_2^{'}, ..., u_n^{'}, w_1^{'}, w_2^{'}, ..., w_n^{'}$, we can construct the mycielski graph of triple star graph with vertex set $V(\mu(K_{1,n,n,n})) = \{v\} \cup \{v^{'}\} \cup \{w\} \cup \{v_i : 1 \leq i \leq n\} \cup \{u_i : 1 \leq i \leq n\} \cup \{w_i; 1 \leq i \leq n\} \cup \{v_i^{'} : 1 \leq i \leq n\} \cup \{u_i^{'} : 1 \leq i \leq n\} \cup \{w_i^{'} : 1 \leq i \leq n\}$ and the edge set $E(\mu(K_{1,n})) = \{vv_i : 1 \leq i \leq n\} \cup \{v_iu_i : 1 \leq i \leq n\} \cup \{u_iw_i : 1 \leq i \leq n\} \cup \{v_iv_i^{'}; 1 \leq i \leq n\} \cup \{v_iu_i^{'}; 1 \leq i \leq n\} \cup \{u_iw_i^{'}; 1 \leq i \leq n\} \cup \{v^{'}v_i; 1 \leq i \leq n\} \cup \{v_i^{'}u_i; 1 \leq i \leq n\} \cup \{u_i^{'}w_i; 1 \leq i \leq n\} \cup \{wv^{'}\} \cup \{wv_i^{'}; 1 \leq i \leq n\} \cup \{wu_i^{'}; 1 \leq i \leq n\} \cup \{ww_i^{'}; 1 \leq i \leq n\}$.
Define the coloring function $\tau : V(\mu(K_{1,n,n,n})) \longrightarrow \{1, 2\}$ as follows
$\tau(v) = 1; \tau(v^{'}) = 2; \tau(w) = 1;$
$\tau(v_i) = 2$; for $1 \leq i \leq n$
$\tau(u_i) = 1$; for $1 \leq i \leq n$
$\tau(w_i) = 2$; for $1 \leq i \leq n$
$\tau(v_i^{'}) = 2$; for $1 \leq i \leq n$

$\tau(u_i^{'}) = 2$; for $1 \leq i \leq n$
$\tau(w_i^{'}) = 2$; for $1 \leq i \leq n$
From the above procedure, it is obvious that all the pairwise adjacent vertices gets distinct neighbourhood sum. Assume τ is a $\sigma-$ coloring with $\sigma(\mu(K_{1,n,n,n})) \leq 2$.
Assume τ is a $\sigma-$ coloring with $\sigma(\mu(K_{1,n,n,n})) \leq 2$. Then consider $\sigma(\mu(K_{1,n,n,n})) = 1$, then v_i and u_i $(1 \leq i \leq n)$ are adjacent and are of equal degree receives same neighbourhood sum. Since by lemma 2.1, So, $\sigma(\mu(K_{1,n,n,n}) \neq 1$. Therefore $\sigma(\mu(K_{1,n,n,n})) = 2$.

Theorem 2.4
For $n \in \mathbb{N}$, where $(n \geq 2)$, the sigma chromatic number of mycielski graph of comb graph $\mu(P_n \odot K_1)$ is determined to be $\sigma(\mu(P_n \odot K_1)) = 2$

Proof:
Let the vertex set of comb graph $P_n \odot K_1$ be $V(P_n \odot K_1) = \{u_i; 1 \leq i \leq n\} \cup \{v_i; 1 \leq i \leq n\}$. By introducing new vertices $w, u_1^{'}, u_2^{'}, ..., u_n^{'}, v_1^{'}, v_2^{'}, ..., v_n^{'}$, we can construct mycielski graph of comb graph and its vertex set $V(\mu(P_n \odot K_1)) = \{u_i; 1 \leq i \leq n\} \cup \{v_i; 1 \leq i \leq n\} \cup \{u_i^{'}; 1 \leq i \leq n\} \cup \{v_i^{'}; 1 \leq i \leq n\} \cup \{w\}$ and the edge set $E(\mu(P_n \odot K_1)) = \{u_i u_{i+1}; 1 \leq i \leq n-1\} \cup \{u_i v_i : 1 \leq i \leq n\} \cup \{u_i u_{i+1}^{'}; 1 \leq i \leq n-1\} \cup \{u_i v_i^{'}; 1 \leq i \leq n\} \cup \{u_i^{'} u_{i+1}; 1 \leq i \leq n-1\} \cup \{u_i^{'} v_i; 1 \leq i \leq n\} \cup \{v_i^{'} w; 1 \leq i \leq n\} \cup \{v_i^{'} w; 1 \leq i \leq n\}$.
Define the coloring function $\tau : V(\mu(P_n \odot K_1)) \longrightarrow \{1, 2\}$ as follows

