

AN ALTERNATIVE GENERATING FUNCTION FOR k -REGULAR PARTITIONS

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MSC 2010 Classifications: Primary 05A17, 05A15, 11P84.

Keywords and phrases: integer partition, partition generating function, 2-regular partitions.

We thank Dennis Eichhorn for correcting an embarrassing mistake. We also thank the anonymous referee for the careful reading of the manuscript.

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Abstract We construct a k -fold q -series as a generating function of $(k+1)$ -regular partitions for each positive integer k . The $k=1$ case is one of Euler's q -series identities pertaining to the partitions into distinct parts. The construction is combinatorial. Although we find a connection to Bessel polynomials in the $k=2$ case, this note is certainly not a study of Bessel polynomials and their q -analogs.

1 Introduction and the statement of results

An integer partition of a non-negative integer n is a non-decreasing sequence of positive integers whose sum is n [1]. For example, 4 has the following five partitions.

$$4, \quad 2+2, \quad 1+3, \quad 1+1+2, \quad 1+1+1+1.$$

Among these, 4 and $1+3$ are partitions into distinct parts. If we denote the number of partitions of n satisfying condition A by $p(n|A)$, then

$$\sum_{n \geq 0} p(n|A)q^n$$

is called a partition generating function. It is well known [1] that

$$\sum_{n \geq 0} p(n)q^n = \frac{1}{(q; q)_{\infty}}, \quad \text{and} \quad \sum_{n \geq 0} p(n|\text{distinct parts})q^n = (-q; q)_{\infty},$$

where

$$(a; q)_n = \prod_{j=1}^n (1 - aq^{j-1}), \quad \text{and} \quad (a; q)_{\infty} = \lim_{n \rightarrow \infty} (a; q)_n.$$

Throughout this paper the infinite products converge absolutely when $|q| < 1$ [1].

It is often necessary to work with a second parameter and consider the number of partitions of n into m parts satisfying condition A , denoted by $p(m, n|A)$. Again, it is well known [1] that

$$\sum_{m, n \geq 0} p(m, n)x^m q^n = \frac{1}{(xq; q)_{\infty}}, \quad \text{and} \quad \sum_{m, n \geq 0} p(m, n|\text{distinct parts})x^m q^n = (-xq; q)_{\infty}.$$

k -regular partitions are those in which no part is repeated k or more times. 2-regular partitions are partitions into distinct parts. The seven 3-regular partitions of 6 are

$$6, \quad 1+5, \quad 2+4, \quad 1+1+4, \quad 3+3, \quad 1+2+3, \quad 1+1+2+2.$$

Depending on the purpose, it is possible to represent partitions in different ways. For instance, we can drop the plus signs for brevity, or stack the repeated parts on top of each other for emphasis on k -regularity. So, the last displayed partition may be written as 1 1 2 2, or $\begin{smallmatrix} 1 & 2 \\ 1 & 2 \end{smallmatrix}$.

k -regular partitions are widely studied either for their combinatorial properties (e.g. [2]), or their arithmetic properties (e.g. [2, 3]). Our main goal is to prove the following theorem (Theorem 1.1) for 3-regular partitions, and then to generalize it to k -regular partitions for any integer $k \geq 2$ (Theorem 1.3). We will also note a connection between the polynomials $b(m, n)$ in Theorem 1.1 and the coefficient of Bessel polynomials [5, 12]. However, our aim is not studying the Bessel polynomials here.

Theorem 1.1.

$$\frac{(x^3 q^3; q^3)_\infty}{(xq; q)_\infty} = \sum_{m, n \geq 0} \frac{q^{\binom{m+n+1}{2} + \binom{m+1}{2}} (1-q)^m x^{2m+n}}{(q; q)_{2m+n}} b(m, n),$$

