

ON n -TUPLE PRODUCT RULES FOR THE q -GAMMA FUNCTION $\Gamma_q(z)$ AND REPRESENTATION OF THIS AND RELATED FUNCTIONS VIA OTHER ANALYTIC FUNCTION

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Abstract Recently, we have considered inverse applications of the generalized Littlewood theorem about integrals of the logarithm of analytic functions, and using it proved and re-proved a number of transformation and n -tuple product rules for different elliptic functions. Here we continue this approach, and apply it to the q -gamma function $\Gamma_q(z)$. Among others, we prove (or reprove) a few n -tuple rules expressing $\Gamma_q(nz)$ via the products of $\Gamma_{q^n}(z + \frac{l}{n})$ or $\Gamma_q(z + \frac{l}{n} + \frac{j\tau}{n})$, where $\tau = -\frac{2\pi i}{\ln q}$, and present systematic relations between the product $\Gamma_q(z)\Gamma_q(1-z)$ and elliptic theta-functions.

1 Introduction

The q -gamma function $\Gamma_q(z)$ was introduced by rev. Jackson at the beginning of the XX-th century [1], and nowadays it is a well-known important q -analogue of the usual gamma function $\Gamma(z)$. For real positive $0 < q < 1$, it is defined as (see e.g., [1, 2, 3, 4, 5] for the main definitions and properties of this function and other related q -functions):

$$\Gamma_q(z) = (1-q)^{1-z} \prod_{k=0}^{\infty} \frac{1-q^{k+1}}{1-q^{k+z}}. \quad (1.1)$$

In the terminology of q -series, $\prod_{j=0}^{\infty} (1-xq^j)$ can be written as $(x; q)$, provided that this limit exists, we have $\Gamma_q(z) = (1-q)^{1-z} \frac{(q; q)}{(q^z; q)}$.

Immediately from the definition it is clear that this function has no zeroes and has simple poles at the points $z_{k,l} = -k \pm \frac{2\pi il}{\ln q}$ where, unless specifically stated to the contrary, here and below in this paper, k, l are *non-negative* integers. The writing $\ln z$ always implies the main branch of the logarithm function.

Correspondingly, $\Gamma_q^{-1}(z)$ is an entire function having simple zeroes at the aforementioned $z_{k,l}$. This regularity of zeroes location makes this function quite resembling the elliptic theta-functions, or Weiersstrass sigma-function, and, in particular, enables to establish a few n -tuple product rules, some of which are well known, see e.g. [3]. In this paper we study this question systematically and, in particular, present a couple of apparently new n -tuple product rules.

2 n -tuple product rules

Our main ingredient is the following simple Lemma which recently have been applied to elliptic functions [6, 7].

Lemma 2.1. *Let C denote the rectangle bounded by the lines $x = X_1$, $x = X_2$, $y = Y_1$, $y = Y_2$, where $z = x + iy$, x and y real, $X_1 < X_2$, $Y_1 < Y_2$. Let the functions $f(z)$ and $g(z)$ be analytic*

and meromorphic on the complex plane, and let for some integer n there exist a sequence of contours C_i tending to infinity, that is X_1 and Y_1 tend to minus infinity while X_2 and Y_2 tend to plus infinity, such that

$$\int_{C_i} \frac{\ln f(z)}{(z-a)^{n+1}} dz \text{ and } \int_{C_i} \frac{\ln g(z)}{(z-a)^{n+1}} dz$$

tend to zero. Here a is an arbitrary complex number not coinciding with any zero or pole of the functions $f(z)$ and $g(z)$. Let also the poles and zeroes of these functions, taking into account their multiplicities, coincide. Then

$$\frac{d^n}{dz^n} \ln f(z) = \frac{d^n}{dz^n} \ln g(z).$$

Here we do not discuss the connection of this Lemma with the generalized Littlewood theorem about the contour integrals of analytic functions (for this Theorem and its applications see e.g. [8, 9, 10]), this material and easy proof of the Lemma can be found in [6].

To begin with, recall, see e.g. [2]:

$$\frac{d}{dz} \ln \Gamma_q(1+z) = -\gamma_q + \sum_{k=1}^{\infty} \left(\frac{1-q}{1-q^k} + \frac{\ln q}{q^{-(z+k)} - 1} \right), \quad (2.1)$$

where

$$\gamma_q = \lim_{n \rightarrow \infty} \left(1 + \frac{1-q^2}{1-q} + \frac{1-q^3}{1-q} + \dots + \frac{1-q^n}{1-q} - \ln n \right), \quad (2.2)$$

and the functional equation

$$\Gamma_q(z+1) = \frac{1-q^z}{1-q} \Gamma_q(z). \quad (2.3)$$

Thus for small z

$$\Gamma_q(z) = -\frac{1-q}{z \ln q} + O(1) \quad (2.4)$$

and

$$\Gamma_q(z) \Gamma_q(1-z) = -\frac{1-q}{z \ln q} + O(1). \quad (2.5)$$

Also,

$$\frac{d}{dz} \ln \Gamma_q(1+z)|_{z=0} = -\gamma_q \sum_{k=1}^{\infty} \frac{1-q+q^k \ln q}{1-q^k}. \quad (2.6)$$

Our first aim is the proof of the following known Proposition, see [3]:

Proposition 2.2. *For real $0 < q < 1$ and $n = 2, 3, 4, \dots$, we have:*

$$\Gamma_q(nz) = C \left(\frac{1-q^n}{1-q} \right)^{nz-1} \prod_{j=0}^{n-1} \Gamma_{q^n} \left(z + \frac{j}{n} \right), \quad (2.7)$$

with

$$C^{-1} = \prod_{j=1}^{n-1} \Gamma_{q^n} \left(\frac{j}{n} \right). \quad (2.8)$$

Proof. First, repeating the easily anticipated hint of rev. Jackson [1], we introduce the “exponent free” modified functions

$$\tilde{\Gamma}_q(z) := (1-q)^{z-1} \Gamma_q(z) = \prod_{k=0}^{\infty} \frac{1-q^{k+1}}{1-q^{k+z}}. \quad (2.9)$$

The entire function $\Gamma_q^{-1}(nz)$ has simple zeroes at the points $z = -\frac{k}{n} \pm \frac{2\pi il}{n \ln q} = -\frac{k}{n} \pm \frac{2\pi il}{\ln q^n}$, again k, l are non-negative integers, from which it immediately follows that this function and

the entire function $\prod_{j=0}^{n-1} \Gamma_{q^n}^{-1} \left(z + \frac{j}{n} \right)$ have the same simple zeroes. The same is valid for the functions $\tilde{\Gamma}_q^{-1}(nz)$ and $\prod_{j=0}^{n-1} \tilde{\Gamma}_{q^n}^{-1} \left(z + \frac{j}{n} \right)$. Thus we can apply our Lemma 2.1 with certain n , and to determine the value of this n we need to know the asymptotic of the q -gamma function at large $|z|$, or order of this function. For large by module and negative $\text{Re}(z)$, from the definition we see that the order of magnitude of $\ln \Gamma_q(z)$ is $O(C|z|^2)$ with some constant C (asymptotic for $\text{Re}(z) > 0$ is smaller and well established in Moak's paper [11]), so that the integrals

$$\int_{C_i} \frac{\ln \Gamma_q(z)}{(z-a)^{n+1}} dz \rightarrow 0, \quad \int_{C_i} \frac{\ln \tilde{\Gamma}_q(z)}{(z-a)^{n+1}} dz \rightarrow 0$$

for $n \geq 3$. Thus,

$$\tilde{\Gamma}_q(nz) = C(q, n) \exp(C_1(q, n)z + C_2(q, n)z^2) \prod_{j=0}^{n-1} \tilde{\Gamma}_{q^n} \left(z + \frac{j}{n} \right).$$