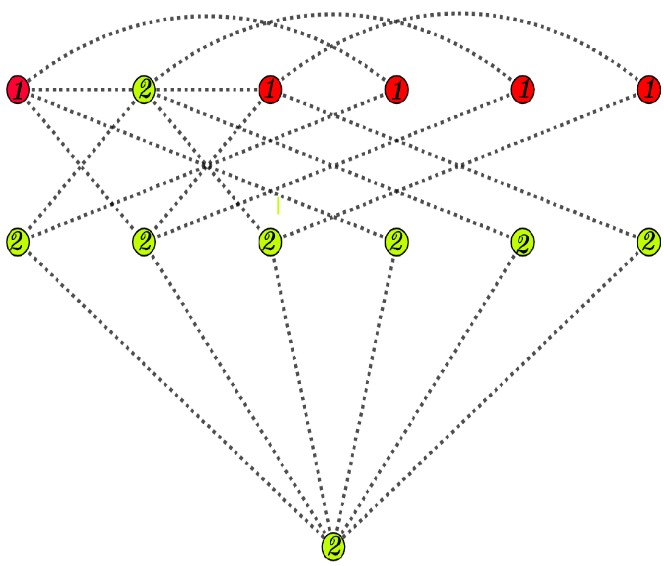

Figure 1. Mycielski graph of comb graph $\mu(P_3 \odot K_1)$

Case(i): If n=2, then
$\tau(u_i) = i$; for $1 \leq i \leq n$
$\tau(v_i) = 1$; for $1 \leq i \leq n$
$\tau(u_i^{'}) = 2$; for $1 \leq i \leq n$
$\tau(v_i^{'}) = 2$; for $1 \leq i \leq n$
$\tau(w) = 1$.
Case(ii): If $n \geq 3$, then

$$\tau(u_i) = \begin{cases} 1 \text{ if i is odd} \\ 2 \text{ if i is even} \end{cases} \quad \text{for } 1 \leq i \leq n$$

subcase(i):If n is odd, then
$\tau(v_i) = 1$; for $1 \leq i \leq n$

$\tau(u_i^{'}) = 2$; for $1 \leq i \leq n$
$\tau(v_i^{'}) = 2$; for $1 \leq i \leq n$
$\tau(w) = 2$.
subcase(ii): If n is even, then
$\tau(v_i) = 2$; for $1 \leq i \leq n$
$\tau(u_i^{'}) = 1$; for $1 \leq i \leq n$
$\tau(v_i^{'}) = 2$; for $1 \leq i \leq n$
$\tau(w) = 1$.
From the above, it is obvious that no two adjacent vertices gets same neighbourhood sum. Assume τ is a $\sigma-$ coloring with $\sigma(\mu(P_n \odot K_1)) \leq 2$. Suppose $\sigma(\mu(P_n \odot K_1)) = 1$, then
(i) For $n = 2$, u_1 and u_2 are adjacent and of same degree. so $\sigma(u_1) = \sigma(u_2)$.
(ii) For $n \geq 3$, u_i and u_{i+1} $(2 \leq i \leq n-2)$ are adjacent and are of equal degree. So $\sigma(u_i) = \sigma(u_{i+1})$ which contradicts the rule of sigma coloring. Since by lemma 2.1, so $\sigma(\mu(P_n \odot K_1)) \neq 1$. Therefore $\sigma(\mu(P_n \odot K_1)) = 2$. The sigma coloring of $\mu(P_3 \odot K_1)$ is illustrated in figure1, which clearly demonstrates how 2 colors are used to avoid neighborhood sum conflicts.

