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# On $r$ - Dynamic $k$ Coloring of Central graph of Ladder Graph Families

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**Abstract**

*The concept of $r$ - dynamic $k$ coloring of a graph $G$ refers to a proper vertex $k$ coloring where the number of colors assigned to the vertices in the neighborhood of each vertex $v$ is greater than or equal to the $min\{r, \Delta(G)\}$. This study focuses on the $r$ - dynamic coloring on Central graphs of various types of Ladder Graphs, which are derived from the Cartesian product of $P_2$ and $P_n$.*

## 1 Introduction

*Let $r$ and $k$ be positive integers. In 2001[4], Montgomery introduced a profound extension to the classical graph coloring problem, termed the 'r - dynamic proper $k$ - coloring'. For a given graph G, we utilize the notations $\delta(G)$ and $\Delta(G)$ to respectively represent the minimum and maximum degrees present within G. In essence, an $r$ - dynamic coloring of a graph involves a mapping function $c$ that assigns colors from a predefined set to each vertex in $V(G)$. This mapping follows two essential rules:*

- *If $u, v \in V(G)$ are adjacent vertices in G, then $c(u) = c(v)$ and*
- *for any $v \in V(G), |c(N(G(v)))|min|N(G(v))|, r.$*

*These conditions can be broken down into the proper coloring condition and the $r$ - adjacency condition. Consequently, the $r$ - dynamic chromatic number of a graph G, denoted as $\chi_r(G)$, is defined as the smallest positive integer $k$ for which G can be appropriately colored according to these conditions. Furthermore, it's important to note that the 1 - dynamic chromatic number is equivalent to the graph's chromatic number.*

*Additionally, we delve into the study of the 2 - dynamic chromatic number, also known as the dynamic chromatic number within references[7, 8, 9, 10, 14, 20, 21]. Extensive exploration has been conducted to establish upper and lower bounds for $\chi_d(G)$, where d represents the dynamic chromatic number. For instances where the maximum degree $\Delta(G)$ of a graph G exceeds or equals 3, Lai et al. have demonstrated that $\chi_d(G)$ is bounded above by $\Delta(G) + 1$, with the exception of the cycle graph $C5$.*

*In, an upper bound has been proposed for the dynamic chromatic number of a regular graph G in relation to the graph's independence number $(\alpha(G))$[1]. The bound is expressed as $\chi_2(G) \leq \chi(G) + 2log2\alpha(G) + O(1)$. Furthermore, Taherkhani[2] has contributed an upper bound for $\chi_2(G)$ by considering the chromatic number, maximum degree $(\Delta)$, and minimum degree $(\delta)$ of graph G. Specifically, the bound is represented as $\chi_2(G) - \chi(G) \leq [\Delta e/\delta log(2e(\Delta 2+1))]$, where G is a $d$ - regular graph.*

*Recent researchers focus more on $r$ - dynamic coloring[6, 7, 13, 16, 17, 19, 22, 23, 24, 25] where the range of $r$ varies from 2 to $\Delta(G)$ and also central graph on various graph and coloring are studied by many. In order to explore more on the variants of ladder graph[3, 15] this paper focuses on the study of central graph[11, 12, 18] of ladder graph families.*

## 2 Preliminaries

*To lay a groundwork for this paper, this section immerses itself in pivotal concepts and consequential outcomes of Graph Theory. These concepts will be referenced throughout the paper, and for an in-depth understanding of the topic, we recommend referring to[5].*

*Any Graph $G = (V(G), E(G))$ is formed by a set $V(G)$ of vertices and a set $E(G)$ of edges so that each edge contains two vertices, which are called adjacent. The subset of vertices in $V(G)$ that are adjacent to a vertex $v \in V(G)$ is its Neighborhood $N(v)$. The complete graph of order $n$ is denoted $K_n$. A Path $P_n$ between two distinct vertices $v, w \in V(G)$ is any ordered sequence of $n$ adjacent and pairwise distinct vertices $v_0 = v, v_1, \ldots, v_{n-2}, v_{n-1} = w_i$ in $V(G)$, with $n > 2$. A Ladder graph $L_n$ is defined by $L_n = P_n \times K_2$ where $P_n$ is a path with $n$ vertices and $\times$ denotes the Cartesian product and $K_2$ is a complete graph with two vertices.*

*A Diagonal ladder graph $DL_n$[15], is a ladder graph with $2n$ vertices and $5n-4$ edges and is got from a closed ladder graph $L_n$ with the additional edges $u_iv_{i+1}$ and $u_{i+1}v_i$ for $1 \leq i \leq n-1$. A Open Diagonal ladder graph $ODL_n$[15], is a ladder graph with $2n$ vertices and $5n-2$ edges and is got from a open ladder graph $OL_n$ with the additional edges $u_iv_{i+1}$ and $u_{i+1}v_i$ for $1 \leq i \leq n-1$. A Triangular ladder $TL_n$[15], is a ladder graph with $2n$ vertices and is got from a closed ladder graph $L_n$ with the additional edges $u_iv_{i+1}$ for $1 \leq i \leq n-1$. A Open Triangular ladder $OTL_n$[15], is a ladder graph with $2n$ vertices and is got from an open ladder $OL_n$ with the additional edges $u_iv_{i+1}$ for $1 \leq i \leq n-1$. A Slanting ladder graph $SL_n$[15], is a ladder graph with $2n$ vertices and is obtained from two paths of length $n-1$ with additional edges $u_iv_{i+1}$ for $1 \leq i \leq n-1$. The central graph $C(G)$[11] of a graph G is obtained from G by adding an extra vertex on each edge of G, and then joining each pair of vertices of the original graph which were previously non-adjacent.*

**Lemma 2.1.** *[8] $\chi_r(G) \geq min\{r, \Delta(G)\} + 1$*

**Lemma 2.2.** *[9] Let $n$ and $r$ be two positive integers such that $n > 2$. Then,*

*(i)* $$\chi_r(P_n) = \begin{cases} 2, & \text{if } r = 1, \\ 3, & otherwise. \end{cases}$$

*(ii)* $$\chi_r(C_n) = \begin{cases} 2, & \text{if } r = 1 \text{ and } n \text{ is even}, \\ 3, & if \begin{cases} r = 1 \text{ and } n \text{ is odd}, \\ r > 1 \text{ and } n \equiv 0 \bmod 3, \end{cases} \\ 4, & \text{if } r > 1 \text{ and } n \neq 5, \\ 5, & otherwise. \end{cases}$$

*(iii)* $\chi_r(K_n) = n$.

