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COMPUTATION OF DOMINATOR AND TOTAL DOMINATOR CHROMATIC NUMBER OF M-SPLITTING GRAPHS

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Abstract Let $\mathcal{G} = \{\mathcal{V}(\mathcal{G}), \mathcal{E}(\mathcal{G})\}$ be a simple, undirected and finite graph. The notions $\mathcal{V}(\mathcal{G})$ and $\mathcal{E}(\mathcal{G})$ represent the vertex set and edge set of the graph $\mathcal{G}$ respectively. A Dominator Coloring of a graph $\mathcal{G}$ is a proper coloring utilized in such a way that every vertex in $\mathcal{G}$ is adjacent to all the vertices of atleast a color class. A Total Dominator Coloring is also a dominator coloring, but with the additional requirement that each vertex must dominate at least one color class excluding its own. This paper explores the dominator as well as total dominator coloring of m-splitting graphs determining the exact values of both the chromatic numbers of m-splitting graphs derived from any graph.

1 Introduction

A simple graph is formally defined as an ordered triple $\mathcal{G} = \{\mathcal{V}(\mathcal{G}), \mathcal{E}(\mathcal{G}), \mathcal{I}_{\mathcal{G}}\}$ [8, 2]. In this study, we focus on finite, undirected, and connected graphs. The vertex set and edge set of the graph $\mathcal{G}$ are denoted by $\mathcal{V}(\mathcal{G})$ and $\mathcal{E}(\mathcal{G})$, respectively. Let n represent the number of vertices (the order of $\mathcal{G}$). A subset S of the vertex set $\mathcal{V}(\mathcal{G})$ is called a dominating set for the graph $\mathcal{G}$ if every vertex in the complement set $\mathcal{V}(\mathcal{G}) - S$ is adjacent to at least one vertex in S. In this context, we say that S serves as a dominator for the vertices in $\mathcal{V}(\mathcal{G}) - S$. The domination number $\gamma(\mathcal{G})$ is defined as the minimum size of a dominating set within the graph $\mathcal{G}$. Similarly, A set T of vertices in a graph $\mathcal{G}$ is called a total dominating set if every vertex $v \in \mathcal{V}(\mathcal{G})$ is adjacent to an element of T. Respectively the total domination number of a graph $\mathcal{G}$ denoted by $\gamma_t(\mathcal{G})$ is the minimum cardinality of a total dominating set in $\mathcal{G}$. For all parameters related to the concept of domination, refer [9].

Graph coloring, a well-established concept, involves assigning colors to vertices such that no two adjacent vertices share the same color. The set of all vertices of a specific color forms the corresponding color class, and the minimum count of colors needed for a proper coloring is referred to as the chromatic number. One notable coloring is Dominator Coloring, introduced in [7] and further developed in [4, 5]. This led to the concept of Total Dominator Coloring, which was introduced by Kazemi in [12]. The combination of Dominator and Total Dominator Colorings was explored in [10]. Additionally, [1] analyzed the algorithmic aspects of Dominator Coloring, while [3] provided a polynomial-time algorithm for computing the Dominator Chromatic Number for certain types of graphs. Determining both the dominator and total dominator chromatic numbers is an NP-complete problem. Various researchers have applied these colorings to different graphs and operations. In [11], the comparison between the dominator and total dominator chromatic number of m-shadow graphs of cycle, complete graph and wheel graph were studied. Specifically, dominator coloring has been investigated for the m-shadow and m-splitting graphs of paths in [13].

Graph coloring has long been a subject of study due to its applicability in modeling a wide range of practical problems, such as map coloring, register allocation, frequency assignment, vehicle routing, and cryptography. As new types of graph coloring are discovered and implemented, they provide effective tools for graph modeling. Similarly, dominator coloring has important applications across various fields, including network analysis, optimization problems, communication networks, and so on. This approach is particularly valuable in scenarios where minimizing resource consumption while maximizing profitability is crucial.

Extending this coloring approach to various graphs and their operations can introduce new insights, allowing us to apply theoretical graph concepts in practical scenarios. The operations and products of certain graphs are especially beneficial in areas like networking, chemical compositions, biological systems, and more. Therefore, integrating this coloring method with graph operations could enhance its utility significantly. In this study, we aim to apply both the Dominator and Total Dominator coloring for m-splitting graphs and calculate the exact chromatic numbers.