Theorem 2.5
For $n \in \mathbb{N}$, where $(n \geq 3)$, the sigma chromatic number of mycielski graph of sunlet graph $\mu(C_n \odot K_1)$ is determined to be $\sigma(\mu(C_n \odot K_1)) = \begin{cases} 2 \text{ if } n \text{ is even} \\ 3 \text{ if } n \text{ is odd} \end{cases}$

Proof:
Let the vertex set of sunlet graph $C_n \odot K_1$ be $V(C_n \odot K_1) = \{u_i; 1 \leq i \leq n\} \cup \{v_i; 1 \leq i \leq n\}$. By introducing new vertices $w, u_1^{'}, u_2^{'}, ..., u_n^{'}, v_1^{'}, v_2^{'}, ..., v_n^{'}$, we can construct mycielski graph ofsunlet graph and its vertex set $V(\mu(C_n \odot K_1)) = \{u_i; 1 \leq i \leq n\} \cup \{v_i; 1 \leq i \leq n\} \cup \{u_i^{'}; 1 \leq i \leq n\} \cup \{v_i^{'}; 1 \leq i \leq n\} \cup \{w\}$ and the edge set $E(\mu(C_n \odot K_1)) = \{u_i u_{i+1}; 1 \leq i \leq n-1\} \cup \{u_i v_i : 1 \leq i \leq n\} \cup \{u_i u_{i+1}^{'}; 1 \leq i \leq n-1\} \cup \{u_i v_i^{'}; 1 \leq i \leq n\} \cup \{u_i^{'} u_{i+1}; 1 \leq i \leq n-1\} \cup \{u_i^{'} v_i; 1 \leq i \leq n\} \cup \{v_i^{'} w; 1 \leq i \leq n\} \cup \{v_i^{'} w; 1 \leq i \leq n\} \cup \{u_1 u_n^{'}\} \cup \{u_n u_1^{'}\}$.
Case(i): If n=3, then
Define the coloring function $\tau : V(\mu(C_n \odot K_1)) \longrightarrow \{1, 2, 3\}$ as follows
$\tau(u_i) = i$; for $1 \leq i \leq n$
$\tau(v_i) = 3$; for $1 \leq i \leq n$
$\tau(u_i^{'}) = 2$; for $1 \leq i \leq n$
$\tau(v_i^{'}) = 2$; for $1 \leq i \leq n$
$\tau(w) = 1$.
From the above, it is obvious that no two adjacent vertices gets same neighbourhood sum. Assume τ is a $\sigma-$ coloring with $\sigma(\mu(C_n \odot K_1)) \leq 2$. Suppose $\sigma(\mu(C_n \odot K_1)) = 1$, then the vertices u_i and u_{i+1} $(1 \leq i \leq n-1)$ are adjacent and are of equal degree. So $\sigma(u_i) = \sigma(u_{i+1})$ which contradicts the rule of sigma coloring. Since by lemma 2.1, so $\sigma(\mu(C_n \odot K_1)) \neq 1$. Next, assume $\sigma(\mu(C_n \odot K_1)) = 2$, then atleast two vertices which are adjacent in the graph receives same neighbourhood sum. Therefore $\sigma(\mu(C_n \odot K_1)) \neq 1$. So $\sigma(C_n \odot K_1) = 3$.
Case(ii): If n is even, then
Define the coloring function $\tau : V(\mu(C_n \odot K_1)) \longrightarrow \{1, 2\}$ as follows
$\tau(u_i) = \begin{cases} 1 \text{ if } i \text{ is odd} \\ 2 \text{ if } i \text{ is even} \end{cases}$ for $1 \leq i \leq n$
$\tau(v_i) = 2$; for $1 \leq i \leq n$
$\tau(u_i^{'}) = 2$; for $1 \leq i \leq n$
$\tau(v_i^{'}) = 2$; for $1 \leq i \leq n$
$\tau(w) = 1$.
From the above, it is obvious that no two adjacent vertices gets same neighbourhood sum. Assume τ is a $\sigma-$ coloring with $\sigma(\mu(C_n \odot K_1)) \leq 2$. Suppose $\sigma(\mu(C_n \odot K_1)) = 1$, then the vertices u_i and u_{i+1} $(1 \leq i \leq n-1)$ are adjacent and are of equal degree. So $\sigma(u_i) = \sigma(u_{i+1})$ which contradicts the rule of sigma coloring. Since by lemma 2.1, so $\sigma(\mu(C_n \odot K_1)) \neq 1$.So $\sigma(C_n \odot K_1) = 2$.
Case(iii):If n is odd, then
Define the coloring function $\tau : V(\mu(C_n \odot K_1)) \longrightarrow \{1, 2, 3\}$ as follows