## 3 Main Results

**Lemma 3.1.** *The lower bound of $r$ - Dynamic Chromatic number of Central graph of Diagonal Ladder Graph $C(DL_n)$ is,*

$$\chi_r C(DL_n) \geq \begin{cases} \Delta(G) + 1, & 2 \leq r \leq \Delta(G) - 2 \\ r + 4, & for \;\; r = \Delta(G) - 1, \Delta(G). \end{cases}$$

*Proof.* The vertex and edge set of Central graph of Diagonal Ladder graph is represented as follows,

$$V(C(DL_n)) = \{u_i : 1 \leq i \leq n\} \cup \{v_i : 1 \leq i \leq n\} \cup \{a_i : 1 \leq i \leq n-1\} \cup \{b_i : 1 \leq i \leq n-1\}$$

$$\cup\{d_i : 1 \le i \le n\} \cup \{e_i : 1 \le i \le n+k, k \in N\}$$
$$E(C(DL_n)) = \{u_iu_j, u_iv_j, v_iv_j, v_iu_j : 1 \le i \le n, i+2 \le j \le n\}$$
$$\cup\{u_ia_j, v_ib_j : 1 \le i \le n, j = i-1, i\} \cup \{u_id_j, v_id_j : 1 \le i \le n, 1 \le j \le n\}$$
$$\cup\{u_ie_{\lceil\frac{j+2}{2}\rceil}, u_ie_{\lceil\frac{j+3}{2}\rceil}, v_ie_{\lceil\frac{j+2}{2}\rceil}, v_ie_{\lceil\frac{j+3}{2}\rceil} : 1 \le i \le n, 1 \le j \le n+k, k \in N\}$$

When $2 \le r \le \Delta(G) - 2$

It can be observed that the original vertices of the given graph $u_i, v_i$ are of same degree.

$$(i.e.,)d(u_i) = d(v_i) = \Delta(G)$$

Moreover from the edge set

$$E(C(DL_n)) = \{u_iu_j, u_iv_j, v_iv_j, v_i, u_j : 1 \le i \le n, i+2 \le j \le n\}$$

it can be observed that the vertices are directly connected to each other. So, in order to obtain the proper coloring we are in need of more colors. We prove this case using the method of contradiction. Let us assume that the coloring is done using $\Delta(G)$ colors. Consider $C(DL_4)$ the vertices $u_i$ are colored as $1, 5, 7, 4$ and let the vertices $v_i$ be colored as $2, 6, 5, 3$ but by the above coloring we can see that the edge $u_2v_3$ are colored by the same color 5 which is a contradicition. Therefore we are in need of $\Delta(G) + 1$ colors to satisfy the condition. When $r \ge \Delta(G) - 1$

This case is also proved using the technique of contradicition. In this case the vertices $a_i, b_i$ are introduced with a new color $r+3, r+4$. Suppose if the vertices $a_i, b_i$ are not colored with the new color it can be observed the vertices $u_i, v_i$ does not satisfy the r - adjacency. For example, in $C(DL_4)$ if the vertices of $a_i, b_i$ are colored as $3, 2, 1$ it can be seen that the vertex $u_1$ has r - adjacency less than $\Delta(G) - 1$. This is a contradiction, so we need $r + 4$ colors to satisfy the r - adjacency. □

**Theorem 3.2.** *The r - dynamic Chromatic number of Central graph of Diagonal Ladder Graph $C(DL_n)$ is,*

$$\chi_r C(DL_n) = \begin{cases} \Delta(G)+1, & 2 \le r \le \Delta(G)-2 \\ r+4, & for \;\; r = \Delta(G)-1, \Delta(G). \end{cases}$$

*Proof.* The cardinality of the vertex set and the edge set of the Cental graph of Diagonal Ladder Graph is $6n + k$, where $k = \{-1, 0, 1, 2, \ldots\}$ and $(\Delta(G) \times l) + m$, where $l = \{5, 6, 7, \ldots\}$ and $m \in R$ respectively. The Adjacency matrix is of order $(6n + k) \times (6n + k)$. Let us consider the first row of the adjacency matrix which corresponds to the vertex $u_1$ which comprises of $6n + k$ columns.

cccccccccccc

c(cccccccccccc) $u_1u_2 \cdots u_nv_1v_2 \cdots v_na_1a_2 \cdots a_{n-1}$
$00 \cdots 100 \cdots 110 \cdots 0$
$u_1b_1b_2 \cdots b_{n-1}d_1d_2 \cdots d_ne_1e_2 \cdots e_{n+k}$
$00 \cdots 010 \cdots 010 \cdots 0$

From the adjacency matrix we derive the neighbourhood set of each and every vertices such as

$$N(u_1) = \{a_1, d_1, e_1, u_3, u_4, \ldots, u_n, v_3, v_4, \ldots, v_n\}$$
$$N(u_2) = \{a_1, a_2, d_2, e_2, e_3, u_4, u_5, \ldots, u_n, v_4, v_5, \ldots, v_n\}$$
$$\vdots$$
$$N(u_n) = \{a_{n-1}, d_n, e_{n+k}, u_1, u_2, \ldots, u_{n-2}, v_1, v_2, \ldots, v_{n-2}\}$$
$$N(v_1) = \{b_1, d_1, e_2, v_3, v_4, \ldots, v_n, u_3, u_4, \ldots, u_n\}$$
$$N(v_2) = \{b_1, b_2, d_2, e_1, e_4, v_4, v_5, \ldots, v_n, u_4, u_5, \ldots, u_n\}$$
$$\vdots$$
$$N(v_n) = \{b_{n-1}, d_n, e_{n+(k-1)}, v_1, v_2, \ldots, v_{n-2}, u_1, u_2, \ldots, u_{n-2}\}$$

$$\begin{array}{c|c} N(a_1)=\{u_1,u_2\} & N(b_1)=\{v_1,v_2\} \\ N(a_2)=\{u_2,u_3\} & N(b_2)=\{v_2,v_3\} \\ \vdots & \vdots \\ N(a_{n-1})=\{u_{n-1},u_n\} & N(b_{n-1})=\{v_{n-1},v_n\} \end{array}$$

$$\begin{array}{c|c} N(e_1)=\{u_1,v_2\} & N(d_1)=\{u_1,v_1\} \\ N(e_2)=\{u_2,v_1\} & N(d_2)=\{u_2,v_2\} \\ \vdots & \vdots \\ N(e_{n+(k-1)})=\{u_{n-1},v_n\} & N(d_{n-1})=\{u_{n-1},v_{n-1}\} \\ N(e_{n+k})=\{u_n,v_{n-1}\} & N(d_n)=\{u_n,v_n\} \end{array}$$

When $2 \leq r \leq \Delta(G)-2$ In order to obtain a proper $r$ - dynamic coloring the mapping of the color set to the vertex set of $C(DL_n)$. Consider the mapping $f_1$,
$f_1 : V(C(DL_n)) \mapsto \{1, 2, \ldots, \Delta(G)+1\}$

$$f_1(u_i) = \begin{cases} 1, & when \;\; i=1 \\ 4, & when \;\; i=n \\ 5,7,9,\ldots,\Delta(G), & otherwise. \end{cases}, f_1(v_i) = \begin{cases} 2, & when \;\; i=1 \\ 3, & when \;\; i=n \\ 6,8,10,\ldots,\Delta(G)+1, & otherwise. \end{cases}$$

$$f_1(d_i) = \begin{cases} \Delta(G)+1, & when \;\; i=1 \\ 1,5,7,\ldots & otherwise. \end{cases}$$

Since each $d_i$ has only two neighbours $u_i, v_i$ the color of $u_i$ is applied to the vertex $d_{i+1}$ except the color 3 because the cardinality of the vertices $u_i, d_i$ is $n$, there is no $n+1$ and the vertex $d_1$ is colored with last color.

$$f_1(a_i, b_i) = \begin{cases} 4, & when \;\; i=1 \\ 3, & when \;\; i=2 \\ 2, & when \;\; i=3 \\ 1, & when \;\; i=4 \\ i, & for \;\; i \geq 5 \end{cases}$$

A special case exist for $n=4$ as there is only three vertices of $a_i, b_i$ exist and so the vertices are colored with the colors $3, 2, 1$.

$$f_1(e_i) = \begin{cases} i+4, & for \;\; 1 \leq i \leq n+k-2 \\ 4, & when \;\; i=n+k-1 \\ 3, & when \;\; i=n+k \end{cases}$$