2 Preliminaries

Basic definitions and results regarding the coloring are referred from [3, 7, 12, 6].

Definition 2.1. [14] The *m-splitting graph* $Spl_m(\mathcal{G})$ of a graph $\mathcal{G}$ is obtained by adding to each vertex v of $\mathcal{G}$, new m vertices, say $v_1, v_2, v_3, ..., v_m$ such that $v_i, 1 \leq i \leq m$ is adjacent to each vertex that is adjacent to v in $\mathcal{G}$.

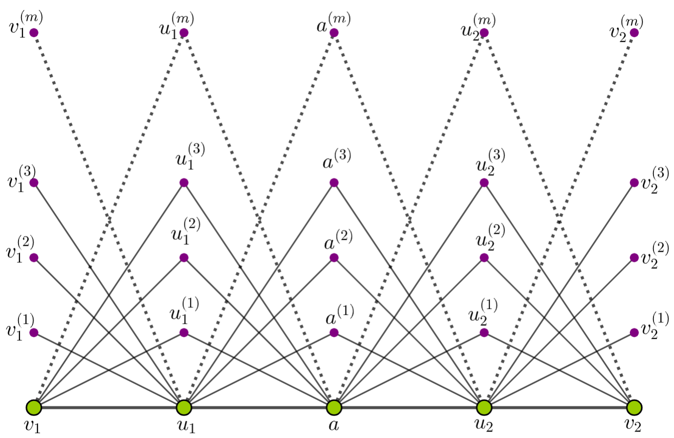

Figure 1. *Illustration of m-splitting graph of Path P_5.*

Definition 2.2. [7] A *Dominator Coloring* of a graph $\mathcal{G}$ is a proper coloring where each vertex is adjacent to every vertex in at least one color class. The minimum number of colors needed for this Dominator Coloring is referred to as the *Dominator Chromatic number*, denoted by $\chi_d(\mathcal{G})$.

Definition 2.3. [12] A *Total Dominator Coloring* of a graph $\mathcal{G}$ is a proper coloring in which every vertex dominates at least one color class besides its own. The minimum number of colors necessary for this Total Dominator Coloring is known as the *Total Dominator Chromatic number*, denoted by $\chi_d^t(\mathcal{G})$.

Lemma 2.4. *[6] Let G be a connected graph. Then*

$$max\{\chi(G), \gamma(G)\} \leq \chi_d(G) \leq \chi(G) + \gamma(G)$$

Lemma 2.5. *[10] For any graph G with $\delta(G) \geq 1$,*

$$max\{\chi(G), \gamma_t(G)\} \leq \chi_d^t(G) \leq \chi(G) + \gamma_t(G)$$

3 Main Results

Lemma 3.1. *For any graph $\mathcal{G}$, the total dominator chromatic number of its m-splitting graph is given by*

$$\chi_d^t(Spl_m(\mathcal{G})) = \chi_d^t(G) + 1$$

Proof. For a graph $\mathcal{G}$ with order n, its vertex set is defined by $\mathcal{V}(\mathcal{G}) = \{v_s : 1 \leq s \leq n\}$. The vertex set of the m-splitting graph is defined by

$$\mathcal{V}(Spl_m(\mathcal{G})) = \{v_s, v_s^{(t)} : 1 \leq s \leq n, 1 \leq t \leq m\}$$

Since $|\mathcal{V}(\mathcal{G})| = n$, the cardinality of the m-splitting graph of $\mathcal{G}$ is given by $|\mathcal{V}(Spl_m(\mathcal{G}))| = n(m+1)$. To derive the m-splitting graph of $\mathcal{G}$, follow the definition 2.1.

Associate a coloring transformation

$$\varphi : \mathcal{V}(Spl_m(\mathcal{G})) \to \{1, 2, 3, ..., \chi_d^t(Spl_m(\mathcal{G}))\}.$$

Let $\{\mathcal{C}_i : 1 \leq i \leq \chi_d^t(Spl_m(\mathcal{G}))\}$ represent the set of color classes generated by this coloring.