$\tau(u_1) = 2;$
$\tau(u_i) = \begin{cases} 1 \text{ if } i \text{ is even} \\ 3 \text{ if } i \text{ is odd} \end{cases}$ for $2 \leq i \leq n$
$\tau(v_i) = 3$; for $1 \leq i \leq n$
$\tau(u_i^{'}) = 2$; for $1 \leq i \leq n$
$\tau(v_i^{'}) = 2$; for $1 \leq i \leq n$
$\tau(w) = 1.$
From the above, it is clear that no two adjacent vertices gets same neighbourhood sum. Assume τ is a $\sigma-$ coloring with $\sigma(\mu(C_n \odot K_1)) \leq 2$. Suppose $\sigma(\mu(C_n \odot K_1)) = 1$, then the vertices u_i and u_{i+1} $(1 \leq i \leq n-1)$ are adjacent and are of equal degree. So $\sigma(u_i) = \sigma(u_{i+1})$ which contradicts the rule of sigma coloring. Since by lemma 2.1, so $\sigma(\mu(C_n \odot K_1)) \neq 1$. Next, assume $\sigma(\mu(C_n \odot K_1)) = 2$, then atleast two vertices which are adjacent in the graph receives same neighbourhood sum. Therefore $\sigma(\mu(C_n \odot K_1)) \neq 1$. So $\sigma(C_n \odot K_1) = 3$.

Theorem 2.6
For $n \in \mathbb{N}$, where $(n \geq 3)$, the sigma chromatic number of mycielski graph of ladder graph $\mu(L_n)$ is determined to be $\sigma(\mu(L_n)) = 2$