where $b(0, 0) = 1$, $b(m, n) = 0$ if $m < 0$ or $n < 0$, and

$$b(m, n) = (1 + q + q^2 + \cdots + q^{2m+n-2}) b(m-1, n) + q^m b(m, n-1).$$

It is well known [1] that

$$\sum_{n \geq 0} \frac{q^{\binom{n+1}{2}}}{(q; q)_n} \quad (1.1)$$

generates partitions into distinct parts. For any non-negative integer n , we can think of $q^{\binom{n+1}{2}}$ as generating a base partition into exactly n parts with the smallest weight, that is $1 + 2 + \cdots + n$. Then, for the same n , $\frac{1}{(q; q)_n}$ generates partitions into n parts, zeros allowed. Take $0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ as a generic one. Finally, their combination given by

$$(1 + \lambda_1) + (2 + \lambda_2) + \cdots + (n + \lambda_n)$$

generates a partition into exactly n distinct parts. Conversely, given a partition into n distinct parts $\mu_1 + \mu_2 + \cdots + \mu_n$ where $0 < \mu_1 < \mu_2 < \cdots < \mu_n$, we have

$$0 \leq \mu_1 - 1 \leq \mu_2 - 2 \leq \cdots \leq \mu_n - n.$$

Setting $\lambda_j = \mu_j - j$ for $j = 1, 2, \dots, n$ will give us a partition $0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ into n parts, zeros allowed, along with the base partition into n parts with minimal weight $1 + 2 + \cdots + n$.

We can also imagine parts of the base partition moving forward or backward. For example, part j in the base partition moves forward λ_j times for $j = n, (n-1), \dots, 1$, in this order.

$$\begin{array}{lcl} 1 & 2 & \cdots & (n-1) & \mathbf{n} & \text{(the highlighted part moves forward } \lambda_n \text{ times),} \\ \rightsquigarrow & 1 & 2 & \cdots & (\mathbf{n}-1) & (n + \lambda_n) & \text{(the highlighted part moves forward } \lambda_{n-1} \text{ times),} \\ & & & & \vdots & \\ & & & & (1 + \lambda_1) & (2 + \lambda_2) & \cdots & (n + \lambda_n) & = & \mu_1 & \mu_2 & \cdots & \mu_n. \end{array}$$

For any non-negative integer n , this gives a bijection between partitions into n distinct parts ($0 < \mu_1 < \mu_2 < \cdots < \mu_n$) and pairs consisting of the base partition $1 + 2 + \cdots + n$ and a partition into n parts $0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$, zeros allowed. This is the most direct combinatorial explanation of the fact that (1.1) generates partitions into distinct parts. We can also consider

$$\sum_{n \geq 0} \frac{q^{\binom{n+1}{2}} x^n}{(q; q)_n}.$$

The same bijection works. The exponent of x keeps track of the number of parts, while the exponent of q is the number being partitioned.

For our purposes, we will let $\frac{1}{(q;q)_n}$ generate the conjugate of $\lambda_1 + \lambda_2 + \dots + \lambda_n$. This is a partition into parts which are at most n , with f_1 1's, f_2 2's, $\dots$, f_n n 's. In this case, each j moves the j largest parts in the intermediate partition forward, one each, f_j times. This is done for each $j = 1, 2, \dots, n$. By the intermediate partition, we mean any partition encountered in constructing any partition $\mu_1 + \mu_2 + \dots + \mu_n$ into distinct parts from the base partition $1 + 2 + \dots + n$ with these steps.

We want to imitate the above bijection for 3-regular partitions, that is the partitions into parts that repeat at most twice. This was attempted in [9, 11] also, while studying 3-regular partitions that arose naturally in a class of partitions due to Kanade and Russell [6]. We will discuss the differences between approaches after the construction.

Lemma 1.2.

$$\begin{aligned} \frac{(x^3 q^3; q^3)_\infty}{(xq; q)_\infty} &= \prod_{n \geq 1} (1 + xq^n + x^2 q^{2n}) \\ &= \sum_{\substack{m, n \geq 0 \\ 1 \leq i_1 < i_2 < \dots < i_m \leq m+n}} \frac{q^{\binom{m+n+1}{2} + i_1 + i_2 + \dots + i_m} x^{2m+n}}{(q; q)_{2m+n}} \dots \\ &\quad \times (1 - q^{2m+n+1-i_1-1})(1 - q^{2m+n+1-i_2-2}) \dots (1 - q^{2m+n+1-i_m-m}). \end{aligned}$$

Proof. It is clear that the infinite products in the lemma generate 3-regular partitions. The exponent of q keeps track of the weight, and the exponent of x keeps track of the length, that is the number of parts.