To show that exponential factor in the right hand side here is identically 1, we use Moak's asymptotic of the q -gamma function for real positive $\text{Re}(z)$ tending to plus infinity, see Theorem 2 in [11]. (This is not cyclic: Moak's proof does not use any kind of the n -tuple rules). WE prioritize the first term:

$$\ln \Gamma_q(z) \sim (z - 1/2) \ln \frac{1 - q^z}{1 - q} + o(z),$$

that is, even more roughly,

$$\ln \Gamma_q(z) \sim -z \ln(1 - q) + o(z).$$

We have with $o(z)$ precision:

$$\ln \Gamma_q(nz) \sim -nz \ln(1 - q)$$

and

$$\ln \Gamma_{q^n} \left(z + \frac{j}{n} \right) \sim -z \ln(1 - q^n),$$

that is

$$\ln \tilde{\Gamma}_q(nz) = o(z)$$

and similarly

$$\ln \tilde{\Gamma}_{q^n}(z) = o(z),$$

from what $C_1(q, n) = 0$ as well as $C_2(q, n) = 0$ immediately follow. Returning to the q -gamma functions, we have that $\Gamma_q(nz)$ is equal to :

$$C(n, q)(1-q)^{1-nz}(1-q^n)^{z-1}(1-q^n)^{z+\frac{1}{n}-1}(1-q^n)^{z+\frac{2}{n}-1} \dots (1-q^n)^{z+\frac{n-1}{n}-1} \prod_{j=0}^{n-1} \Gamma_{q^n} \left(z + \frac{j}{n} \right)$$

which is equal to

$$C_{\text{mod}}(n, q) \left(\frac{1 - q^n}{1 - q} \right)^{nz} \prod_{j=0}^{n-1} \Gamma_{q^n} \left(z + \frac{j}{n} \right).$$

(Constants C and C_{mod} are different). Comparing the values when z tends to zero using (2.4), we get:

$$-\frac{1-q}{nz \ln q} = -C_{\text{mod}}(n, q) \frac{1-q^n}{nz \ln q} \prod_{j=1}^{n-1} \Gamma_{q^n} \left(z + \frac{j}{n} \right),$$

hence finally we obtain the "standard form" (2.7). $\square$

Remark 2.3. Remind that the Stirling asymptotic formula $\ln \Gamma(z) = z \ln z - z + o(z)$ contains also the $O(z \ln z)$ term lacking in the asymptotic of the q -gamma function. Thus $\ln \Gamma(nz) = nz \ln z - nz + nz \ln n + o(z)$, which explains the origin of the $e^{-nz \ln n}$ exponential term in $\prod_{k=0}^{n-1} \Gamma(z + \frac{k}{n}) = (2\pi)^{\frac{n-1}{2}} n^{\frac{1}{2}} e^{-nz \ln n} \Gamma(nz)$ [5].

Remark 2.4. Actually, rev. Jackson [1] showed that the ratio

$$\begin{aligned} \varphi(z) &:= \frac{1}{\tilde{\Gamma}_q(nz)} \prod_{j=0}^{n-1} \tilde{\Gamma}_{q^n} \left(z + \frac{j}{n} \right) \\ &= \prod_{k=0}^{\infty} f(q, k) \frac{1 - q^{k+nz}}{(1 - q^{nk+nz})(1 - q^{nk+nz+1}) \dots (1 - q^{nk+nz+(n-1)})}, \end{aligned}$$

where the function $f(q, k)$ does not depend on z , is not only (evidently) $\frac{2\pi i}{\ln q}$ -periodic but also 1-periodic. The latter can be seen as follows. If we consider the function $\varphi(z+1)$, we observe that in comparison with the function $\varphi(z)$, in the nominator we miss the product $(1 - q^{nz})(1 - q^{nz+1}) \dots (1 - q^{nz+(n-1)})$ - and this is exactly the factor which becomes missing also in the denominator (it corresponds to the case $k=0$ there), that is $\varphi(z+1) = \varphi(z)$. Thus this function, i.e. the ratio $\prod_{j=0}^{n-1} \frac{\tilde{\Gamma}_{q^n}(z + \frac{j}{n})}{\tilde{\Gamma}_{q^n}(nz)}$, is a double-periodic analytic function and, having no poles, it is a constant.

The proof via manipulation of the q -series (more precisely, the statement that the proof is “the same as given for $\Gamma_{q^2}(2x)\Gamma_{q^2}(1/2) = \Gamma_{q^2}(x)\Gamma_{q^2}(x+1/2)(1+q)^{2x+1}$ ”; in the author’s opinion, however, this is quite difficult to pass in this way) can be found in [2].

Applying (2.1) to the logarithmical derivative of (2.7) with the notation $\Psi_q(z) = \frac{d \ln \Gamma_q(z)}{dz}$, we obtain

$$\begin{aligned} &-n\gamma_q + n \sum_{k=1}^{\infty} \frac{1-q+q^k}{1-q^k} \ln q - \frac{(n-1)}{2} \ln q \\ &= \sum_{j=1}^{n-1} \Psi_{q^n} \left(\frac{j}{n} \right) + n \ln \left(\frac{1-q^n}{1-q} \right) - \gamma_{q^n} - \sum_{k=1}^{\infty} \frac{1-q^n + nq^{kn} \ln q}{1-q^{kn}}. \end{aligned}$$

When $q \rightarrow 1-$, the sums over k tend to zero and $n \ln \left(\frac{1-q^n}{1-q} \right)$ tends to $n \ln n$ so that the Gauss identity for the digamma function [5]

$$\sum_{k=1}^n \Psi \left(\frac{k}{n} \right) = -n(\gamma + \ln n)$$

is retrieved.

Next we, being inspired by the corresponding n -tuple rule for the first elliptic theta-functions [6], prove the following Theorem.

Theorem 2.5. For all integer $n = 2, 3, 4, \dots$, and real $0 < q < 1$ the following n -tuple product rule holds:

$$\Gamma_q(nz) = C(1-q)^{(n^2-n)z} \prod_{l=0}^{n-1} \prod_{j=0}^{n-1} \Gamma_q \left(z + \frac{l}{n} + \frac{j\tau}{n} \right) \quad (2.10)$$

with

$$C^{-1} = n \prod_{l=0}^{n-1} \prod_{\substack{j=0 \\ |l|+|j| \neq 0}}^{n-1} \Gamma_q \left(\frac{l}{n} + \frac{j\tau}{n} \right). \quad (2.11)$$

Here $\tau = -\frac{2\pi i}{\ln q}$.