Thus the color assigned using the above mentioned mapping yields $\Delta(G)+1$ independent color classes namely $C_1, C_2, \ldots C_{\Delta(G)+1}$.

$$C_1 = \{u_1, a_4, b_4, d_2\}$$
$$C_2 = \{v_1, a_3, b_3\}$$
$$C_3 = \{v_n, a_2, b_2, e_{n+k}\}$$
$$\vdots$$
$$C_{\Delta(G)+1} = \{v_{n-1}, d_1, e_{n+k-2}\}$$

By correlating the vertices in the coloring set with the neighbourhood set we can observe that every vertex have $r$ or $r+1$ variant colors of adjacent vertices. When $r \geq \Delta(G)-1$

The coloring of the vertices $u_i, v_i$ and $e_i$ remains the same. Only the vertices of $a_i, b_i$ and $d_i$ are colored with new colors.
When $r = \Delta(G)-1$, the vertices $a_i, b_i$ are colored with new colors

$$\Delta(G)+2, \;\; for \;\; i = odd$$
$$\Delta(G)+3, \;\; for \;\; i = even$$

When $r = \Delta(G)$ adjacency of $u_1, v_1$ remains $r-1$ since it has two neighbours with the color $\Delta(G)+1$ and so the vertex $d_1$ is colored with new color $r+4$.

Thus the color assigned using the above mentioned mapping yields $r+4$ independent color classes namely $C_1, C_2, \ldots, C_{r+3}, C_{r+4}$.

$$C_1 = \{u_1, d_2\}$$
$$C_2 = \{v_1\}$$
$$C_3 = \{v_n, e_{n+k}\}$$
$$\vdots$$
$$C_{r+3} = \{a_2, a_4, \ldots, a_{n-2}, b_2, b_4, \ldots, b_{n-2}\}$$
$$C_{r+4} = \{d_1\}$$

By correlating the vertices in the coloring set with the neighbourhood set we can observe that every vertex have $r$ or $r+1$ variant colors of adjacent vertices.
It can be observed that Case 1 and Case 2 deals with the upper bound,

$$\chi_r C(DL_n) \leq \begin{cases} \Delta(G)+1, & 2 \leq r \leq \Delta(G)-2 \\ r+4, & for \;\; r = \Delta(G)-1, \Delta(G). \end{cases}$$

The lower bound has been discussed through the lemma 3.1 which leads to acqiure the

$$\chi_r C(DL_n) = \begin{cases} \Delta(G)+1, & 2 \le r \le \Delta(G)-2 \\ r+4, & for \ \ r = \Delta(G)-1, \Delta(G). \end{cases}$$

□

**Lemma 3.3.** *The lower bound of r - Dynamic Chromatic number of Central graph of Open Diagonal Ladder Graph $C(ODL_n)$ is,*

$$\chi_r C(ODL_n) \ge \begin{cases} \Delta(G)+1, & 2 \le r \le \Delta(G)-2 \\ r+4, & for \ \ r = \Delta(G)-1, \Delta(G). \end{cases}$$

*Proof.* The vertex and edge set of Central graph of Open Diagonal Ladder graph is represented as follows,

$$V(C(ODL_n)) = \{u_i : 1 \le i \le n\} \cup \{v_i : 1 \le i \le n\} \cup \{a_i : 1 \le i \le n-1\} \cup \{b_i : 1 \le i \le n-1\}$$
$$\cup \{d_i : 1 \le i \le n-2\} \cup \{e_i : 1 \le i \le n+k, k \in N\}$$
$$E(C(ODL_n)) = \{u_1v_1, u_nv_n, u_iu_j, u_iv_j, v_iv_j, v_iu_j : 1 \le i \le n, i+2 \le j \le n\}$$
$$\cup \{u_ia_j, v_ib_j : 1 \le i \le n, j = i-1, i\} \cup \{u_id_j, v_id_j : 2 \le i \le n-1, 1 \le j \le n-2\}$$
$$\cup \{u_ie_{\lceil \frac{j+2}{2} \rceil}, u_ie_{\lceil \frac{j+3}{2} \rceil}, v_ie_{\lceil \frac{j+2}{2} \rceil}, v_ie_{\lceil \frac{j+3}{2} \rceil} : 1 \le i \le n, 1 \le j \le n+k, k \in N\}$$

The proof follows the same as lemma 3.1 □

**Theorem 3.4.** *The r - dynamic Chromatic number of Central graph of Open Diagonal Ladder Graph $C(ODL_n)$ is,*

$$\chi_r C(ODL_n) = \begin{cases} \Delta(G)+1, & 2 \le r \le \Delta(G)-2 \\ r+4, & for \ \ r = \Delta(G)-1, \Delta(G). \end{cases}$$

*Proof.* The cardinality of the vertex set and the edge set of the Central graph of Open Diagonal Ladder Graph is $6n+k$, where $k = \{-3,-2,-1,0,1,2,\ldots\}$ and $(\Delta(G) \times l) + m$, where $l = \{5,6,7,\ldots\}$ and $m = \{-1,0,1,2,\ldots\}$ respectively. The Adjacency matrix is of order $(6n+k) \times (6n+k)$ . Let us consider the first row of the adjacency matrix which corresponds to the vertex $u_1$ which comprises of $6n+k$ columns.

$cccccccccccc$

c(cccccccccccc) u$_1u_2 \cdots u_nv_1v_2 \cdots v_na_1a_2 \cdots a_{n-1}$
$00 \cdots 110 \cdots 110 \cdots 0$
$u_1b_1b_2 \cdots b_{n-1}d_1d_2 \cdots d_{n-2}e_1e_2 \cdots e_{n+k}$
$00 \cdots 000 \cdots 010 \cdots 0$

From the adjacency matrix we derive the neighbourhood set of each and every vertices such as

$$N(u_1) = \{a_1, e_1, u_3, u_4, \ldots, u_n, v_1, v_3, \ldots, v_n\}$$
$$N(u_2) = \{a_1, a_2, d_1, e_2, e_3, u_4, u_5, \ldots, u_n, v_4, v_5, \ldots, v_n\}$$
$$\vdots$$
$$N(u_n) = \{a_{n-1}, e_{n+k}, u_1, u_2, \ldots, u_{n-2}, v_1, v_2, \ldots, v_{n-2}, v_n\}$$

$$N(v_1) = \{b_1, e_2, v_3, v_4, \ldots, v_n, u_1, u_3, \ldots, u_n\}$$
$$N(v_2) = \{b_1, b_2, d_1, e_1, e_4, v_4, v_5, \ldots, v_n, u_4, u_5, \ldots, u_n\}$$
$$\vdots$$
$$N(v_n) = \{b_{n-1}, e_{n+(k-1)}, v_1, v_2, \ldots, v_{n-2}, u_1, u_2, \ldots, u_{n-2}, u_n\}$$

| | |
|---|---|
| $N(a_1) = \{u_1, u_2\}$ | $N(b_1) = \{v_1, v_2\}$ |
| $N(a_2) = \{u_2, u_3\}$ | $N(b_2) = \{v_2, v_3\}$ |
| $\vdots$ | $\vdots$ |
| $N(a_{n-1}) = \{u_{n-1}, u_n\}$ | $N(b_{n-1}) = \{v_{n-1}, v_n\}$ |
| $N(e_1) = \{u_1, v_2\}$ | $N(d_1) = \{u_2, v_2\}$ |
| $N(e_2) = \{u_2, v_1\}$ | $N(d_2) = \{u_3, v_3\}$ |
| $\vdots$ | $\vdots$ |
| $N(e_{n+(k-1)}) = \{u_{n-1}, v_n\}$ | $N(d_{n-3}) = \{u_{n-2}, v_{n-2}\}$ |
| $N(e_{n+k}) = \{u_n, v_{n-1}\}$ | $N(d_{n-2}) = \{u_{n-1}, v_{n-1}\}$ |