This transformation φ is defined so that the vertices $\{v_s\}$ of the graph $\mathcal{G}$ are initially colored using its χ_d^t-coloring mapping $\mathcal{V}(\mathcal{G})$ to $\{1, 2, 3, ..., \chi_d^t(\mathcal{G})\}$. Since, a χ_d^t-coloring of any graph $\mathcal{G}$ guarantees that each vertex totally dominates at least one color class, this property naturally extends to its m-splitting graph.

To preserve the total dominator coloring property across the m copies, it suffices to introduce just one additional color beyond $\chi_d^t(G)$. By assigning the remaining m copies of each vertex a single color, say k, it suffices to achieve both dominance and total dominance. These vertices $\{v_s^{(t)}\}$ will dominate at least one color class $\mathcal{C}_i$ used among the parent vertices $\{v_s\}$. This ensures that all m copies of each vertex can be properly and dominantly colored without conflict.

This conclusion follows immediately from Lemma 2.5. Since the m-splitting graph is, by construction, an extension of the parent vertices, the associated parameters satisfy

$$\chi(Spl_m(\mathcal{G})) = \chi(\mathcal{G}) + 1, \quad \gamma_t(Spl_m(\mathcal{G})) = \gamma_t(\mathcal{G}).$$

Therefore,

$$\max\{\chi(Spl_m(\mathcal{G})), \gamma_t(Spl_m(\mathcal{G}))\} \leq \chi_d^t(Spl_m(\mathcal{G})) \leq \chi(Spl_m(\mathcal{G})) + \gamma_t(Spl_m(\mathcal{G})),$$

which reduces to

$$\max\{\chi(\mathcal{G}) + 1, \gamma_t(\mathcal{G})\} \leq \chi_d^t(\mathcal{G}) + 1 \leq \chi(\mathcal{G}) + 1 + \gamma_t(\mathcal{G}).$$

Thus, the assertion is verified. □

3.1 Dominator Coloring of m-splitting graphs with $\chi_d(\mathcal{G}) = \chi_d^t(\mathcal{G})$

Lemma 3.2. *For all graphs $\mathcal{G}$ with $\chi_d(\mathcal{G}) = \chi_d^t(\mathcal{G})$, the following statements are valid.*

(i) $\chi_d(Spl_m(\mathcal{G})) = \chi_d(G) + 1$

(ii) $\chi_d(Spl_m(\mathcal{G})) = \chi_d^t(Spl_m(\mathcal{G}))$

Proof. Consider a graph $\mathcal{G}$ with a similar dominator and total dominator coloring such that $\chi_d(\mathcal{G}) = \chi_d^t(\mathcal{G})$. The vertex set of the graph follows directly from the Lemma 3.1.

Associate a coloring transformation $\varphi : \mathcal{V}(Spl_m(\mathcal{G})) \to \{1, 2, 3, ..., \chi_d(Spl_m(\mathcal{G}))\}$. Let $\{\mathcal{C}_i : 1 \leq i \leq \chi_d(Spl_m(\mathcal{G}))\}$ represent the set of color classes generated by this coloring. Define this transformation φ in such a way that the vertices $\{v_s\}$ of the graph $\mathcal{G}$ are initially colored using its χ_d-coloring mapping $\mathcal{V}(\mathcal{G})$ to $\{1, 2, 3, ..., \chi_d(\mathcal{G})\}$.

Given that $\chi_d(\mathcal{G}) = \chi_d^t(\mathcal{G})$, it follows that all the vertices $\{v_s\}$ are dependent on each other (interdependent) for the purpose of total dominance. The interdependence among these vertices allows for a χ_d as well as χ_d^t-coloring of the m-splitting graph with just one additional color.

Trivially, from the previous Lemma 3.1, each vertex $\{v_s^{(t)}\}$ will dominate at least one color class $\mathcal{C}_i$ assigned among the original vertices $\{v_s\}$, thereby ensuring a coloring that satisfies both the dominator and total dominator conditions.