Proof. In a close analogy with the first theta-function [6], this is straightforward to see that the functions $\Gamma_q(nz)$ and double product $\prod_{l=0}^{n-1} \prod_{j=0}^{n-1} \Gamma_q\left(z + \frac{l}{n} + \frac{j\tau}{n}\right)$ both have no zeroes and the same simple poles lying at $z_{k,m} = -\frac{k}{n} \pm \frac{2\pi mi}{n \ln q}$. The same is valid for the “exponent-free” modified functions, see (2.9), $\tilde{\Gamma}_q(nz)$ and $\prod_{l=0}^{n-1} \prod_{j=0}^{n-1} \tilde{\Gamma}_q\left(z + \frac{l}{n} + \frac{j\tau}{n}\right)$. Thus Lemma 2.1 gives

$$\tilde{\Gamma}_q(nz) = C \exp(C_1 z + C_2 z^2) \prod_{l=0}^{n-1} \prod_{j=0}^{n-1} \tilde{\Gamma}_q\left(z + \frac{l}{n} + \frac{j\tau}{n}\right).$$

The exponent in the right hand side here is identically one as can be shown by the same approach as above implying the Moak’s asymptotic [11]: for large real positive $Re(z)$, $\ln \tilde{\Gamma}_q(nz) \sim o(z)$ and $\ln \tilde{\Gamma}_q\left(z + \frac{l}{n} + \frac{j\tau}{n}\right) \sim o(z)$. Thus we have for the q -gamma functions:

$$\Gamma_q(nz) = C(n, q)(1-q)^{1-nz} \prod_{l=0}^{n-1} \prod_{j=0}^{n-1} (1-q)^{z + \frac{l}{n} + \frac{j\tau}{n} - 1} \prod_{l=0}^{n-1} \prod_{j=0}^{n-1} \Gamma_q\left(z + \frac{l}{n} + \frac{j\tau}{n}\right),$$

i.e.

$$\Gamma_q(nz) = C_{\text{mod}}(n, q)(1-q)^{(n^2-n)z} \prod_{l=0}^{n-1} \prod_{j=0}^{n-1} \Gamma_q\left(z + \frac{l}{n} + \frac{j\tau}{n}\right)$$

(constants C and C_{mod} are different), and equating the values when z tends to zero, we get (2.11). $\square$

Now remind one more Theorem from [7], viz. the Theorem 1 there. We noted that the functions $\theta_1(z | \tau)$ and $\prod_{k=0}^{n-1} \theta_1(z + k\pi\tau | n\tau)$ are both entire and have the same zeroes which leads to the n -tuple product rule relating them. The same is valid for the functions $\Gamma_q(z)$ and $\prod_{k=0}^{n-1} \Gamma_{q^{1/n}}\left(z + \frac{2\pi ik}{\ln q}\right)$ as well as for the modified functions $\tilde{\Gamma}_q(z)$ and $\prod_{k=0}^{n-1} \tilde{\Gamma}_{q^{1/n}}\left(z + \frac{2\pi ik}{\ln q}\right)$. And again, Moak’s asymptotic excludes the exponential factors in

$$\tilde{\Gamma}_q(z) = C \exp(C_1 z + C_2 z^2) \prod_{k=0}^{n-1} \tilde{\Gamma}_{q^{1/n}}\left(z + \frac{2\pi ik}{\ln q}\right).$$

Thus we have

$$\Gamma_q(z) = C(1-q)^{1-z} \prod_{k=0}^{n-1} (1-q)^{z + \frac{2\pi ik}{\ln q} - 1} \Gamma_{q^{1/n}}\left(z + \frac{2\pi ik}{\ln q}\right)$$

and, finally

$$\Gamma_q(z) = \tilde{C}(1-q)^{(n-1)z} \prod_{k=0}^{n-1} \Gamma_{q^{1/n}}\left(z + \frac{2\pi ik}{\ln q}\right).$$

At zero, we have in the leading $O(1/z)$ term:

$$-\frac{1-q}{z \ln q} = -\tilde{C}n \frac{1-q^{1/n}}{z \ln q} \prod_{k=1}^{n-1} \Gamma_{q^{1/n}}\left(\frac{2\pi ik}{\ln q}\right)$$

so that

$$\tilde{C}^{-1} = n \frac{1-q^{1/n}}{1-q} \prod_{k=1}^{n-1} \Gamma_{q^{1/n}}\left(\frac{2\pi ik}{\ln q}\right).$$

Thus we have proven the following Theorem.

Theorem 2.6. For all integer $n = 2, 3, 4, \dots$ and real $0 < q < 1$, the following equation holds:

$$\Gamma_q(z) = \tilde{C}(1-q)^{(n-1)z} \prod_{k=0}^{n-1} \Gamma_{q^{1/n}} \left(z + \frac{2\pi i k}{\ln q} \right), \quad (2.12)$$

with

$$\tilde{C}^{-1} = n \frac{1 - q^{1/n}}{1 - q} \prod_{k=1}^{n-1} \Gamma_{q^{1/n}} \left(\frac{2\pi i k}{\ln q} \right). \quad (2.13)$$

To finish this Section, for completeness let us present two more elementary “transformation rules”. For $q > 1$, the q -gamma function is defined as [2]

$$\Gamma_q(z) = (q-1)^{1-z} q^{z(z-1)/2} \prod_{n=0}^{\infty} \frac{1 - q^{-(n+1)}}{1 - q^{-(n+z)}}. \quad (2.14)$$

It certainly has the same simple poles as $\Gamma_{1/q}(z)$ function (and both have no zeroes), but already in the definition the factor $\exp(Cz^2)$, viz. $q^{z^2/2}$, is present – and this leads to qualitatively different asymptotic. Finally, we obtain $\Gamma_{1/q}(z) = q^{-\frac{(z-1)(z-2)}{2}} \Gamma_q(z)$, see e.g. again Moak’s paper [11].

Next we note that simple poles of $\sinh(z \ln q/2) \Gamma_q(z)$ and $\Gamma_q(z+1)$ are the same. We might use the generalized Littlewood theorem to deal with this circumstance and move further, but the relation between the aforementioned functions is established by elementary means. From

$$\Gamma_q(z+1) = \frac{1 - q^z}{1 - q} \Gamma_q(z)$$

and

$$1 - q^z = q^{z/2} (q^{-z/2} - q^{z/2}) = -2q^{z/2} \sinh(z \ln q/2),$$

it trivially follows that

$$\Gamma_q(z+1) = -\frac{2}{1-q} q^{z/2} \sinh\left(\frac{z \ln q}{2}\right) \Gamma_q(z).$$

Further, from

$$\Gamma_q(z+2) = \frac{1 - q^{z+1}}{1 - q} \Gamma_q(z+1),$$

we get

$$\Gamma_q(z+2) = \frac{4}{(1-q)^2} q^{z/2} q^{z/2+1/2} \sinh\left(\frac{z \ln q}{2}\right) \sinh\left(\frac{(z+1) \ln q}{2}\right) \Gamma_q(z),$$

and so forth:

$$\Gamma_q(z+n) = \frac{2^n (-1)^n}{(1-q)^n} q^{nz/2} q^{n(n-1)/4} \left(\prod_{k=0}^{n-1} \sinh\left(\frac{z+k}{2} \ln q\right) \right) \Gamma_q(z). \quad (2.15)$$