Given a 3-regular partition, we will move parts backward so as to minimize the weight, and record the number of moves in an auxiliary partition (c.f. [7, 8]). However, we will keep the repeated parts (hereafter *pairs*) as well as the relative positions of pairs and non-repeated parts (hereafter *singletons*). Let $\mu_1, \mu_2, \dots, \mu_{m+n}$ be the part sizes of an arbitrary but fixed 3-regular partition having m pairs and n singletons such that

$$0 < \mu_1 < \mu_2 < \dots < \mu_{m+n}.$$

This 3-regular partition has $2m + n$ parts, because exactly m of the μ_j 's repeat by assumption. Call them $\mu_{i_1}, \mu_{i_2}, \dots, \mu_{i_m}$, where

$$1 \leq i_1 < i_2 < \dots < i_m \leq m + n.$$

Observe that the 3-regular partition at hand has $2m + n - i_j - j$ parts which are $> \mu_{i_j}$ and $2m + n + 2 - i_j - j$ parts which are $\geq \mu_{i_j}$ for each $j = 1, 2, \dots, m$.

$$\begin{array}{ccc} \text{(parts } < \mu_{i_j} \text{)} & \begin{array}{c} \mu_{i_j} \\ \mu_{i_j} \end{array} & \begin{array}{c} \overbrace{\text{(parts } > \mu_{i_j} \text{)}}^{(m-j) \text{ of them repeat}} \\ \underbrace{\hspace{1cm}}_{(m+n-i_j) \text{ part sizes}} \end{array} \end{array} \quad (1.2)$$

Unless the 3-regular partition is the empty partition (generated by $q^0 = 1$), we run the following procedure.

- 1b -** Set $k = 1$, the auxiliary partition λ = the empty partition ε , and r = the number of parts that are $\geq \mu_k$.
- 2b -** Subtract $(\mu_k - k)$ from all parts that are $\geq \mu_k$, and append $(\mu_k - k)$ r 's to the auxiliary partition λ . (Part sizes μ_j 's are dynamically updated here, μ is the intermediate partition.)
- 3b -** Increment k by 1. If $k > m + n$, then stop.

Now we have the base partition with part sizes $1, 2, \dots, (m + n)$, in which $i_1, i_2, \dots, i_m$ repeat, namely

$$\begin{array}{ccccccc} & & i_1 & & & i_2 & \\ 1 & 2 & \dots & i_1 & (i_1 + 1) & \dots & (i_2 - 1) & i_2 & (i_2 + 1) & \dots & (m + n) \end{array},$$

and an auxiliary partition λ into parts that are at most $(2m + n)$. By the observation just above the procedure, λ cannot contain parts that are equal to

$$(2m + n + 1 - i_1 - 1), \quad (2m + n + 1 - i_2 - 2), \quad \dots \quad (2m + n + 1 - i_m - m).$$

The base partition has weight $\binom{m+n+1}{2} + i_1 + i_2 + \dots + i_m$. Therefore, the base partition and the auxiliary partition pairs are generated by

$$\frac{q^{\binom{m+n+1}{2} + i_1 + i_2 + \dots + i_m} (1 - q^{2m+n+1-i_1-1}) (1 - q^{2m+n+1-i_2-2}) \dots (1 - q^{2m+n+1-i_m-m})}{(q; q)_{2m+n}}. \quad (1.3)$$

The procedure 1b-3b also preserves the relative placement of the singletons and the pairs.