When $2 \le r \le \Delta(G)-2$ In order to obtain a proper $r$ - dynamic coloring the mapping of the color set to the vertex set of $C(ODL_n)$. Consider the mapping $f_1$,
$f_1 : V(C(ODL_n)) \mapsto \{1, 2, \ldots, \Delta(G)+1\}$

$$f_1(u_i) = \begin{cases} 1, & when \ \ i = 1 \\ 4, & when \ \ i = n \\ 5,7,9,\ldots,\Delta(G), & otherwise. \end{cases}, f_1(v_i) = \begin{cases} 2, & when \ \ i = 1 \\ 3, & when \ \ i = n \\ 6,8,10,\ldots,\Delta(G)+1, & otherwise. \end{cases}$$

$$f_1(d_i) = 1, 5, 7, \ldots$$

Since each $d_i$ has only two neighbours $u_{i+1}, v_{i+1}$ the color of $u_i$ is applied to the vertex $d_i$ except the color 3 because 3 is mapped to $v_n$.

$$f_1(a_i, b_i) = \begin{cases} 4, & when \;\; i = 1 \\ 3, & when \;\; i = 2 \\ 2, & when \;\; i = 3 \\ 1, & when \;\; i = 4 \\ i, & for \;\; i \geq 5 \end{cases}$$

A special case exist for $n = 4$ as there is only three vertices of $a_i, b_i$ exist and so the vertices are colored with the colors $3, 2, 1$.

$$f_1(e_i) = \begin{cases} i+4, & for \;\; 1 \leq i \leq n+k-2 \\ 4, & when \;\; i = n+k-1 \\ 3, & when \;\; i = n+k \end{cases}$$

Thus the color assigned using the above mentioned mapping yields $\Delta(G)+1$ independent color classes namely $C_1, C_2, \ldots C_{\Delta(G)+1}$.

$$C_1 = \{u_1, a_4, b_4, d_1\}$$
$$C_2 = \{v_1, a_3, b_3\}$$
$$C_3 = \{v_n, a_2, b_2, e_{n+k}\}$$
$$\vdots$$
$$C_{\Delta(G)+1} = \{v_{n-1}, d_1, e_{n+k-2}\}$$

By correlating the vertices in the coloring set with the neighbourhood set we can observe that every vertex have $r$ or $r+1$ variant colors of adjacent vertices. When $r \geq \Delta(G) - 1$

The coloring of the vertices $u_i, v_i$ and $d_i$ remains the same. Only the vertices of $a_i, b_i$ and $e_i$ are colored with new colors.

When $r = \Delta(G) - 1$, the vertices $a_i, b_i$ are colored with new colors

$$\Delta(G) + 2, \;\; for \;\; i = odd$$
$$\Delta(G) + 3, \;\; for \;\; i = even$$

When $r = \Delta(G)$ adjacency of $u_n, v_n$ remains $r - 1$ since it has two neighbours with the color 3, 4 respectively and so the vertices $e_{n+k}, e_{n+k-1}$ is colored with new color $r + 4$.

Thus the color assigned using the above mentioned mapping yields $r+4$ independent color classes namely $C_1, C_2, \ldots, C_{r+3}, C_{r+4}$.

$$C_1 = \{u_1, d_1\}$$
$$C_2 = \{v_1\}$$
$$C_3 = \{v_n\}$$
$$\vdots$$
$$C_{r+3} = \{a_2, a_4, \ldots, a_{n-2}, b_2, b_4, \ldots, b_{n-2}\}$$
$$C_{r+4} = \{e_{n+k}, e_{n+k-1}\}$$

By correlating the vertices in the coloring set with the neighbourhood set we can observe that every vertex have $r$ or $r+1$ variant colors of adjacent vertices.

It can be observed that Case 1 and Case 2 deals with the upper bound,

$$\chi_r C(ODL_n) \leq \begin{cases} \Delta(G) + 1, & 2 \leq r \leq \Delta(G) - 2 \\ r + 4, & for \;\; r = \Delta(G) - 1, \Delta(G). \end{cases}$$

The lower bound has been discussed through the lemma 3.3 which leads to acqiure the

$$\chi_r C(ODL_n) = \begin{cases} \Delta(G) + 1, & 2 \leq r \leq \Delta(G) - 2 \\ r + 4, & for \;\; r = \Delta(G) - 1, \Delta(G). \end{cases}$$

□

**Lemma 3.5.** *The lower bound of r - Dynamic Chromatic number of Central graph of Triangular Ladder Graph $C(TL_n)$ is,*

$$\chi_r C(TL_n) \geq \begin{cases} \Delta(G) + 1, & 2 \leq r \leq \Delta(G) - 3 \\ r + 4, & \Delta(G) - 2 \leq r \leq \Delta(G). \end{cases}$$

*Proof.* The vertex and edge set of Cental graph of Triangular Ladder graph is represented as follows,

$$V(C(TL_n)) = \{u_i : 1 \leq i \leq n\} \cup \{v_i : 1 \leq i \leq n\} \cup \{a_i : 1 \leq i \leq n-1\} \cup \{b_i : 1 \leq i \leq n-1\}$$
$$\cup \{d_i : 1 \leq i \leq n\} \cup \{e_i : 1 \leq i \leq n-1\}$$
$$E(C(TL_n)) = \{u_i u_j, u_i v_j, v_i v_j : 1 \leq i \leq n, i+2 \leq j \leq n\} \cup \{v_i u_j : 1 \leq i \leq n, i+1 \leq j \leq n\}$$
$$\{u_i a_j, v_i b_j : 1 \leq i \leq n, j = i-1, i\} \cup \{u_i d_j, v_i d_j : 1 \leq i \leq n, 1 \leq j \leq n\}$$
$$\cup \{u_i e_i, v_i e_{i-1} : 1 \leq i \leq n\}$$

The proof follows the same as lemma 3.1 □

**Theorem 3.6.** *The r - dynamic Chromatic number of Central graph of Triangular Ladder Graph $C(TL_n)$ is,*

$$\chi_r C(TL_n) = \begin{cases} \Delta(G)+1, & 2 \le r \le \Delta(G)-3 \\ r+4, & \Delta(G)-2 \le r \le \Delta(G). \end{cases}$$

*Proof.* The cardinality of the vertex set and the edge set of the Central graph of Triangular Ladder Graph is $6n-3$ and $(\Delta(G) \times l) + m$, where $l = \{4,5,6,\ldots\}$ and $m = \{4,6,8,\ldots\}$ respectively. The Adjacency matrix is of order $(6n-3) \times (6n-3)$ . Let us consider the first row of the adjacency matrix which corresponds to the vertex $u_1$ which comprises of $6n-3$ columns.