This result follows directly from Lemma 2.4. Since the m-splitting graph is, by definition, an extension of the parent vertices, the corresponding parameters are given by,

$$\chi(Spl_m(\mathcal{G})) = \chi(\mathcal{G}) + 1, \quad \gamma(Spl_m(\mathcal{G})) = \gamma(\mathcal{G}).$$

Consequently, the bounds satisfy

$$\max\{\chi(Spl_m(\mathcal{G})), \gamma(Spl_m(\mathcal{G}))\} \leq \chi_d(Spl_m(\mathcal{G})) \leq \chi(Spl_m(\mathcal{G})) + \gamma(Spl_m(\mathcal{G})),$$

which simplifies to

$$\max\{\chi(\mathcal{G}) + 1, \gamma(\mathcal{G})\} \leq \chi_d(\mathcal{G}) + 1 \leq \chi(\mathcal{G}) + 1 + \gamma(\mathcal{G}).$$

Thus, the claim is established, and conditions (i) and (ii) are satisfied.

□

Example 3.3. Consider the cycle C_3 where $\chi_d(\mathcal{G}) = \chi_d^t(\mathcal{G})$. The dominator coloring for the m-splitting graph of C_3 is illustrated in the figure 2. Initially $\chi_d(C_3) = \chi_d^t(C_3) = 3$. In accordance with lemma 3.2, an additional color $\chi_d(C_3) + 1$ is used to achieve dominance in $\chi_d(Spl_m(C_3))$. It is also observed that part (ii) of lemma 3.2 holds true.

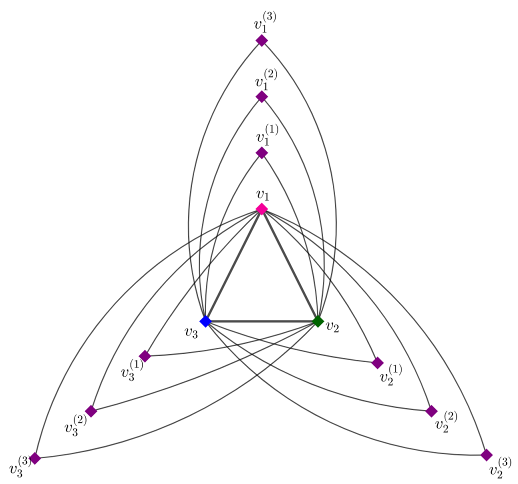

Figure 2. *Dominator coloring of the 3-splitting graph of cycle C_3 using 4 colors to ensure each vertex dominates at least one color class.*

The dominator and total dominator chromatic numbers of the m-splitting graphs of the following specific graphs are established as a consequence of lemma 3.1 and 3.2.

Corollary 3.1.

- *For Complete graph K_n,*

$$\chi_d(Spl_m(K_n)) = \chi_d^t(Spl_m(K_n)) = \chi_d(K_n) + 1$$

- *For Wheel graph $W_{1,n}$,*

$$\chi_d(Spl_m(W_{1,n})) = \chi_d^t(Spl_m(W_{1,n})) = \chi_d(W_{1,n}) + 1$$

- *For Star graph $K_{1,n}$,*

$$\chi_d(Spl_m(K_{1,n})) = \chi_d^t(Spl_m(K_{1,n})) = \chi_d(K_{1,n}) + 1$$

- *For Complete Bipartite graph $K_{m,n}$,*

$$\chi_d(Spl_m(K_{m,n})) = \chi_d^t(Spl_m(K_{m,n})) = \chi_d(K_{m,n}) + 1$$

For graphs with $\chi_d(\mathcal{G}) \neq \chi_d^t(\mathcal{G})$, the dominator coloring employed on their m-splitting graphs yields interesting results which are explored in the succeeding subsection.

3.2 Dominator Coloring of m-splitting graphs with $\chi_d(\mathcal{G}) \neq \chi_d^t(\mathcal{G})$

Lemma 3.4. *For all graphs with $\chi_d(\mathcal{G}) \neq \chi_d^t(\mathcal{G})$, the following statements are true.*

(i) $\chi_d(Spl_m(\mathcal{G})) = \chi_d^t(G) + 1$

(ii) $\chi_d(Spl_m(\mathcal{G})) = \chi_d^t(Spl_m(\mathcal{G}))$

Proof. Consider a graph $\mathcal{G}$ with distinct dominator and total dominator colorings such that $\chi_d(\mathcal{G}) = \chi_d^t(\mathcal{G})$. The vertex set of the graph follows directly from the Lemma 3.1.