Conversely, suppose that we are given the base partition with part sizes $1, 2, \dots, (m + n)$ in which $i_1, i_2, \dots, i_m$ occur twice ($1 \leq i_1 < i_2 < \dots < i_m \leq m + n$), and all others occur once; and a partition λ into parts that are at most $(2m + n)$ having no parts equal to $(2m + n + 1 - i_1 - 1), (2m + n + 1 - i_2 - 2), \dots, (2m + n + 1 - i_m - m)$. First; similar to (1.2), we can argue that the base partition has $2m + n - i_j - j$ parts which are strictly greater than i_j , and $2m + n + 2 - i_j - j$ parts which are at least i_j for $j = 1, 2, \dots, m$.

$$\begin{array}{ccccccc} & & & & \overbrace{(m-j) \text{ of them repeat}} & & \\ & & & & \underbrace{i_j \quad (i_j + 1) \quad \dots \quad (i_{j+1} - 1) \quad i_{j+1} \quad (i_{j+1} - 1) \quad \dots \quad (m + n)}_{(m+n-i_j) \text{ part sizes}} & & \\ 1 & 2 & \dots & i_j & & & \end{array} \quad (1.4)$$

We run the following procedure.

- 1f -** Set $r = 1$ and the intermediate partition $\mu =$ the base partition in (1.4).
- 2f -** Let f_r be the number of parts in λ that are equal to r . Increment the largest r parts in μ f_r times.
- 3f -** Increment r by 1. If $r > 2m + n$, then stop.

Now we have a 3-regular partition μ which has $(m + n)$ part sizes m of which repeat. By the observation just above (1.4), the base partition and the constructed μ have the same relative ordering of singletons and pairs. Again, by the same observation, there is no ambiguity in applying step 2f. This is because some part sizes in λ are restricted, so that the largest r parts never contain exactly one copy of a repeated part. The base partition has weight $\binom{m+n+1}{2} + i_1 + \dots + i_m$, so that the base partition and auxiliary partition pairs are generated by (1.3). $\square$

Let us give an example to the backward phase of the proof of Lemma 1.2. Consider the following 3-regular partition.

$$\begin{array}{ccccc} & 10 & & 19 & \\ 3 & 6 & 10 & 15 & 19 \end{array}.$$

The part sizes are

$$\mu_1 = 3 < \mu_2 = 6 < \mu_3 = 10 < \mu_4 = 15 < \mu_5 = 19.$$

We set $k = 1$, $\lambda = \varepsilon$, the empty partition. There are 7 parts greater than or equal to μ_1 . We subtract $\mu_1 - 1 = 2$ from all of them, and append 2 7's to λ . Now, the 3-regular partition becomes

$$\begin{array}{ccccc} & 8 & & 17 & \\ 1 & 4 & 8 & 13 & 17 \end{array},$$

and $\lambda = 7 \ 7$. We increment k , it becomes $k = 2$. There are 6 parts $\geq \mu_2 = 4$. We subtract $\mu_2 - 2 = 2$ from all of them, and append 2 6's to λ . The 3-regular partition becomes

$$\begin{array}{ccccc} & 6 & & 15 & \\ 1 & 2 & 6 & 11 & 15 \end{array},$$

and $\lambda = 6\ 6\ 7\ 7$. The same operation for $k = 3$ makes the intermediate 3-regular partition

$$\begin{array}{ccccccc} & & 3 & & 12 & & \\ 1 & 2 & 3 & 8 & 12 & & \end{array},$$

and $\lambda = 5\ 5\ 5\ 6\ 6\ 7\ 7$. Because μ_3 is a repeated part, λ will skip a part. It will not have any 4's. In the end, the 3-regular partition will be the base partition

$$\begin{array}{ccccccc} & & 3 & & 5 & & \\ 1 & 2 & 3 & 4 & 5 & & \end{array},$$

and $\lambda = 2\ 2\ 2\ 3\ 3\ 3\ 3\ 5\ 5\ 5\ 6\ 6\ 7\ 7$. λ will not have any 1's, either. We invite the reader to run the forward phase of the algorithm in the proof of Lemma 1.2 on the last displayed 3-regular base partition and λ .

The combinatorial approach in Lemma 1.2 is more straightforward than in [9, 11]. In contrast, the index range in the multiple sum in Lemma 1.2 is more intricate than the multiple sum that is constructed in [9, 11], due to more relaxed conditions on the base partition. Another difference is that the construction in [9, 11] works for 3-regular partitions only.

