

ON THE DENSITY OF 2-CLASS GROUPS OF PURE QUARTIC NUMBER FIELDS $\mathbb{Q}(\sqrt[4]{2d^2})$

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Abstract Let $K = \mathbb{Q}(\sqrt[4]{2d^2})$ be a real pure quartic number field, where $d = q_1 q_2 \cdots q_u$ is the product of u distinct odd primes. In this paper, we analyze the densities of certain 2-class groups associated with K , demonstrating that their distribution exhibits an orderly and deterministic pattern. Our approach combines analytical techniques with computational methods. To support our theoretical results, we perform numerical simulations using the Pari/GP computer algebra system, providing a comparative analysis of the observed and predicted behaviors.

1 Introduction

Let p be a prime number, and let K be an algebraic number field. Denote by $Cl_p(K)$ the p -Sylow subgroup of the class group $Cl(K)$ of K , and define the p^k -rank $r_{p^k}(K)$ of $Cl_p(K)$ as the number of cyclic components of $Cl_p(K)$ of order divisible by p^k . In other words,

$$r_{p^k}(K) = \dim_{\mathbb{F}_p} \left((Cl_p(K))^{p^{k-1}} / (Cl_p(K))^{p^k} \right).$$

The study of the density of number fields possessing a predetermined p -class group is a central problem in arithmetic statistics. For quadratic number fields, this topic has attracted considerable attention, particularly in light of the Cohen–Lenstra conjectures [1], which predict the probabilistic distribution of odd p -class groups, with a focus on the p^k -rank. These conjectures were later extended to the case $p = 2$ by Gerth [7, 9]. For quartic number fields, the situation is more intricate. The Cohen–Martinet heuristics [2, 3] provide a framework for understanding the distribution of class groups, particularly for "good" primes p -those not dividing the degree of the Galois closure of the field. Conversely, "bad" primes are those that divide this degree, introducing additional complexities. Gerth further explored the densities of cyclic p -extensions of $\mathbb{Q}$ with specified p^n -rank in [8, 10]. Other significant contributions include the works of Fouvry, Klüners, Stevenhagen, and Milovic [4, 5, 18, 17, 14].

This paper investigates the distribution of real pure quartic fields, specifically those having $\mathbb{Q}(\sqrt{2})$ as the unique quadratic subfield with a specified 2-class group. Notably, we exclude cases where the pure quartic field contains $\mathbb{Q}(\sqrt{p})$ as a quadratic subfield, as the density of such fields is negligible compared to fields where $\mathbb{Q}(\sqrt{p})$ is absent. These cases, corresponding to non-Galois fields K whose Galois closure $\mathbb{Q}(\sqrt[4]{2d^2}, i)$ has degree 8 divisible by 2, are deemed "bad".

Let $K = \mathbb{Q}(\sqrt[4]{2d^2})$ be a real pure quartic field where d is an odd squarefree integer. Then $K/\mathbb{Q}(\sqrt{2})$ is a quadratic extension. Using results from [11], the order of the 2-class group of K can be determined via the 2-Sylow subgroup of the ambiguous class group of $K/\mathbb{Q}(\sqrt{2})$.

Specifically, the possible forms of d for which the 2-class group is cyclic, trivial, or of type $(2^{n_1}, 2^{n_2})$ are characterized. These results are summarized in Theorem 1. In particular, we can deduce the following result from [11, Corollaries 4.1–4.3].

Theorem 1.1 ([11]). *Let $K = \mathbb{Q}(\sqrt[4]{2d^2})$ be a real pure quartic number field, such that d is an odd square-free integer and n_1, n_2 are two positive integers. Then*

(i) *The 2-class group of K is trivial if and only if d take one of the following forms:*

- a. $d = q \equiv 3 \pmod{8}$,
- b. $d = 1$.

(ii) *The 2-class group of K is cyclic if and only if d take one of the following forms:*

- a. $d = q \equiv 5, 7 \pmod{8}$,
- b. $d = q \equiv 1 \pmod{8}$, $\left(\frac{2}{q}\right)_4 (-1)^{\frac{q-1}{8}} = -1$,
- c. $d = q_1 q_2$, $(q_1, q_2) \equiv (3, 5), (3, 3) \pmod{8}$.

(iii) *The 2-class group of K is of the type $(2^{n_1}, 2^{n_2})$ if and only if d take one of the following forms:*

- a. $d = q_1 q_2$, $(q_1, q_2) \equiv (7, 3), (7, 5), (1, 3), (5, 5) \pmod{8}$,
- b. $d = q_1 q_2$, $(q_1, q_2) \equiv (1, 5) \pmod{8}$ and $\left(\frac{2}{q_1}\right)_4 (-1)^{\frac{q_1-1}{8}} = -1$,
- c. $d = q \equiv 1 \pmod{8}$, $\left(\frac{2}{q}\right)_4 (-1)^{\frac{q-1}{8}} = 1$,
- d. $d = q_1 q_2 q_3$, $(q_1, q_2, q_3) \equiv (3, 3, 3), (3, 3, 5), (3, 5, 5) \pmod{8}$.