Putting here (2.4), we have for $n = 2, 3, 4, \dots$

$$\Gamma_q(n) = \frac{2^{n-1} (-1)^{n-1}}{(1-q)^{n-1}} q^{n(n-1)/4} \sinh\left(\frac{1}{2} \ln q\right) \dots \sinh\left(\frac{n-1}{2} \ln q\right). \quad (2.16)$$

3 Representation of $\Gamma_q(z)$ via double gamma-function

Let us now demonstrate the representation of the q -gamma function via the double Gamma function, see [12, 13, 14] for the properties of the latter. Double Gamma function may be defined as

$$G(z; \tau) = \frac{z}{\tau} e^{\alpha(\tau) \frac{z}{\tau} + \beta(\tau) \frac{z^2}{2\tau}} \prod_{m \geq 0} \prod_{n \geq 0} \left(1 + \frac{z}{m\tau + n} \right) e^{-\frac{z}{m\tau + n} + \frac{z^2}{(m\tau + n)^2}},$$

and it is an analytic function if $\tau \in C/(-\infty, 0]$. Alternatively, one can use the “symmetric version”, see e.g. Proposition 8.3 in Spreafico paper [14],

$$\Gamma_2(z | a, b) = \frac{1}{z} e^{\alpha(a,b)z + \beta(a,b)\frac{z^2}{2}} \prod_{m \geq 0} \prod_{n \geq 0} \left(1 + \frac{z}{am + bn}\right) e^{-\frac{z}{am+bn} + \frac{z^2}{(am+bn)^2}},$$

and here we will follow this; definitions of α, β also can be found in [13, 14]. For us only $\Gamma_2(z | 1, b)$ with purely imaginary b is sufficient, so that the difference between the definitions is minimal.

We have the following theorem (despite the widespread claim of the close connection between q -gamma function and double gamma function, the author was unable to find exact relation of this type in the literature).

Theorem 3.1. *For real $0 < q < l$, we have*

$$\Gamma_q(z) = \frac{(1-q)^{1-z} \Gamma_2\left(z | 1, \frac{2\pi i}{\ln q}\right) \Gamma_2\left(z | 1, -\frac{2\pi i}{\ln q}\right)}{\ln(1/q) \Gamma(z)}. \quad (3.1)$$

Proof. From the definition, it immediately follows that the double gamma function $\Gamma_2(z | 1, -\frac{2\pi i}{\ln q})$ has no zeroes and has simple poles at $z_{k,l} = -k + \frac{2\pi il}{\ln q}$, where k, l are non-negative integers. The function $\Gamma_2(z | 1, \frac{2\pi i}{\ln q})$ has no zeroes and has simple poles at $z_{k,l} = -k - \frac{2\pi il}{\ln q}$, and the product $\Gamma_2(z | 1, \frac{2\pi i}{\ln q}) \Gamma_2(z | 1, -\frac{2\pi i}{\ln q})$ has no zeroes and simple poles at $z_{k,l} = -k \pm \frac{2\pi il}{\ln q}$ for $l \neq 0$, and has double poles at $z_{k,l} = -k$. The latter can be converted into single poles by division on the gamma function $\Gamma(z)$, and we see that the functions $\Gamma_2(z | 1, \frac{2\pi i}{\ln q}) \Gamma_2(z | 1, -\frac{2\pi i}{\ln q}) / \Gamma(z)$ and $\Gamma_q(z)$ have the same simple poles and (no) zeroes. Thus the inverse application of the generalized Littlewood theorem to them would be possible provided we know the asymptotic of the logarithms of these functions at large $|z|$. For the double gamma-function, we have by Proposition 8.11 in [14]:

$$\begin{aligned} \ln \Gamma_2(z | a, b) &\sim -\frac{1}{2ab} z^2 \ln z + \frac{1}{2} \left(\frac{1}{a} + \frac{1}{b} \right) z \ln z \\ &\quad - \left(\frac{1}{4} + \frac{1}{12} \left(\frac{a}{b} + \frac{b}{a} \right) \right) \ln z + \frac{3}{4ab} z^2 - \frac{1}{2} \left(\frac{1}{a} + \frac{1}{b} \right) z + O(1) \end{aligned}$$

Thus

$$\ln \Gamma_2(z | 1, b) + \ln \Gamma_2(z | 1, -b) \sim z \ln z - \frac{1}{2} \ln z + O(1).$$

(Spreafico works under the condition that $\operatorname{Re} a, \operatorname{Re} b > 0$ but this asymptotic holds for our case as well). Thus we use $n = 3$ in the conditions of our Lemma 2.1. Then we need also the aforementioned Moak’s asymptotic [11] $\ln \Gamma_q(z) \sim -z \ln(1-q)$, and classic Stirling’s result $\ln \Gamma(z) \sim (z - 1/2) \ln z - z + O(1)$. Thus for the relation

$$\Gamma_q(z) = C \exp(C_1 z + c_2 z^2) \Gamma_2\left(z | 1, \frac{2\pi i}{\ln q}\right) \Gamma_2\left(z | 1, -\frac{2\pi i}{\ln q}\right) / \Gamma(z),$$

we immediately get $C_2 = 0$ and $C_1 = -\ln(1-q)$ whence

$$\Gamma_q(z) = C(1-q)^{-z} \Gamma_2\left(z | 1, \frac{2\pi i}{\ln q}\right) \Gamma_2\left(z | 1, -\frac{2\pi i}{\ln q}\right) / \Gamma(z).$$

Coefficient C is determined comparing asymptotic at z tending to zero. From the definition we have $\Gamma_2(z | a, b) = \frac{1}{z} + O(1)$ thus $1 = -C \frac{1-q}{\ln q}$ and (3.1) follows. $\square$

4 Representation of $\Gamma_q(z) \Gamma_q(1-z)$ via elliptic theta-functions

The product $\Gamma_q^{-1}(z) \Gamma_q^{-1}(1-z)$ is an entire function having simple zeroes at $z = \pm k \pm \frac{2\pi il}{\ln q}$. Thus, it has exactly the same zeroes and (no) poles as the elliptic theta function $\theta_1(\pi z | \tau)$ with

$$\tau = -\frac{2\pi i}{\ln q} \quad (4.1)$$

For the definition and discussion of the main properties of theta-functions, see e.g. [15, 16, 17]; see also [6, 7]. Note, that the formula (4.1) is different from the standard relation between q and τ in the theory of elliptic theta-functions, which is $q = e^{i\pi\tau}$. We believe that this cannot lead to confusion because in the paper we will not use the q -representation of the elliptic theta-functions.

The following Theorem holds.