*cccccccccccc*

c(cccccccccccc) $u_1 u_2 \cdots u_n v_1 v_2 \cdots v_n a_1 a_2 \cdots a_{n-1}$
$00 \cdots 100 \cdots 110 \cdots 0$
$u_1 b_1 b_2 \cdots b_{n-1} d_1 d_2 \cdots d_n e_1 e_2 \cdots e_{n-1}$
$00 \cdots 010 \cdots 010 \cdots 0$

From the adjacency matrix we derive the neighbourhood set of each and every vertices such as

$$N(u_1) = \{a_1, d_1, e_1, u_3, u_4, \ldots, u_n, v_3, v_4, \ldots, v_n\}$$
$$N(u_2) = \{a_1, a_2, d_2, e_2, u_4, u_5, \ldots, u_n, v_1, v_4, v_5, \ldots, v_n\}$$
$$\vdots$$
$$N(u_n) = \{a_{n-1}, d_n, u_1, u_2, \ldots, u_{n-2}, v_1, v_2, \ldots, v_{n-1}\}$$

$$N(v_1) = \{b_1, d_1, u_2, u_3, u_4, \ldots, u_n, v_3, v_4, \ldots, v_n\}$$
$$N(v_2) = \{b_1, b_2, d_2, e_1, u_3, u_4, u_5, \ldots, u_n, v_4, v_5, \ldots, v_n\}$$
$$\vdots$$
$$N(v_n) = \{b_{n-1}, d_n, e_{n-1}, u_1, u_2, \ldots, u_{n-2}, v_1, v_2, \ldots, v_{n-2}\}$$

| | |
|---|---|
| $N(a_1) = \{u_1, u_2\}$ | $N(b_1) = \{v_1, v_2\}$ |
| $N(a_2) = \{u_2, u_3\}$ | $N(b_2) = \{v_2, v_3\}$ |
| $\vdots$ | $\vdots$ |
| $N(a_{n-1}) = \{u_{n-1}, u_n\}$ | $N(b_{n-1}) = \{v_{n-1}, v_n\}$ |
| $N(e_1) = \{u_1, v_2\}$ | $N(d_1) = \{u_1, v_1\}$ |
| $N(e_2) = \{u_2, v_3\}$ | $N(d_2) = \{u_2, v_2\}$ |
| $\vdots$ | $\vdots$ |
| $N(e_{n-1}) = \{u_{n-1}, v_n\}$ | $N(d_n) = \{u_n, v_n\}$ |

When $2 \le r \le \Delta(G) - 3$

In order to obtain a proper $r$ - dynamic coloring the mapping is taken from the color set to the vertex set of$C(TL_n)$. Consider the mapping $f_1$,
$f_1 : V(C(TL_n)) \mapsto \{1, 2, \ldots, \Delta(G)+1\}$

$$f_1(u_i) = \begin{cases} 1, & when \;\; i=1 \\ 4, & when \;\; i=n \\ 5,7,9,\ldots,\Delta(G) & otherwise. \end{cases}, \quad f_1(v_i) = \begin{cases} 2, & when \;\; i=1 \\ 3, & when \;\; i=n \\ 6,8,10,\ldots,\Delta(G) & otherwise. \end{cases}$$

$$f_1(d_i) = \begin{cases} 6, & when \;\; i=1 \\ 1,5,7,\ldots, & otherwise. \end{cases}$$

Since each $d_i$ has only two neighbours $u_i, v_i$ the color of $u_i$ is applied to the vertex $d_{i+1}$ except the color 3 because the cardinality of the vertices $u_i, d_i$ is $n$, there is no $n+1$ and so the vertex $d_1$ is colored with the color 6.

$$f_1(a_i, b_i, e_i) = \begin{cases} 4, & when \;\; i=1 \\ 3, & when \;\; i=2 \\ 2, & when \;\; i=3 \\ 1, & when \;\; i=4 \\ i, & for \;\; i \ge 5 \end{cases}$$

A special case exist for $n = 4$ since order of the vertex set is only three and so they are colored with 3, 2, 1.

Thus the color assigned using the above mentioned mapping yields $\Delta(G)+1$ independent color classes namely $C_1, C_2, \ldots C_{\Delta(G)+1}$.

$$C_1 = \{u_1, a_4, b_4, d_2\}$$
$$C_2 = \{v_1, a_3, b_3\}$$
$$C_3 = \{v_n, a_2, b_2, e_{n+k}\}$$
$$\vdots$$
$$C_{\Delta(G)+1} = \{v_{n-1}, d_1, e_{n+k-2}\}$$

By correlating the vertices in the coloring set with the neighbourhood set we can observe that every vertex have $r$ or $r+1$ variant colors of adjacent vertices. When $r \geq \Delta(G) - 2$

The coloring of the vertices $u_i$ and $v_i$ remains the same. According to the $r$ - adjacency new colors are included to meet out the condition of $r$ - dynamic coloring. When $r = \Delta(G) - 2$, the vertices of $e_i$ , $r = \Delta(G) - 1$, the odd vertices of $a_i, b_i$ and when $r = \Delta$, the even vertices are colored with the color $r + 4$.

Thus the color assigned using the above mentioned mapping yields $r+4$ independent color classes namely $C_1, C_2, \ldots C_{r+4}$.

$$C_1 = \{u_1, d_2\}$$
$$C_2 = \{v_1\}$$
$$C_3 = \{v_n\}$$
$$\vdots$$
$$C_{r+4} = \{a_2, a_4, \ldots, a_{n-2}, b_2, b_4, \ldots, b_{n-2}\}$$

By correlating the vertices in the coloring set with the neighbourhood set we can observe that every vertex have $r$ or $r+1$ variant colors of adjacent vertices. It can be observed that Case 1 and Case 2 deals with the upper bound,

$$\chi_r C(TL_n) \leq \begin{cases} \Delta(G) + 1, & 2 \leq r \leq \Delta(G) - 3 \\ r + 4, & \Delta(G) - 2 \leq r \leq \Delta(G). \end{cases}$$

The lower bound has been discussed through the lemma 3.5 which leads to acqiure the

$$\chi_r C(TL_n) = \begin{cases} \Delta(G) + 1, & 2 \leq r \leq \Delta(G) - 3 \\ r + 4, & \Delta(G) - 2 \leq r \leq \Delta(G). \end{cases}$$

□

**Lemma 3.7.** *The lower bound of r - Dynamic Chromatic number of Central graph of Open Triangular Ladder Graph $C(OTL_n)$ is,*

$$\chi_r C(OTL_n) \geq \begin{cases} \Delta(G) + 1, & 2 \leq r \leq \Delta(G) - 3 \\ r + 4, & \Delta(G) - 2 \leq r \leq \Delta(G). \end{cases}$$

*Proof.* The vertex and edge set of Central graph of Open Triangular Ladder graph is represented as follows,

$$V(C(OTL_n)) = \{u_i : 1 \leq i \leq n\} \cup \{v_i : 1 \leq i \leq n\} \cup \{a_i : 1 \leq i \leq n-1\} \cup \{b_i : 1 \leq i \leq n-1\}$$
$$\cup \{d_i : 1 \leq i \leq n-2\} \cup \{e_i : 1 \leq i \leq n-1\}$$
$$E(C(OTL_n)) = \{u_1v_1, u_nv_n, u_iu_j, u_iv_j, v_iv_j : 1 \leq i \leq n, i+2 \leq j \leq n\} \cup \{v_iu_j : 1 \leq i \leq n, i+1 \leq j \leq n\}$$
$$\{u_ia_j, v_ib_j : 1 \leq i \leq n, j = i-1, i\} \cup \{u_id_j, v_id_j : 2 \leq i \leq n-2, 1 \leq j \leq n-2\}$$
$$\cup \{u_ie_i, v_ie_{i-1} : 1 \leq i \leq n\}$$

The proof follows the same as lemma 3.1 □

**Theorem 3.8.** *The r - dynamic Chromatic number of Central graph of Open Triangular Ladder Graph $C(OTL_n)$ is,*

$$\chi_r C(OTL_n) = \begin{cases} \Delta(G) + 1, & 2 \leq r \leq \Delta(G) - 3 \\ r + 4, & \Delta(G) - 2 \leq r \leq \Delta(G). \end{cases}$$

*Proof.* The cardinality of the vertex set and the edge set of the Central graph of Open Triangular Ladder Graph is $6n - 5$ and $(\Delta(G) \times (n+1)) + l$, where $l = \{2, 4, 6, \ldots\}$ respectively. The Adjacency matrix is of order $(6n-5) \times (6n-5)$ . Let us consider the first row of the adjacency matrix which corresponds to the vertex $u_1$ which comprises of $6n - 5$ columns.