Define a coloring function $\varphi : \mathcal{V}(Spl_m(\mathcal{G})) \to \{1, 2, 3, ..., \chi_d(Spl_m(\mathcal{G}))\}$, where the set of color classes induced by this coloring is denoted as $\{\mathcal{C}_i : 1 \leq i \leq \chi_d(Spl_m(\mathcal{G}))\}$.

Similar to Lemma 3.2, assume that the transformation φ is defined such that the vertices $\{v_s\}$ of the graph $\mathcal{G}$ are initially assigned colors based on its χ_d - coloring, mapping $\mathcal{V}(\mathcal{G})$ to $\{1, 2, 3, ..., \chi_d(\mathcal{G})\}$. Since a χ_d - coloring does not necessarily guarantee total dominance, certain vertices remain self-dominating in this coloring. Consequently, it becomes necessary to ensure that the colors assigned to these self-dominating vertices are not repeated. To preserve the self-dominating property of the parent graph, each corresponding split vertex must be assigned a unique color in every copy.

If a graph $\mathcal{G}$ contains t number of self-dominating vertices, then each m^{th} copy will require t distinct colors in addition to the repeated colors. Therefore, a minimum of $\chi_d(\mathcal{G}) + 1 + (m \times t)$ colors is necessary to properly produce a dominator coloring of the m-splitting graph.

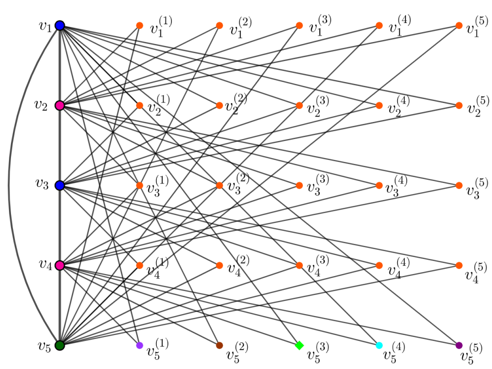

Figure 3. $\chi_d(Spl_5(C_5)) = 9$ ✗

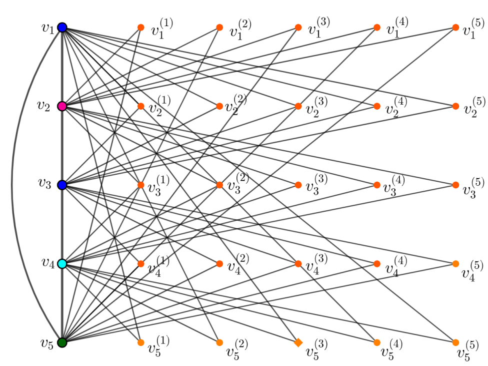

Figure 4. $\chi_d(Spl_5(C_5)) = 5$ ✓

Comparison of two coloring methods for the 5-splitting graph of the cycle C_5 illustrating that the total dominator coloring approach on the parent graph uses the fewest colors.

For instance, consider the cycle C_5 in which the vertex v_5 is self-dominating and assigned a unique color. Consequently, when applying the χ_d - coloring of C_5 to its m-splitting graph, the color assigned to v_5 cannot be reused. As a result, the corresponding split vertices must each receive distinct colors in every copy, leading to a total of $\chi_d(C_5) + 1 + (5 \times 1) = 3 + 1 + 5 = 9$ colors. Refer Figure 3.

In analogy with the previous lemmas, the parameters of the m-splitting graph satisfy

$$\chi(Spl_m(\mathcal{G})) = \chi(\mathcal{G}) + 1, \quad \gamma(Spl_m(\mathcal{G})) = \gamma(\mathcal{G}).$$

Accordingly, the bound from Lemma 2.4 takes the form

$$\max\{\chi(Spl_m(\mathcal{G})), \gamma(Spl_m(\mathcal{G}))\} \leq \chi_d(Spl_m(\mathcal{G})) \leq \chi(Spl_m(\mathcal{G})) + \gamma(Spl_m(\mathcal{G})),$$

which reduces to

$$\max\{\chi(\mathcal{G}) + 1, \gamma(\mathcal{G})\} \leq \chi_d(\mathcal{G}) + mt + 1 \leq \chi(\mathcal{G}) + 1 + \gamma(\mathcal{G}).$$

However, in this relation, the upper bound is not always satisfied. This indicates that the construction employs the maximum possible number of colors, ultimately violating the general bound stated in Lemma 2.4.