The series in Lemma 1.2 has evidently positive coefficients, too. After the proof of Lemma 1.2, the proof of Theorem 1.1 becomes a straightforward q -series manipulation.

proof of Theorem 1.1. Rewrite the series in Lemma 1.2 as

$$\sum_{m,n \geq 0} \frac{q^{\binom{m+n+1}{2}} x^{2m+n}}{(q; q)_{2m+n}} \sum_{1 \leq i_1 < \dots < i_m \leq m+n} q^{i_1 + \dots + i_m} (1 - q^{2m+n+1-i_1-1}) \dots (1 - q^{2m+n+1-i_m-m}),$$

and call the inner sum $a(m, n)$. We can calculate

$$\begin{aligned} a(m, n) &= q(1 - q^{2m+n-1}) \sum_{1 < i_2 < \dots < i_m \leq m+n} q^{i_2 + \dots + i_m} (1 - q^{2m+n+1-i_2-2}) \dots (1 - q^{2m+n+1-i_m-m}) \\ &\quad + \sum_{1 < i_1 < \dots < i_m \leq m+n} q^{i_1 + \dots + i_m} (1 - q^{2m+n+1-i_1-1}) \dots (1 - q^{2m+n+1-i_m-m}), \end{aligned}$$

where the separated sums are for the cases $i_1 = 1$ and $i_1 > 1$, respectively.

$$\begin{aligned} &= q^m (1 - q^{2m+n-1}) \sum_{1 \leq i_2-1 < \dots < i_m-1 \leq (m-1)+n} q^{(i_2-1) + \dots + (i_m-1)} (1 - q^{2(m-1)+n+1-(i_2-1)-1}) \dots \\ &\quad \times (1 - q^{2(m-1)+n+1-(i_m-1)-(m-1)}) \\ &+ q^m \sum_{1 \leq i_1-1 < \dots < i_m-1 \leq m+(n-1)} q^{(i_1-1) + \dots + (i_m-1)} (1 - q^{2m+(n-1)+1-(i_1-1)-1}) \dots \\ &\quad \times (1 - q^{2m+(n-1)+1-(i_m-1)-m}) \\ &= q^m (1 - q^{2m+n-1}) a(m-1, n) + q^m a(m, n-1). \end{aligned} \tag{1.5}$$

The recurrence (1.5) uniquely defines $a(m, n)$ for integers m and n once we declare that $a(0, 0) = 1$ and $a(m, n) = 0$ for $m < 0$ or $n < 0$. These initial conditions are also apparent in Lemma 1.2. The proof follows once we set

$$a(m, n) = q^{\binom{m+1}{2}} (1 - q)^m b(m, n)$$

for all integers m and n . □

For example, $a(0, n) = 1$ for all $n \geq 0$ and

$$a(m, 0) = q^{\binom{m+1}{2}} (-q; q^2)_m$$

for all $m \geq 0$. Values of $a(m, n)$ for small m and n , in particular for $m + n \leq 4$, are shown in the following table.

$n \backslash m$	0	1	2	3	4
0	1	1	1	1	1
1	$q - q^2$	$q + q^2 - 2q^3$	$q + q^2 + q^3 - 3q^4$	$q + q^2 + q^3 + q^4 - 4q^5$	
2	$q^3 - q^4 - q^6 + q^7$	$q^3 + q^4 - q^5 - q^6 - q^7 - 2q^8 + 3q^9$	$q^3 + q^4 + 2q^5 - 2q^6 - q^7 - 2q^8 - 2q^9 - 3q^{10} + 6q^{11}$		
3	$q^6 - q^7 - q^9 + q^{10} - q^{11} + q^{12} + q^{14} - q^{15}$	$q^6 + q^7 - q^8 - 2q^{10} - 2q^{11} + q^{12} + 2q^{15} + q^{16} + 3q^{17} - 4q^{18}$			
4	$q^{10} - q^{11} - q^{13} + q^{14} - q^{15} + q^{16} - q^{17} + 2q^{18} - q^{19} + q^{20} - q^{21} + q^{22} - q^{23} - q^{25} + q^{26}$				

Values of $b(m, n)$ for small m and n , in particular, for $m + n \leq 4$ are shown in the following table.