*cccccccccccc*

c(cccccccccccc) $u_1 u_2 \cdots u_n v_1 v_2 \cdots v_n a_1 a_2 \cdots a_{n-1}$
$00 \cdots 110 \cdots 110 \cdots 0$
$u_1 b_1 b_2 \cdots b_{n-1} d_1 d_2 \cdots d_n e_1 e_2 \cdots e_{n+k}$
$00 \cdots 000 \cdots 010 \cdots 0$

From the adjacency matrix we derive the neighbourhood set of each and every vertices such as

$$N(u_1) = \{a_1, e_1, u_3, u_4, \ldots, u_n, v_1, v_3, v_4, \ldots, v_n\}$$
$$N(u_2) = \{a_1, a_2, d_1, e_2, u_4, u_5, \ldots, u_n, v_1, v_4, v_5, \ldots, v_n\}$$
$$\vdots$$
$$N(u_n) = \{a_{n-1}, u_1, u_2, \ldots, u_{n-2}, v_1, v_2, \ldots, v_{n-1}\}$$
$$N(v_1) = \{b_1, u_1, u_2, u_3, \ldots, u_n, v_3, v_4, \ldots, v_n\}$$
$$N(v_2) = \{b_1, b_2, d_1, e_1, u_3, u_4, \ldots, u_n, v_4, v_5, \ldots, v_n\}$$
$$\vdots$$
$$N(v_n) = \{b_{n-1}, e_{n-1}, u_1, u_2, \ldots, u_{n-2}, v_1, v_2, \ldots, v_{n-2}\}$$

$$\begin{array}{c|c} N(a_1) = u_1, u_2 & N(b_1) = v_1, v_2 \\ N(a_2) = u_2, u_3 & N(b_2) = v_2, v_3 \\ \vdots & \vdots \\ N(a_{n-1}) = u_{n-1}, u_n & N(b_{n-1}) = v_{n-1}, v_n \end{array}$$

$$\begin{array}{c|c} N(e_1) = \{u_1, v_2\} & N(d_1) = \{u_2, v_2\} \\ N(e_2) = \{u_2, v_3\} & N(d_2) = \{u_3, v_3\} \\ \vdots & \vdots \\ N(e_{n-1}) = \{u_{n-1}, v_n\} & N(d_{n-2}) = \{u_{n-1}, v_{n-1}\} \end{array}$$

When $2 \leq r \leq \Delta(G) - 3$

In order to obtain a proper $r$ - dynamic coloring the mapping is taken from the color set to the vertex set of$C(OTL_n)$. Consider the mapping $f_1$,
$f_1 : V(C(OTL_n)) \mapsto \{1, 2, \ldots, \Delta(G) + 1\}$

$$f_1(u_i) = \begin{cases} 1, & when \;\; i = 1 \\ 4, & when \;\; i = n \\ 5, 7, 9, \ldots, \Delta(G) & otherwise. \end{cases}, f_1(v_i) = \begin{cases} 2, & when \;\; i = 1 \\ 3, & when \;\; i = n \\ 6, 8, 10, \ldots, \Delta(G) + 1 & otherwise. \end{cases}$$

$$f_1(d_i) = \begin{cases} 1, & when \;\; i = 1 \\ 5, 7, 9, \ldots, & otherwise. \end{cases}$$

Since each $d_i$ has only two neighbours $u_i, v_i$ the color of $u_i$ is applied to the vertex $d_i$ except the color 3 because the color is applied to the vertex $v_n$.

$$f_1(a_i, b_i, e_i) = \begin{cases} 4, & when \;\; i = 1 \\ 3, & when \;\; i = 2 \\ 2, & when \;\; i = 3 \\ 1, & when \;\; i = 4 \\ i, & for \;\; i \geq 5 \end{cases}$$

A special case exist for $n = 4$ since order of the vertex set is only three and so they are colored with 3, 2, 1.

Thus the color assigned using the above mentioned mapping yields $\Delta(G)+1$ independent color classes namely $C_1, C_2, \ldots C_{\Delta(G)+1}$.

$$\begin{aligned} C_1 &= \{u_1, a_4, b_4, d_2\} \\ C_2 &= \{v_1, a_3, b_3\} \\ C_3 &= \{v_n, a_2, b_2, e_{n+k}\} \\ &\vdots \\ C_{\Delta(G)+1} &= \{v_{n-1}, d_1, e_{n+k-2}\} \end{aligned}$$

By correlating the vertices in the coloring set with the neighbourhood set we can observe that every vertex have $r$ or $r+1$ variant colors of adjacent vertices. When $r \geq \Delta(G) - 2$

The coloring of the vertices $u_i$ and $v_i$ remains the same. According to the $r$ - adjacency new colors are included to meet out the condition of $r$ - dynamic coloring. When $r = \Delta(G) - 2$, the vertices of $e_i$ , $r = \Delta(G) - 1$, the odd vertices of $a_i, b_i$ and when $r = \Delta$, the even vertices are colored with the color $r + 4$.

Thus the color assigned using the above mentioned mapping yields $r+4$ independent color classes namely $C_1, C_2, \ldots C_{r+4}$.

$$\begin{aligned} C_1 &= \{u_1, d_1\} \\ C_2 &= \{u_n\} \\ C_3 &= \{v_n\} \\ &\vdots \\ C_{r+4} &= \{a_2, a_4, \ldots, a_{n-2}, b_2, b_4, \ldots, b_{n-2}\} \end{aligned}$$

By correlating the vertices in the coloring set with the neighbourhood set we can observe that every vertex have $r$ or $r+1$ variant colors of adjacent vertices. It can be observed that Case 1 and Case 2 deals with the upper bound,

$$\chi_r C(OTL_n) \leq \begin{cases} \Delta(G) + 1, & 2 \leq r \leq \Delta(G) - 3 \\ r + 4, & \Delta(G) - 2 \leq r \leq \Delta(G). \end{cases}$$

The lower bound has been discussed through the lemma 3.7 which leads to acqiure the

$$\chi_r C(OTL_n) = \begin{cases} \Delta(G) + 1, & 2 \leq r \leq \Delta(G) - 3 \\ r + 4, & \Delta(G) - 2 \leq r \leq \Delta(G). \end{cases}$$

□

**Lemma 3.9.** *The lower bound of r - Dynamic Chromatic number of Central graph of Slanting Ladder Graph $C(SL_n)$ is,*

$$\chi_r C(SL_n) \geq \begin{cases} \Delta(G) + 1, & 2 \leq r \leq \Delta(G) - 3 \\ \Delta(G) + 3, & r = \Delta(G) - 2, \Delta(G) - 1 \\ r + 4, & r = \Delta(G). \end{cases}$$