Theorem 4.1. *For real $0 < q < 1$ and with $\tau = -\frac{2\pi i}{\ln q}$*

$$\Gamma_q^{-1}(z)\Gamma_q^{-1}(1-z) = \frac{2i}{(1-q)\theta_1'(0 | -\frac{i\ln q}{2\pi})} q^{z/2} \theta_1\left(-\frac{iz\ln q}{2} \middle| -\frac{i\ln q}{2\pi}\right) \quad (4.2)$$

or

$$\Gamma_q^{-1}(z)\Gamma_q^{-1}(1-z) = \frac{q^{-1/4}}{\Gamma_q^2(\frac{1}{2})\theta_1(-\frac{i\ln q}{4} | -\frac{i\ln q}{2\pi})} q^{z/2} \theta_1\left(-\frac{iz\ln q}{2} \middle| -\frac{i\ln q}{2\pi}\right); \quad (4.3)$$

and

$$\Gamma_q^{-1}(z)\Gamma_q^{-1}(1-z) = \frac{\ln(1/q)}{\pi(1-q)\theta_1'(0 | -\frac{2\pi i}{\ln q})} q^{\frac{z-z^2}{2}} \theta_1\left(\pi z \middle| -\frac{2\pi i}{\ln q}\right), \quad (4.4)$$

or

$$\Gamma_q^{-1}(z)\Gamma_q^{-1}(1-z) = \frac{q^{-1/8}}{\Gamma_q^2(\frac{1}{2})\theta_2(0 | -\frac{2\pi i}{\ln q})} q^{\frac{z-z^2}{2}} \theta_1\left(\pi z \middle| -\frac{2\pi i}{\ln q}\right). \quad (4.5)$$

Proof. Let us first try a standard “brute force” approach. Given the aforementioned asymptotic of the q -gamma function and that of the theta-function for large $|z|$, see [6], we have from Lemma 2.1: $[\Gamma_q(z)\Gamma_q(1-z)]^{-1} = C \exp(C_2 z^2 + C_1 z) \theta_1(\pi z | \tau)$, that is

$$[\Gamma_q(z)\Gamma_q(1-z)]^{-1} = C \exp(C_2 z^2 + C_1 z) \theta_1(\pi z | \tau)/z. \quad (4.6)$$

Unfortunately, z -periods of the product of q -gamma functions and $\theta_1(\pi z | \tau)$ are incommensurable (see below), so that we have no other option than to analyze the Taylor expressions of the logarithms of both sides of eq. (4.6) to find the exponential factor. We know

$$\ln[\theta_1(\pi z | \tau)/(\pi z \theta_1'(0 | \tau))] = -\sum_{j=1}^{\infty} \frac{1}{2j} \delta_{2j}(z^{2j}), \quad (4.7)$$

where

$$\delta_{2j}(\tau) = \sum_{n=-\infty}^{\infty} \sum_{\substack{m=-\infty \\ |m|+|n|\neq 0}}^{\infty} \frac{1}{(m+n\tau)^{2j}}, \quad (4.8)$$

the order of summation is important for $j = 1$ [15]. We establish the Taylor expansion of the l.h.s exploiting the relations (2.3), (2.4) supplemented with the elementary

$$\ln(1-q^z) = \ln z + \ln(-\ln q) + z \frac{1}{2} \ln q + \frac{z^2}{24} \ln^2 q + O(z^4). \quad (4.9)$$

We have first, from (2.1) by differentiation

$$\frac{d^2}{dz^2} \ln \Gamma_q(1+z)|_{z=0} = \ln^2 q \sum_{k=1}^{\infty} \frac{q^k}{(1-q^k)^2},$$

and also

$$\begin{aligned} \ln(z\Gamma_q(z)\Gamma_q(1-z)) &= \ln z + \ln(\Gamma_q(1+z)\Gamma_q(1-z)) + \ln \frac{1-q}{1-qz} \\ &= \ln z + \ln(1-q) - \ln(1-q^z) + z^2 \ln^2 q \sum_{k=1}^{\infty} \frac{q^k}{(1-q^k)^2} \\ &= \ln(1-q) - \ln(\ln(1/q)) - \frac{\ln q}{2} z + z^2 \ln^2 q \left(\sum_{k=1}^{\infty} \frac{q^k}{(1-q^k)^2} - \frac{1}{24} \right) \\ &\quad + O(z^4). \end{aligned} \quad (4.10)$$

Equating the Taylor coefficients of logarithms in the both sides of (4.6), we get

$$\ln(1 - q) + \ln(\ln(1/q)) = \ln C + \ln(\pi\theta'_1(0 | \tau)), \quad \frac{1}{2} \ln q = C_1,$$

$$-\ln^2 q \left(\sum_{k=1}^{\infty} \frac{q^k}{(1 - q^k)^2} - \frac{1}{24} \right) = C_2 + \frac{\delta_2}{2},$$

therefore

$$C = \frac{\ln(1/q)}{(1 - q)\pi\theta'_1(0 | \tau)}, \quad (4.11)$$

and

$$C_2 = \frac{\delta_2}{2} - \ln^2 q \left(\sum_{k=1}^{\infty} \frac{q^k}{(1 - q^k)^2} - \frac{1}{24} \right), \quad (4.12)$$

so that

$$\Gamma_q^{-1}(z)\Gamma_q^{-1}(1 - z) = \frac{\ln(1/q)q^{z/2}}{(1 - q)\pi\theta'_1(0 | -2\pi i/\ln q)} \exp(C_2 z^2)\theta_1(\pi z | -2\pi i/\ln q).$$

As we see, these formulae do not look easy to use (how to calculate explicitly the value of C_2 ? – but see below), so we prefer a different approach. From the definition we have

$$\Gamma_q(z)\Gamma_q(1 - z) = (1 - q) \prod_{n=0}^{\infty} \frac{(1 - q^{n+1})^2}{(1 - q^{n+z})(1 - q^{n+1-z})},$$

and this function is clearly $-\frac{2\pi i}{\ln q}$ -periodic (i.e. τ - and hence also 2τ -periodic) in z . At the same time, the function $\theta_1(\pi z | \tau)$ is 2-periodic, and this already mentioned above incommensurability of the periods complicates the situation. To make it easier, it is convenient to consider, being inspired by the first fundamental modular transformation of the theta-functions [15, 16, 17, 6], the transformed function $\theta_1(\pi z/\tau | -1/\tau)$. It has the same zeroes as $\theta_1(\pi z | \tau)$ but is 2τ -periodic in z . This helps us to answer the question concerning the coefficient C_2 in the formula

$$[\Gamma_q(z)\Gamma_q(1 - z)]^{-1} = C \exp(C_2 z^2 + C_1 z)\theta_1\left(\frac{\pi z}{\tau} \mid -\frac{1}{\tau}\right)$$

(this one we can immediately write): only $C_2 = 0$ is compatible with the aforementioned 2τ -periodicity in z of both sides; cf. similar reasoning in [6, 7].