$n \backslash m$	0	1	2	3	4
0	1	1	1	1	1
1	1	$1 + 2q$	$1 + 2q + 3q^2$	$1 + 2q + 3q^2 + 4q^3$	
2	$1 + q + q^2$	$1 + 3q + 4q^2 + 4q^3 + 3q^4$	$1 + 3q + 7q^2 + 9q^3 + 10q^4 + 9q^5 + 6q^6$		
3	$1 + 2q + 3q^2 + 3q^3 + 3q^4 + 2q^5 + q^6$	$1 + 4q + 8q^2 + 13q^3 + 17q^4 + 18q^5 + 17q^6 + 14q^7 + 9q^8 + 4q^9$			
4	$1 + 3q + 6q^2 + 9q^3 + 12q^4 + 14q^5 + 15q^6 + 14q^7 + 12q^8 + 9q^9 + 6q^{10} + 3q^{11} + q^{12}$				

Observe that the polynomials in the latter table are all unimodal. A unimodal polynomial $p(q) = c_0 + c_1q + \dots + c_rq^r$ has coefficients satisfying

$$c_0 \leq c_1 \leq \dots \leq c_{s-1} \leq c_s \geq c_{s+1} \geq \dots \geq c_r$$

for some $0 \leq s \leq r$.

Letting $q \rightarrow 1$ makes $a(m, n)$'s all 1's and 0's. However, letting $q \rightarrow 1$ makes $b(m, n)$'s coefficients of Bessel polynomials [5, 12], because the closed formula

$$\lim_{q \rightarrow 1} b(m, n) = \frac{(2m + n)!}{m!(m + n)!2^m}$$

satisfies the recurrence in Theorem 1.1 upon $q \rightarrow 1$, as well as the initial values. These are the coefficients of the Bessel polynomials [5, 12]

$$y_n(x) = \sum_{k=0}^n \frac{(n + k)!}{n!k!} \left(\frac{x}{2}\right)^k$$

after the change of parameters $n \leftarrow m + n$ and $k \leftarrow m$. However, the q -analog of the coefficients of Bessel polynomials [12]

$$y_{n,q}(x) = \sum_{k=0}^n \frac{[n + k]_q!}{[n]_q![k]_q!} \left(\frac{x}{2}\right)^k$$

are different from $b(m, n)$ in Theorem 1.1, because they do not satisfy the recurrence in the theorem under any change of parameters. Here,

$$[n]_q = 1 + q + \dots + q^{n-1}, \quad \text{and} \quad [n]_q! = [n]_q[n-1]_q \dots [1]_q.$$

In this sense, $b(m, n)$'s are another q -analog of the coefficients of Bessel polynomials. We failed to find a closed formula for $b(m, n)$'s.

It is routine to adapt the machinery in Lemma 1.2 and Theorem 1.1 for 4-regular partitions. One finds that

$$\frac{(x^4 q^4; q^4)_\infty}{(xq; q)_\infty} = \sum_{l, m, n \geq 0} \frac{q^{\binom{l+m+n+1}{2} + \binom{l+m+1}{2} + \binom{l+1}{2}} (1-q)^{2l+m} x^{3l+2m+n}}{(q; q)_{3l+2m+n}} b(l, m, n)$$

where $b(0, 0, 0) = 1$, $b(l, m, n) = 0$ if $l < 0$ or $m < 0$ or $n < 0$, and

$$\begin{aligned} b(l, m, n) &= (1 + q + q^2 + \dots + q^{3l+2m+n-2}) (1 + q + q^2 + \dots + q^{3l+2m+n-3}) b(l-1, m, n) \\ &\quad + (1 + q + q^2 + \dots + q^{3l+2m+n-2}) q^l b(l, m-1, n) \\ &\quad + q^{2l+m} b(l, m, n-1). \end{aligned}$$

In fact, we can go even further, and do it for $(k+1)$ -regular partitions for any integer $k \geq 1$.