Interestingly, by choosing an alternative approach, the number of required colors can be significantly reduced, leading to a more optimized dominator coloring.

Rather than using dominator coloring, consider applying the total dominator coloring to the parent graph $\mathcal{G}$. i.e., employ $\chi_d^t(\mathcal{G})$. Since total dominator coloring ensures that all vertices $\{v_s\}$ are mutually dependent for total dominance, this approach provides a more structured coloring strategy.

According to Lemma 3.2, this interdependence among vertices enables the m-splitting graph to be χ_d-colored using only one additional color. By assigning all m copies of each vertex a single color, say k, both dominance and total dominance can be ensured. The vertices $\{v_s^{(t)}\}$ will dominate at least one color class $\mathcal{C}_i$ used among the original vertices $\{v_s\}$, guaranteeing a coloring that meets both dominator and total dominator conditions. This approach requires just $\chi_d^t(\mathcal{G}) + 1$ colors, which is notably fewer than the number $\chi_d(\mathcal{G}) + 1 + (m \times t)$, used in the previous method, leading to a more efficient dominator coloring. Refer Figure 4.

Hence, in this approach, the bound from Lemma 2.4 takes the form

$$\max\{\chi(\mathcal{G}) + 1, \gamma(\mathcal{G})\} \leq \chi_d^t(\mathcal{G}) + 1 \leq \chi(\mathcal{G}) + 1 + \gamma(\mathcal{G}).$$

Therefore, this approach establishes the claim, and both conditions (i) and (ii) are satisfied, which completes the proof. □

From lemmas 3.1, 3.2 and 3.4, an important theorem can be formulated as follows:

Theorem 3.5. *For any graph $\mathcal{G}$, the statement is valid.*

$$\chi_d(Spl_m(\mathcal{G})) = \chi_d^t(Spl_m(\mathcal{G})) = \chi_d^t(G) + 1$$

The dominator coloring of a 3-splitting graph of a triple star graph $K_{1,2,2,2}$ is provided as an example for Theorem 3.5.

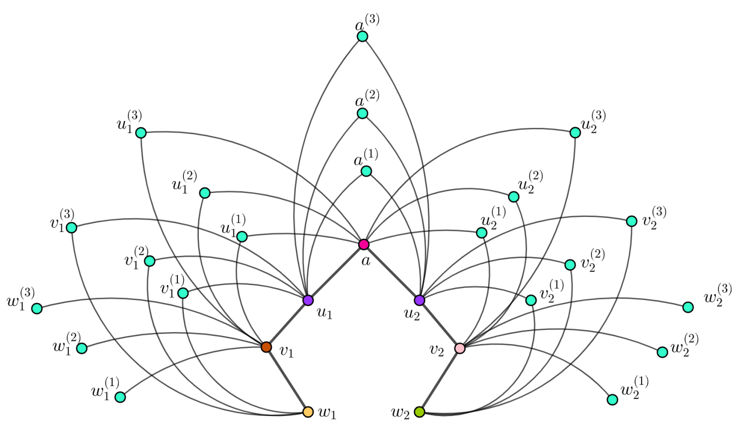

Figure 5. *Illustration of the dominator coloring of the 3-splitting graph of the triple star graph $K_{1,n,n,n}$ showing that $\chi_d(Spl_3(K_{1,2,2,2})) = \chi_d^t(Spl_3(K_{1,2,2,2})) = 7$*

4 Conclusion remarks

In this study, the dominator chromatic number of m-splitting graphs for cases where $\chi_d(\mathcal{G}) = \chi_d^t(\mathcal{G})$ and $\chi_d(\mathcal{G}) \neq \chi_d^t(\mathcal{G})$ has been explored. Additionally, the total dominator chromatic number for m-splitting graphs derived from any graph is calculated. The precise values of the dominator chromatic number for m-splitting graph of any graph has also been determined. This comprehensive analysis not only establishes important numerical results but also contributes to a deeper understanding of the coloring properties of these graph classes.

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