To find C_1 and C we again use the same Taylor expansions of the logarithms of the functions involved, eq. (4.10), and obtain

$$[\Gamma_q(z)\Gamma_q(1 - z)]^{-1} = \frac{\tau \ln q}{\pi(1 - q)\theta'_1(0 | -1/\tau)} q^{z/2} \theta_1\left(\frac{\pi z}{\tau} \mid -\frac{1}{\tau}\right),$$

that is

$$\Gamma_q^{-1}(z)\Gamma_q^{-1}(1 - z) = \frac{2i}{(1 - q)\theta'_1(0 | -\frac{i \ln q}{2\pi})} q^{z/2} \theta_1\left(-\frac{iz \ln q}{2} \mid -\frac{i \ln q}{2\pi}\right). \quad (4.13)$$

Now we explicitly use the first transformation rule [15]

$$\theta_1(z | \tau) = -i(-i\tau)^{-1/2} \exp\left(-\frac{i\pi z^2}{\tau}\right) \theta_1(-z/\tau | -1/\tau),$$

that is

$$\theta_1(-z/\tau | -1/\tau) = i(-i\tau)^{-1/2} \exp\left(\frac{i\pi z^2}{\tau}\right) \theta_1(z | \tau),$$

to write

$$\Gamma_q^{-1}(z)\Gamma_q^{-1}(1 - z) = C \exp\left(-\frac{z^2 \ln q}{2}\right) q^{z/2} \theta_1\left(\pi z \mid -\frac{2\pi i}{\ln q}\right). \quad (4.14)$$

This determines the coefficient C_2 in (4.6): $C_2 = -\frac{\ln q}{2}$, and also explicitly determines the coefficient C there. But to find the latter, actually this is easier to proceed by directly equating expressions in both sides of (4.6) at $z = 0$: $-z\frac{\ln q}{1-q} = C\pi z\theta'_1(0 \mid \tau)$, thus

$$\Gamma_q^{-1}(z)\Gamma_q^{-1}(1-z) = \frac{\ln(1/q)}{\pi(1-q)\theta'_1(0 \mid -\frac{2\pi i}{\ln q})} q^{\frac{z-z^2}{2}} \theta_1\left(\pi z \mid -\frac{2\pi i}{\ln q}\right).$$

Certainly, other than $z = 0$ values can be used to find the value of the constant C . For example, we can use $z = 1/2$ in (4.14) and get

$$\Gamma_1^{-2}\left(\frac{1}{2}\right) = Cq^{1/4}\theta_1\left(-\frac{i\ln q}{4} \mid -\frac{i\ln q}{2\pi}\right),$$

thus

$$C = q^{-1/4}\Gamma_1^{-2}\left(\frac{1}{2}\right)\theta_1^{-1}\left(-\frac{i\ln q}{4} \mid -\frac{i\ln q}{2\pi}\right)$$

and

$$\Gamma_q^{-1}(z)\Gamma_q^{-1}(1-z) = \frac{q^{1/4}}{\Gamma_1^2\left(\frac{1}{2}\right)\theta_1\left(-\frac{i\ln q}{4} \mid -\frac{i\ln q}{2\pi}\right)} q^{z/2}\theta_1\left(-\frac{iz\ln q}{2} \mid -\frac{i\ln q}{2\pi}\right).$$

Also

$$\Gamma_q^{-2}(1/2) = C \exp\left(-\frac{\ln q}{8}\right) q^{1/4}\theta_1\left(\frac{\pi}{2} \mid -\frac{2\pi i}{\ln q}\right),$$

thus

$$C = \Gamma_q^{-2}(1/2)q^{-1/8}\theta_1^{-1}\left(\frac{\pi}{2} \mid -\frac{2\pi i}{\ln q}\right)$$

and

$$\Gamma_q^{-1}(z)\Gamma_q^{-1}(1-z) = \frac{q^{-1/8}}{\Gamma_1^2\left(\frac{1}{2}\right)\theta_2\left(0 \mid -\frac{2\pi i}{\ln q}\right)} q^{\frac{z-z^2}{2}} \theta_1\left(\pi z \mid -\frac{2\pi i}{\ln q}\right).$$

(We used $\theta_1\left(\frac{\pi}{2} \mid \tau\right) = \theta_2(0 \mid \tau)$). □

Note that in a few cases, the values of the $\Gamma_q^2(1/2)$ are known [18], as, for example,

$$\Gamma_{e^{-2\pi}}^2(1/2) = \frac{e^{-7\pi/8}\sqrt{e^{2\pi-1}}}{29/8\pi^{3/4}}\Gamma(1/4).$$

Let us discuss some corollaries of this Theorem.

Corollary 4.2. For real $0 < q < 1$ and with $\tau = -\frac{2\pi i}{\ln q}$,

$$\delta_2(\tau) = 2\ln^2 q \left(\sum_{k=1}^{\infty} \frac{q^k}{(1-q^k)^2} - \frac{1}{24} \right) - \ln q. \quad (4.15)$$

Proof. During the proof of the previous theorem, we have established

$$C_2 = \frac{\delta_2}{2} - \ln^2 q \left(\sum_{k=1}^{\infty} \frac{q^k}{(1-q^k)^2} - \frac{1}{24} \right) \text{ and } C_2 = -\frac{\ln q}{2}.$$

□

For example, for $\delta_2(i) = \pi$ [15], we get

$$\sum_{n=1}^{\infty} \frac{e^{-2\pi n}}{(1-e^{-2\pi n})^2} = \frac{1}{24} - \frac{1}{8\pi}.$$

Or we can rewrite

$$\delta_2(\tau) = -\ln q + \ln^2 q \left(\frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{\sinh^2(k \ln q/2)} - \frac{1}{12} \right),$$

where we used

$$\sum_{k=1}^{\infty} \frac{q^k}{(1-q^k)^2} = \sum_{k=1}^{\infty} \frac{1}{(q^{-k/2} - q^{k/2})^2} = \frac{1}{4} \sum_{k=1}^{\infty} \frac{1}{\sinh^2(k \ln q/2)},$$

and hence

$$\sum_{k=1}^{\infty} \frac{1}{\sinh^2(\pi k)} = \frac{1}{6} - \frac{1}{2\pi},$$

which is the Liang equality [19].

A number of other equalities for similar sums over powers of inverse hyperbolic functions are known, see [19, 20] and references therein. In particular,

$$\sum_{k=1}^{\infty} \frac{1}{\sinh^2(\sqrt{3}\pi k)} = \frac{1}{6} - \frac{\sqrt{3}}{6\pi} - \frac{\nu_1}{8\pi^2}$$

where

$$\nu_1 = [35\delta_6(i\sqrt{3})]^{1/3},$$

and

$$\sum_{k=1}^{\infty} \frac{1}{\sinh^2(\pi k/\sqrt{3})} = \frac{1}{6} - \frac{\sqrt{3}}{2\pi} + \frac{3\nu_2}{8\pi^2}$$

where

$$\nu_2 = [35\delta_6(i/\sqrt{3})]^{1/3};$$

for the definition of $\delta_6(\tau)$ see eq. (4.8). Similarly, for $q = e^{2\sqrt{3}\pi}$, that is $\tau = i/\sqrt{3}$, the following not-without-the-interest relations between δ_2 and δ_6 can be obtained:

$$\delta_2(i/\sqrt{3}) = 2\pi\sqrt{3} + 12\pi^2 \left(\frac{1}{12} - \frac{\sqrt{3}}{12\pi} + \frac{3\nu_2}{16\pi^2} - \frac{1}{12} \right),$$

$$\delta_2(i/\sqrt{3}) = 2\pi\sqrt{3} - \pi\sqrt{3} + \frac{9}{4}\nu_2^2$$

and

$$\delta_2(i/\sqrt{3}) - \frac{9}{4} [35\delta_6(i/\sqrt{3})]^{2/3} = \pi\sqrt{3}.$$