Proof:
Let the vertex set of ladder graph L_n be $V(L_n) = \{u_i; 1 \leq i \leq n\} \cup \{v_i; 1 \leq i \leq n\}$. By introducing new vertices $w, u_1^{'}, u_2^{'}, ..., u_n^{'}, v_1^{'}, v_2^{'}, ..., v_n^{'}$, we can construct mycielski graph of ladder graph and its vertex set $V(\mu(L_n)) = \{u_i; 1 \leq i \leq n\} \cup \{v_i; 1 \leq i \leq n\} \cup \{u_i^{'}; 1 \leq i \leq n\} \cup \{v_i^{'}; 1 \leq i \leq n\} \cup \{w\}$ and the edge set $E(\mu(L_n)) = \{u_i u_{i+1}; 1 \leq i \leq n-1\} \cup \{v_i v_{i+1} : 1 \leq i \leq n-1\} \cup \{u_i v_i; 1 \leq i \leq n\} \cup \{u_i v_i^{'}; 1 \leq i \leq n\} \cup \{v_i u_i^{'}; 1 \leq i \leq n\} \cup \{u_i u_{i+1}^{'}; 1 \leq i \leq n-1\} \cup \{u_i^{'} u_{i+1}; 1 \leq i \leq n-1\} \cup \{v_i v_{i+1}^{'}; 1 \leq i \leq n-1\} \cup \{v_i^{'} v_{i+1}; 1 \leq i \leq n-1\} \cup \{u_i^{'} w; 1 \leq i \leq n-1\} \cup \{v_1^{'} w; 1 \leq i \leq n-1\}$ respectively, where $u_i^{'}$ and $v_i^{'}$ be the set of newly introduced vertices such that $u_i^{'}$ correspond to the vertex u_i such that $u_i^{'}$ is adjacent to u_j and u_k provided u_j and u_k are adjacent to u_i. Likewise $v_i^{'}$ correspond to the vertex v_i such that $v_i^{'}$ is adjacent to v_j and v_k provided v_j and v_k are adjacent to v_i. Next take another vertex $w \neq u_1, v_i, u_i^{'}, v_i^{'}$ for $1 \leq i \leq k$ and add edges from $u_i^{'}$ and $v_i^{'}$ to w for all i.
Case(i): If n is odd, then
Define the coloring function $\tau : V(\mu(L_n)) \longrightarrow \{1, 2\}$ as follows
$\tau(u_i) = 2$; for $1 \leq i \leq n$
$\tau(v_i) = \begin{cases} 2 \text{ if } i \text{ is odd} \\ 1 \text{ if } i \text{ is even} \end{cases}$ for $1 \leq i \leq n$
$\tau(u_i^{'}) = \begin{cases} 1 \text{ if } i \text{ is odd} \\ 2 \text{ if } i \text{ is even} \end{cases}$ for $1 \leq i \leq n$
$\tau(v_i^{'}) = \begin{cases} 1 \text{ if } i \text{ is odd} \\ 2 \text{ if } i \text{ is even} \end{cases}$ for $1 \leq i \leq n$
$\tau(w) = 1.$
From the above, it is true that no two adjacent vertices gets same neighbourhood sum. Assume τ is a $\sigma-$ coloring with $\sigma(\mu(L_n)) \leq 2$. Suppose $\sigma(\mu(L_n)) = 1$, then the vertices u_i and u_{i+1} $(2 \leq i \leq n-2)$ are adjacent and are of equal degree. So $\sigma(u_i) = \sigma(u_{i+1})$ which breaks the rule of sigma coloring. Since by lemma 2.1, so $\sigma(\mu(L_n)) \neq 1$.So $\sigma(L_n) = 2$.
Case(ii): If n is even, then
Define the coloring function $\tau : V(\mu(L_n)) \longrightarrow \{1, 2\}$ as follows
$\tau(u_i) = \begin{cases} 1 \text{ if } i \text{ is odd} \\ 2 \text{ if } i \text{ is even} \end{cases}$ for $1 \leq i \leq n$
$\tau(v_i) = \begin{cases} 2 \text{ if } i \text{ is odd} \\ 1 \text{ if } i \text{ is even} \end{cases}$ for $1 \leq i \leq n$
$\tau(u_i^{'}) = \begin{cases} 2 \text{ if } i \text{ is odd} \\ 1 \text{ if } i \text{ is even} \end{cases}$ for $1 \leq i \leq n$

$\tau(v_i^{'}) = 2$; for $1 \leq i \leq n$
$\tau(w) = 1$.
From the above, it is notable that no two adjacent vertices gets same neighbourhood sum. Assume τ is a $\sigma-$ coloring with $\sigma(\mu(L_n)) \leq 2$. Suppose $\sigma(\mu(L_n)) = 1$, then the vertices v_i and v_{i+1} $(2 \leq i \leq n-2)$ are adjacent and are of equal degree. So $\sigma(v_i) = \sigma(v_{i+1})$ which contradicts the rule of sigma coloring. Therfore,by lemma 2.1, it can concluded that $\sigma(L_n) = 2$.

3 Conclusion

From the above theorems, it can be concluded that the sigma coloring properties are satisfied by mycielski operation of star, double star, triple star, comb graph, sunlet graph and ladder graph. Their respective sigma chromatic number has been successfully determined. Hence, from the above analysis confirms that the inequality $\sigma(G) \leq \chi(G)$ holds true across all considered graph operations.

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