*Proof.* The vertex and edge set of Central graph of Slanting Ladder graph is represented as follows,

$$V(C(SL_n)) = \{u_i : 1 \le i \le n\} \cup \{v_i : 1 \le i \le n\} \cup \{a_i : 1 \le i \le n-1\} \cup \{b_i : 1 \le i \le n-1\}$$
$$\cup \{d_i : 1 \le i \le n-1\}$$
$$E(C(SL_n)) = \{u_iu_j, u_iv_j, v_iv_j : 1 \le i \le n, i+2 \le j \le n\} \cup \{v_iu_j : 1 \le i \le n, i+1 \le j \le n\}$$
$$\cup \{u_iv_i : 1 \le i \le n\} \cup \{u_ia_j, v_ib_j : 1 \le i \le n, j = i-1, i\}$$
$$\cup \{u_id_j, v_id_j : 2 \le i \le n-2, 1 \le j \le n-2\}$$

When $2 \le r \le \Delta(G) - 3$

Let us consider $C(SL_4)$, here the maximun degree is seven and according to the statement we need at least a minimum of eight colors. Suppose if we use seven colors, color the first and last vertices of $u_i, v_i$ as $1, 2, 3, 4$ and the rest of the vertices as $5, 6, 7, 5$ it can be seen that the vertex $d_2$ has only one color 5 in its neighbours which is a contradiction. Hence, a minimum of $\Delta(G) + 1$ colors are required. When $r = \Delta(G) - 2, \Delta(G) - 1$

Considering the same example, to complete the $r$ - adjacency we color the vertices of $a_i, b_i$ with new colors. Suppose if not we can see that the vertex $u_2$ has two neighbours with color 2, 3 respectively and the $r$ - adjacency only upto 4 has been obtained which is a contradiction. Hence, a minimum of $\Delta(G) + 3$ colors are required. When $r = \Delta(G)$

Considering the same example, to complete the $r$ - adjacency we color the vertices of $d_i$ with new colors. Suppose if not we can see that the vertices $u_i$ has two neighbours with same color and the $r$ - adjacency has not been obtained which is a contradiction. Hence, a minimum of $r + 4$ colors are required. □

**Theorem 3.10.** *The r - dynamic Chromatic number of Central graph of Slanting Ladder Graph $C(SL_n)$ is,*

$$\chi_r C(SL_n) = \begin{cases} \Delta(G) + 1, & 2 \le r \le \Delta(G) - 3 \\ \Delta(G) + 3, & r = \Delta(G) - 2, \Delta(G) - 1 \\ r + 4, & r = \Delta(G). \end{cases}$$

*Proof.* The cardinality of the vertex set and the edge set of the Central graph of Slanting Ladder Graph is $5n - 3$ and $(\Delta(G) \times (n+1))$ respectively. The Adjacency matrix is of order $(5n - 3) \times (5n - 3)$ . Let us consider the first row of the adjacency matrix which corresponds to the vertex $u_1$ which comprises of $5n - 3$ columns.

*ccccccccccccc*

c(ccccccccccccc) $u_1u_2 \cdots u_nv_1v_2 \cdots v_na_1a_2 \cdots a_{n-1}$
$00 \cdots 110 \cdots 110 \cdots 0$
$u_1b_1b_2 \cdots b_{n-1}d_1d_2 \cdots d_n$
$00 \cdots 010 \cdots 0$

From the adjacency matrix we derive the neighbourhood set of each and every vertices such as

$$N(u_1) = \{a_1, d_1, u_3, u_4, \ldots, u_n, v_1, v_3, \ldots, v_n\}$$
$$N(u_2) = \{a_1, a_2, d_2, u_4, u_5, \ldots, u_n, v_1, v_2, v_4, v_5, \ldots, v_n\}$$
$$\vdots$$
$$N(u_n) = \{a_{n-1}, u_1, u_2, \ldots, u_{n-2}, v_1, v_2, \ldots, v_n\}$$
$$N(v_1) = \{b_1, v_3, v_4, \ldots, v_n, u_1, u_2, \ldots, u_n\}$$
$$N(v_2) = \{b_1, b_2, d_1, v_4, v_5, \ldots, v_n, u_2, u_3, \ldots, u_n\}$$
$$\vdots$$
$$N(v_n) = \{b_{n-1}, d_{n-1}, v_1, v_2, \ldots, v_{n-2}, u_1, u_2, \ldots, u_{n-2}, u_n\}$$

| | | |
|---|---|---|
| $N(a_1) = u_1, u_2$ | $N(b_1) = v_1, v_2$ | $N(d_1) = \{u_1, v_2\}$ |
| $N(a_2) = u_2, u_3$ | $N(b_2) = v_2, v_3$ | $N(d_2) = \{u_2, v_3\}$ |
| $\vdots$ | $\vdots$ | $\vdots$ |
| $N(a_{n-1}) = u_{n-1}, u_n$ | $N(b_{n-1}) = v_{n-1}, v_n$ | $N(d_{n-1}) = \{u_{n-1}, v_n\}$ |

When $2 \le r \le \Delta(G) - 3$

In order to obtain a proper $r$ - dynamic coloring the mapping is taken from the color set to the vertex set of $C(SL_n)$. Consider the mapping $f_1$,
$f_1 : V(C(SL_n)) \mapsto 1, 2, \ldots, \Delta(G) + 1$

$$f_1(u_i) = \begin{cases} 1, 4, & when \;\; i = 1, n \\ 5, 7, 9, \ldots & otherwise. \end{cases}, f_1(v_i) = \begin{cases} 2, 3, & when \;\; i = 1, n \\ 6, 8, 10, \ldots & otherwise. \end{cases}$$

$$f_1(a_i, b_i, d_i) = \begin{cases} 4, & when \;\; i = 1 \\ 3, & when \;\; i = 2 \\ 2, & when \;\; i = 3 \\ 1, & when \;\; i = 4 \\ i, & for i \ge 5 \end{cases}$$

A special case exist for $n = 4$ as there is only three vertices of $a_i, b_i, d_i$ exist and so the vertices are colored with the color 3, 2, 1.

Thus the color assigned using the above mentioned mapping yields $\Delta(G)+1$ independent color classes namely $C_1, C_2, \ldots C_{\Delta(G)+1}$.

$$C_1 = \{u_1, a_4, b_4, d_4\}$$
$$C_2 = \{v_1, a_3, b_3, d_3\}$$
$$C_3 = \{v_n, a_2, b_2, d_2\}$$
$$\vdots$$
$$C_{\Delta(G)+1} = \{v_{n-1}\}$$

By correlating the vertices in the coloring set with the neighbourhood set we can observe that every vertex have $r$ or $r+1$ variant colors of adjacent vertices. When $r = \Delta(G) - 2, \Delta(G) - 1$

The coloring of all the vertices except the vertices $a_i's, b_i's$ remains the same.

$$f_1(u_i, v_i, d_i) = f_2(u_i, v_i, d_i)$$

$$f_2(a_i, b_i) = \begin{cases} \Delta(G) + 2, & when \quad i \quad is \quad odd \\ \Delta(G) + 3, & when \quad i \quad is \quad even \end{cases}$$

Thus the color assigned using the above mentioned mapping yields $\Delta(G)+3$ independent color classes namely $C_1, C_2, \ldots C_{\Delta(G)+3}$.

$$C_1 = \{u_1\}$$
$$C_2 = \{v_1\}$$
$$C_3 = \{v_n\}$$
$$\vdots$$
$$C_{\Delta(G)+3} = \{a_2, a_4, \ldots, a_{n-2}, b_2, b_4, \ldots, b_{n-2}\}$$

By correlating the vertices in the coloring set with the neighbourhood set we can observe that every vertex have $r$ or $r+1$ variant colors of adjacent vertices. When $r = \Delta(G)$

Atlast to obtain the $r$ - adjacency equal to $\Delta(G)$ only the vertices of $d_i$ are recolored with the color $r + 4$.