Also

$$\delta_2(i\sqrt{3}) = 2\pi/\sqrt{3} + \frac{4\pi^2}{3} \left(\frac{1}{12} - \frac{\sqrt{3}}{12\pi} + \frac{\nu_1}{16\pi^2} - \frac{1}{12} \right),$$

thus

$$\delta_2(i\sqrt{3}) = 2\pi/\sqrt{3} - \frac{\pi}{3\sqrt{3}} + \frac{\nu_1}{12}$$

and

$$\delta_2(i\sqrt{3}) - \frac{1}{12} [35\delta_6(i\sqrt{3})]^{2/3} = \frac{5\pi}{9\sqrt{3}}.$$

For completeness, let us present here one more known related equality [21]:

$$\sum_{n=1}^{\infty} \sigma_1(n) e^{-2\pi n} = \frac{1}{24} - \frac{1}{8\pi}.$$

Here, as usual, $\sigma_1(n) = \sum_{k=1, k|n}^{\infty} k$ is the sum of positive divisors of n . From the general relation $\sum_{n=1}^{\infty} \frac{n^a q^n}{1-q^n} = \sum_{n=1}^{\infty} \sigma_a(n) q^n$, where $\sigma_a(n) = \sum_{k=1, k|n}^{\infty} k^a$ [22], it follows also that

$$\sum_{n=1}^{\infty} \frac{n e^{-2\pi n}}{1 - e^{-2\pi n}} = \frac{1}{24} - \frac{1}{8\pi} = \sum_{n=1}^{\infty} \frac{e^{-2\pi n}}{(1 - e^{-2\pi n})^2}. \quad (4.16)$$

Remark 4.3. It seems that the above representations of the product $\Gamma_q^{-1}(z)\Gamma_q^{-1}(1-z)$ via the first theta-functions, albeit their existence looks evident, somehow are not well known. Still their analogues appeared at certain instances. Cf., for example, Mezo paper [23], where for $q > 1$ it is shown that

$$[\Gamma_{q^2}(z)\Gamma_{q^2}(1-z)]^{-1} = C(q)q^{-2z^2+2z}\vartheta_4\left(\frac{1}{2i}(1-2z)\ln q, \frac{1}{q}\right)$$

with

$$\vartheta_4(z, q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} e^{2niz}$$

and

$$C(q) = \frac{1}{q^2 - 1} \prod_{j=1}^{\infty} (1 - q^{-2j})^{-3}.$$

Moreover, remind [15]

$$\sigma(z, \tau) = \exp\left(\frac{z^2}{2}\delta_2(\tau)\right) \frac{\theta_1(\pi z | \tau)}{\pi\theta_1'(0 | \tau)}$$

(σ, ζ and $\wp$ here and below are the corresponding Weierstrass elliptic functions), and write:

$$\begin{aligned} [\Gamma_q(z)\Gamma_q(1-z)]^{-1} &= -\frac{\ln q}{1-q} q^{\frac{-z^2+z}{2}} \exp\left(-\frac{1}{2}\delta_2 z^2\right) \sigma(z, \tau) \\ &= -\frac{\ln q}{1-q} \exp\left(-\frac{1}{2}[(\delta_2(\tau) + \ln q)z^2 - z \ln q]\right) \sigma(z, \tau). \end{aligned}$$

Logarithmical derivative of this expression is

$$\frac{d}{dz} \ln(\Gamma_q(z)\Gamma_q(1-z)) = -\zeta(z, \tau) + (\delta_2(\tau) + \ln q)z + \frac{1}{2} \ln q,$$

and further:

$$\begin{aligned} \frac{d^2}{dz^2} \ln(\Gamma_q(z)\Gamma_q(1-z)) &= \wp(z, \tau) + \delta_2(\tau) + \ln q, \\ \frac{d^3}{dz^3} \ln(\Gamma_q(z)\Gamma_q(1-z)) &= \wp_z(z, \tau). \end{aligned}$$

In essence, similar relations have been established already by rev. Jackson [1] - so that his results actually can be used as a starting point.

4.1 Proof of Gosper's theorem

Now, mainly to illustrate the utility of the inverse applications of the generalized Littlewood theorem, we prove the following statement concerning the so-called Gosper's q -sine and cosine functions, which he defined, again for real $0 < q < 1$ and in the notation of q -series, as [24]

$$\sin_q(\pi z) = q^{(z-1/2)^2} \frac{(q^{2z}; q^2)(q^{2-2z}; q^2)}{(q; q^2)}$$

and

$$\cos_q(\pi z) = q^{z^2} \frac{(q^{1-2z}; q^2)(q^{2z+1}; q^2)}{(q; q^2)}.$$

Gosper showed that

$$\sin_q(z) = \frac{\theta_1(z, p)}{\theta_2(0, p)} \quad \text{and} \quad \cos_q(z) = \frac{\theta_2(z, p)}{\theta_2(0, p)},$$

where $\ln p \cdot \ln q = \pi^2$. Thus $p = e^{\pi^2/\ln q}$ and, reminding the relation between τ and q -(p -) representation of the elliptical theta functions $p = e^{i\pi\tau}$, we write

$$\sin_q(z) = \frac{\theta_1(z | \tau)}{\theta_2(0 | \tau)} \quad \text{and} \quad \cos_q(z) = \frac{\theta_2(z | \tau)}{\theta_2(0 | \tau)} \quad \text{with } \tau = -\frac{i\pi}{\ln q}.$$

Then he gave very convincing numerical evidences, but not rigorous proofs, of the relations

$$\sin_q(2z) = \text{Const}(q) \sin_{q^2}(z) \cos_{q^2}(z) \quad \text{and} \quad \cos_q(2z) = \cos_{q^2}^2(z) - \sin_{q^2}^2(z).$$

The first of these two was afterwards proven by Mezo in rather lengthy way [25] while the second apparently rests formally unproven.

Proposition 4.4. *For $0 < q < 1$, with the definitions given above*

$$\sin_q(2z) = \frac{2\theta_2(0 | \tau/2)\theta'_1(0 | \tau)}{\theta_2(0 | \tau)\theta'_1(0 | \tau/2)} \sin_{q^2}(z) \cos_{q^2}(z) \quad (4.17)$$

and

$$\cos_q(2z) = \cos_{q^2}^2(z) - \sin_{q^2}^2(z). \quad (4.18)$$

Here $\tau = -\frac{i\pi}{\ln q}$.