Theorem 1.3. *For any positive integer k ,*

$$\begin{aligned} &\frac{(x^{k+1} q^{k+1}; q^{k+1})_\infty}{(xq; q)_\infty} \\ &= \sum_{n_k, n_{k-1}, \dots, n_1 \geq 0} \frac{q^{\binom{n_k+n_{k-1}+\dots+n_1+1}{2} + \binom{n_k+n_{k-1}+\dots+n_2+1}{2} + \dots + \binom{n_k+1}{2}} (1-q)^{(k-1)n_k + (k-2)n_{k-1} + \dots + 2n_3 + n_2}}{(q; q)_{kn_k + (k-1)n_{k-1} + \dots + 2n_2 + n_1}} \\ &\quad \dots \times x^{kn_k + (k-1)n_{k-1} + \dots + 2n_2 + n_1} b(n_k, n_{k-1}, \dots, n_1), \end{aligned}$$

where $b(0, \dots, 0) = 1$, $b(n_k, n_{k-1}, \dots, n_1) = 0$ if $n_j < 0$ for any $j = 1, 2, \dots, k$, and

$$\begin{aligned} &b(n_k, n_{k-1}, \dots, n_1) \\ &= \left(1 + q + q^2 + \dots + q^{kn_k + (k-1)n_{k-1} + \dots + 2n_2 + n_1 - 2}\right) \dots \\ &\quad \times \left(1 + q + q^2 + \dots + q^{kn_k + (k-1)n_{k-1} + \dots + 2n_2 + n_1 - 3}\right) \dots \\ &\quad \vdots \\ &\quad \times \left(1 + q + q^2 + \dots + q^{kn_k + (k-1)n_{k-1} + \dots + 2n_2 + n_1 - k}\right) b(n_k - 1, n_{k-1}, n_{k-2}, \dots, n_1) \\ &\quad + \dots \\ &\quad + \left(1 + q + q^2 + \dots + q^{kn_k + (k-1)n_{k-1} + \dots + 2n_2 + n_1 - 2}\right) \dots \\ &\quad \times \left(1 + q + q^2 + \dots + q^{kn_k + (k-1)n_{k-1} + \dots + 2n_2 + n_1 - 3}\right) \dots \\ &\quad \vdots \\ &\quad \times \left(1 + q + q^2 + \dots + q^{kn_k + (k-1)n_{k-1} + \dots + 2n_2 + n_1 - j}\right) \dots \\ &\quad q^{(k-j)n_k + (k-1-j)n_{k-1} + \dots + 2n_{j+2} + n_{j+1}} b(n_k, \dots, n_{j+1}, n_j - 1, n_{j-1}, \dots, n_1) \\ &\quad + \dots \\ &\quad + q^{(k-1)n_k + (k-2)n_{k-1} + \dots + 2n_3 + n_2} b(n_k, \dots, n_2, n_1 - 1). \end{aligned}$$

The proof is a straightforward extension of the proofs of Lemma 1.2 and Theorem 1.1, and it is omitted. Theorem 1.3 is Euler's formula [1] for $k = 1$, and Theorem 1.1 for $k = 2$. $b(n_1) = 1$ for all $n_1 \geq 0$. Upon $q \rightarrow 1$, $b(n_2, n_1)$ are the coefficients of Bessel polynomials [5, 12], as

discussed above. However; for $q \rightarrow 1$, values of $b(n_3, n_2, n_1)$ do not show up in OEIS, the online encyclopedia of integer sequences [10]. Empirical evidence suggests that $b(n_k, \dots, n_1)$, which are polynomials in q , are always unimodal. We leave it as a conjecture here.

Conjecture: For any integer $k \geq 2$, and any non-negative $n_k, n_{k-1}, \dots, n_1$, $b(n_k, \dots, n_1)$ constructed in Theorem 1.3 is unimodal.

Please notice that this is essentially different from unimodality properties of the coefficients of the Bessel polynomials, e.g. [4].

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