Thus the color assigned using the above mentioned mapping yields $r+4$ independent color classes namely $C_1, C_2, \ldots C_{r+4}$.

$$C_1 = \{u_1\}$$
$$C_2 = \{v_1\}$$
$$C_3 = \{v_n\}$$
$$\vdots$$
$$C_{r+4} = \{d_1, d_2, \ldots, d_{n-1}\}$$

By correlating the vertices in the coloring set with the neighbourhood set we can observe that every vertex have $r$ or $r+1$ variant colors of adjacent vertices. □

## 4 Conclusion Remarks

This paper presents a explorative study of the $r$ - dynamic graph coloring problem, with a specific focus on the central graph of Ladder graph families. Our work is on types of ladder graphs, including the Diagonal Ladder Graph, Open Diagonal Ladder Graph, Triangular Ladder Graph, Open Triangular Ladder Graph, and Slanting Ladder Graph. We establish lower bounds using the coloring conditions and determined the exact $r$ - dynamic chromatic number for these ladder graph families using color set and neighbourhood set. The significance of our study lies in the fact that the research on the $r$ - dynamic chromatic number of ladder graph families and product ladder graph families is relatively limited within the existing literature. Therefore, by pursuing these research directions, we make a notable contribution to the field of graph theory.

## References


[1] A. Dehghan and A. Ahadi, Upper bounds for the 2 - hued chromatic number of graphs in terms of the independence number, Discrete Applied Mathematics, 160(15)(2012), 2142 - 2146.

[2] A. Taherkhani, $r$ - Dynamic chromatic number of graphs, Discrete Applied Mathematics, 201(2016), 222 - 227.

[3] Abdul Rauf Nizami, Mobeen Munir, Amjad Shahbaz Khan, Zakia Shahzadi, On Chromaticity of Ladder Type Graphs, Science International (Lahore), 28(2)(2016), 829 - 836.

[4] B. Montgomery, Dynamic coloring of graphs, ProQuest LLC, Ann Arbor, MI: Ph.D Thesis, West Virginia University, 2001.

[5] F. Harary, Graph Theory, Reading, Massachusetts: Addison Wesley, 1969.

[6] G. Nandini, M. Venkatachalam, Dafik, On $r$ - dynamic coloring of para-line graph of some standard graphs, Palestine Journal of Mathematics, 10(II)(2022), 11 - 22.

[7] G. Nandini, M. Venkatachalam, J. Vernold Vivin, Dafik, On $r$ - Dynamic Coloring of n-Sunlet Graph Families, Proceedings of the Jangjeon Mathematical, 26(1)(2023), 23 - 42.

[8] H. J. Lai, B. Montgomery, H. Poon, Upper bounds of dynamic chromatic number, Ars Combin., 68(2003), 193 - 201.

[9] H. J. Lai, J. Lin, B. Montgomery, T. Shui, S. Fan, Conditional colorings of graphs, Discrete Mathematics, 306(2006), 1997 - 2004.

[10] K. Kaliraj, H. Naresh Kumar and J. Vernold Vivin On dynamic colouring of cartesian product of complete graph with some graphs, Journal of Taibah University for Science, 14(1)(2020), 168 - 171.

[11] K. Kalaiselvi, N. Mohanapriya and J. Vernold Vivin, On $r$ - dynamic coloring of comb graphs, Notes on Number Theory and Discrete Mathematics, 27(2)(2021), 191 - 200.

[12] K. Praveena, M. Venkatachalam, A. Rohini and Dafik, Equitable Coloring Of Prism Graph And It's Central, Middle, Total And Line Graph, International Journal Of Scientific and Technology Research, 8(8)(2019), 706 - 710.

[13] Kristiana, Dafik, Z. R. Ridlo, R. M. Prihandini, and Robiatul Adawiyah, Research Based Learning and STEM Learning Activities: The Use of R - Dynamic Coloring to Improve the Students Meta - literacy in Solving a Tessellation Decoration Problem, European Journal of Education and Pedagogy, 3(4)(2022), 52 - 60.

[14] N. Mohanapriya, J. Vernold Vivin, J. Kok, M. Venkatachalam, On dynamic coloring of certain cycle - related graphs, Arabian Journal Mathematics, 9(2020), 213 - 221.

[15] P. Sumathi and A. Rathi, Quotient Labeling of Some Ladder Graphs, American Journal of Engineering Research, 7(12)(2018), 38 - 42.

[16] R. M. Falcon, M. Venkatachalam, S. Gowri, G. Nandini, On the $r$ - dynamic coloring of the direct product of a path and a $k$ - subdivision of a star graph, Discrete Mathematics, Algorithms and Applications, 14(3)(2022), 2150121.

[17] R. M. Falcon, N. Mohanapriya, V. Aparna, Optimal Shadow Allocations of Secret Sharing Schemes Arisen from the Dynamic Coloring of Extended Neighborhood Coronas, Mathematics, 10(12)(2022), 2018.

[18] R. M. Falcon, M. Venkatachalam, S. Gowri and G. Nandhini , On $r$ - dynamic coloring of some fan graph families, Analele stiintifice ale Universitatii Ovidius Constanta, 29(3)(2021), 151 - 181.

[19] R. Z. Ribah, Dafik, A. I. Kristiana, I. N. Maylisa, Slamin, On the $r$ - dynamic chromatic number of subdivision of wheel graph, Journal of Physics: Conference Series, 2157(2022), 012016.

[20] S. Akbari, M. Ghanbari, S. Jahanbakam, On the dynamic chromatic number of graphs, Contemporary Mathematics, 531(2010), 11 - 18.

[21] S. Akbari, M. Ghanbari, S. Jahanbekam, On the dynamic chromatic number of Cartesian product graphs, Ars Combinatoria, 114(2014), 161 - 167.

[22] S. Jahanabekama, J. Kim, O S, Douglas B. West, On $r$ - dynamic coloring of graphs, 206(2016), 65 - 72.

[23] T. Deepa, M. Venkatachalam, Ismail Naci Cangul, On $r$ - dynamic chromatic number of some brick product graphs $C(2n, 1, p)$, Asian-European Journal of Mathematics, 15(6)(2022), 2250102.

[24] V. Aparna, N. Mohanapriya, $r$ - Dynamic Chromatic Number of Extended Neighbourhood Corona of Complete Graph with Some Graphs, Mathematical Modelling And Computational Intelligence Techniques: ICMMCIT - 2021, Gandhigram, Dindugual, India February 10 - 12, Singapore:Springer Nature Singapore, 235 - 254.

[25] V. Aparna, N. Mohanapriya, S. Broumi, Single Valued Neutrosophic $R$ - dynamic Vertex Coloring of Graphs, Neutrosophic Sets and Systems 48(2022), 306 - 317.

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