Proof. As for the first Gosper's formula $\sin_q(2z) = \text{Const}(q) \sin_{q^2}(z) \cos_{q^2}(z)$, in the theta-function parlance this is $\theta_1(2z | \tau) = C\theta_1(z | \tau/2)\theta_2(z | \tau/2)$ which is extremely well known Landen transformation: for arbitrary τ with $\text{Im}\tau > 0$,

$$\theta_1(2z | 2\tau) = \frac{\theta_1(z | \tau)\theta_2(z | \tau)}{\theta_4(0 | 2\tau)},$$

see e.g. [15]. It is, of course, a particular case of the relation [6]

$$\theta_1(nz | n\tau) = \frac{n\theta'_1(0 | n\tau)}{\theta'_1(0 | \tau) \prod_{k=1}^{n-1} \theta_1(k\pi/n | \tau)} \prod_{k=0}^{n-1} \theta_1\left(z + \frac{k\pi}{n} | \tau\right)$$

together with the relation $\theta_1(z + \pi/2 | \tau) = \theta_2(z | \tau/2)$. Thus for $n = 2$:

$$\theta_1(2z | 2\tau) = \frac{2\theta'_1(0 | 2\tau)\theta_1(z | \tau)\theta_2(z | \tau)}{\theta'_1(0, \tau)\theta_2(0 | \tau)},$$

and by the way this is also a proof of

$$\frac{2\theta'_1(0 | 2\tau)}{\theta'_1(0, \tau)\theta_2(0 | \tau)} = \frac{1}{\theta_4(0 | 2\tau)}.$$

Reminding the formula $\cos_q(z) = \frac{\theta_2(z|\tau)}{\theta_2(0|\tau)}$, we have

$$\sin_q(2z) = \frac{2\theta_2(0 | \tau/2)\theta'_1(0 | \tau)}{\theta_2(0 | \tau)\theta'_1(0 | \tau/2)} \sin_{q^2}(z) \cos_{q^2}(z),$$

or

$$\sin_q(2z) = \frac{\theta_2^2(0 | \tau/2)}{\theta_2(0 | \tau)\theta_4(0 | \tau)} \sin_{q^2}(z) \cos_{q^2}(z),$$

where $\tau = -\frac{i\pi}{\ln q}$. In Mezö's Theorem 3 [25], the value of the constant is given as $C = q^{-1/4} \frac{(q^2; q^4)_\infty}{(q; q^2)_\infty^2}$.

Second Gosper's formula $\cos_q(2z) = \cos_q^2(z) - \sin_q^2(z)$ requires a bit more efforts. In terms of theta-functions, it reads

$$\frac{\theta_2(2z | \tau)}{\theta_2(0 | \tau)} = \frac{1}{\theta_2^2(0 | \tau/2)} (\theta_1(z | \tau/2) - \theta_2(z | \tau/2)) (\theta_1(z | \tau/2) + \theta_2(z | \tau/2)).$$

Let us transform $\tau \rightarrow 2\tau$ to handle

$$\frac{\theta_2(2z | 2\tau)}{\theta_2(0 | 2\tau)} = \frac{1}{\theta_2^2(0 | \tau)} (\theta_1(z | \tau) - \theta_2(z | \tau)) (\theta_1(z | \tau) + \theta_2(z | \tau)).$$

Let τ be arbitrary with $\text{Im}\tau > 0$. We first see that zeroes of $\theta_2(2z | \tau)$ lie at

$$2z = (m + 1/2 + 2n\tau)\pi \Rightarrow z = (m/2 + 1/4 + n\tau)\pi,$$

that is at $(k \pm 1/4 + n\tau)\pi$; here $k, m, n \in \mathbb{Z}$. Zeroes of the $(\theta_1(z | \tau) - \theta_2(z | \tau))(\theta_1(z | \tau) + \theta_2(z | \tau))$ are the solutions of equations $\frac{\theta_1(z | \tau)}{\theta_2(z | \tau)} = \pm 1$ (certainly, z such that $\theta_2(z | \tau) = 0$ are not zeroes of the product at hand). From the definitions [15],

$$\theta_1(z, q) = 2 \sum_{n=0}^{\infty} (-1)^n q^{(n+1/2)^2} \sin[(2n+1)z], \quad \theta_2(z, q) = 2 \sum_{n=0}^{\infty} q^{(n+1/2)^2} \cos[(2n+1)z],$$

it is trivial to see that for any q , and hence for any related τ , $\frac{\theta_1(\pi/4 | \tau)}{\theta_2(\pi/4 | \tau)} = 1$ and $\frac{\theta_1(-\pi/4 | \tau)}{\theta_2(-\pi/4 | \tau)} = -1$. Then the periodicity and quasiperiodicity conditions (translation by τ) [15] immediately show that in the fundamental parallelogram P we have as the roots of equation $\frac{\theta_1(z | \tau)}{\theta_2(z | \tau)} = 1$ the values $z = \pi/4$ and $z = \pi/4 + \pi\tau$ and as the roots of $\frac{\theta_1(z | \tau)}{\theta_2(z | \tau)} = -1$ equation the values $z = 3\pi/4$ and $z = 3\pi/4 + \pi\tau$. There are no more roots of these equations in this parallelogram because for any complex a , the equation $\frac{\theta_1(z | \tau)}{\theta_2(z | \tau)} = a$ has exactly two roots. Namely, the function $\frac{\theta_1(z | \tau)}{\theta_2(z | \tau)}$ is double-periodic (actually it is a rescaling of the quite known (but not such famous as its sn , cn and dn analogues) Jacobi function $sc(z, q)$ [15]), and has in the fundamental parallelogram two simple poles. Thus this statement is the immediate consequence of the argument principle, viz. the application of the

$$\int_P \frac{[\theta_1(z | \tau)/\theta_2(z | \tau) - a]'}{\theta_1(z | \tau)/\theta_2(z | \tau) - a} dz = 0.$$

We have proven that all zeroes of the equations $\frac{\theta_1(z | \tau)}{\theta_2(z | \tau)} = \pm 1$ lie at $(m \pm 1/4 + n\tau)\pi$ whence coincide with those of $\theta_2(2z | 2\tau)$. Our Lemma 2.1 is applicable and gives

$$\theta_2(2z | 2\tau) = C \exp(C_1 z + C_2 z^2) (\theta_2^2(z | \tau) - \theta_1^2(z | \tau)).$$

Exponential factors are excluded by the same reasoning as (almost) always takes place for theta-function equalities (periodicity condition and impossibility to obtain purely imaginary C_1 value from the Taylor expansions of logarithms), and the constant C is calculated by comparison at $z = 0$:

$$\theta_2(2z | 2\tau) = \frac{\theta_2(0 | 2\tau)}{\theta_2^2(0 | \tau)} (\theta_2^2(z | \tau) - \theta_1^2(z | \tau)).$$

This is valid for all τ with $\text{Im}\tau > 0$. Reminding $\sin_q(z) = \frac{\theta_1(z | \tau)}{\theta_2(0 | \tau)}$ and $\cos_q(z) = \frac{\theta_2(z | \tau)}{\theta_2(0 | \tau)}$, we get $\cos_q(2z) = \cos_q^2(z) - \sin_q^2(z)$.

□

5 Conclusion

Thus, in the present paper we apply inverse Littlewood theorem concern contour integrals of the logarithm of an analytic function to the 1-gamma function. A number of n -tuple product rules and representations of this function were proven. The case how the function $\Gamma_q(z)\Gamma_q(1-z)$ is represented via elliptical theta-function is also systematically studied.

Certainly, other applications of this type can be found. For example numerous transformation laws for the double gamma-function, as for instance,

$$G(nz; n\tau) = \prod_{l=0}^{n-1} \frac{G(z + l/n; \tau)}{G(1 + l/n; \tau)},$$

see [13] can be without difficulties proven by our method, but this lies out of the scope of the paper.

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