Where q, q_1, q_2 and q_3 are odd primes. Here $\left(\frac{\cdot}{\cdot}\right)_4$ denotes the rational biquadratic symbol.

A natural question that arises is the following: How are pure quartic number fields distributed when they exhibit some of the specified types of 2-class groups mentioned earlier?

To address this, we focus on the collection of pure quartic fields having $\mathbb{Q}(\sqrt{2})$ as their unique quadratic subfield. We prove in Lemma (2.1) that such quartic fields are of the form $K = \mathbb{Q}(\sqrt[4]{2d^2})$, where d is an odd, square-free positive integer. This classification underpins our main result, presented in Theorem (1.1), which establishes the asymptotic densities for the set of such quartic fields exhibiting specific 2-class groups.

A crucial observation is that the 2-class group $Cl_2(K)$ of these fields is either trivial, cyclic, or of type $(2^{n_1}, 2^{n_2})$, depending on whether its 2-rank r is 0, 1, or 2, respectively (see [11] for further details). This categorization is central to our analysis and makes Theorem (1.1) a key tool in determining the corresponding densities. Additionally, a critical component of our proof involves investigating the asymptotic properties of certain integers satisfying specific arithmetic conditions. In particular, we extend a result of Landau, culminating in Theorem (2.3), which generalizes his classical theorem (see Theorem (2.2) below).

To facilitate this analysis, we define the following sets. Let u be a positive integer, $r \in \{0, 1, 2\}$, and x a positive real number. Define:

$$\mathcal{A}_u = \left\{ \mathbb{Q}(\sqrt[4]{2d^2}) : d = q_1 \cdots q_u, q_i \text{ odd primes for } i = 1, \dots, u \right\}, \quad (1.1)$$

$$\mathcal{A}_{u,x} = \left\{ K \in \mathcal{A}_u : N_{K/\mathbb{Q}}(\Delta_{K/K}) \leq x \right\}, \quad (1.2)$$

$$\mathcal{A}_{u,x}^r = \left\{ K \in \mathcal{A}_{u,x} : r_2(K) = r \right\}. \quad (1.3)$$

Here, $K \in \mathcal{A}_u$ is said to be of type r if $K \in \mathcal{A}_{u,x}^r$ for sufficiently large x . The density of pure quartic fields $K \in \mathcal{A}_u$ of type r is defined as

$$d_{u,r} = \lim_{x \rightarrow +\infty} \frac{\#\mathcal{A}_{u,x}^r}{\#\mathcal{A}_{u,x}}. \quad (1.4)$$

For the trivial case $u = 0$, we set $\mathcal{A}_0 = \{\mathbb{Q}(\sqrt[4]{2})\}$.

Our main results are as follows:

Theorem 1.2. *Let $K = \mathbb{Q}(\sqrt[4]{2d^2})$ be a pure quartic number field where d is an odd squarefree integer with exactly u finite prime factors, and n_1, n_2 are two positive integers. Then*

(i) *The trivial 2-class group of K appears with density $d_{u,0}$ in the set $\mathcal{A}_u$:*

$$d_{u,0} = \begin{cases} 1, & \text{if } u = 0, \\ \frac{1}{4}, & \text{if } u = 1, \\ 0, & \text{if } u \geq 2. \end{cases}$$

(ii) *The cyclic 2-class group of K appears with density $d_{u,1}$ in the set $\mathcal{A}_u$:*

$$d_{u,1} = \begin{cases} 0, & \text{if } u = 0, \\ \frac{5}{8}, & \text{if } u = 1, \\ \frac{1}{8}, & \text{if } u = 2, \\ 0, & \text{if } u \geq 3. \end{cases}$$

(iii) *The 2-class group of K of the type $(2^{n_1}, 2^{n_2})$ appears with density $d_{u,2}$ in the set $\mathcal{A}_u$:*

$$d_{u,2} = \begin{cases} 0, & \text{if } u = 0, \\ \frac{1}{8}, & \text{if } u = 1, \\ \frac{9}{32}, & \text{if } u = 2, \\ \frac{3}{64}, & \text{if } u = 3, \\ 0, & \text{if } u \geq 4. \end{cases}$$

At the end, we make some observations and compare the results obtained with computations performed using Pari/GP [15, version 2.13.0].

These investigations offer promising avenues for future implications, particularly concerning the generalization of our results to encompass all pure quartic number fields with distinct fixed subfields denoted as $\mathbb{Q}(\sqrt{p})$. This expansion of our research lays the foundation for an upcoming comparative analysis. This forthcoming comparison may involve contrasting the findings presented in this paper with those obtainable by substituting the field K with either a biquadratic or cyclic field, both of which share the same quadratic subfield.

Crucially, the outcomes of our contributions play a pivotal role in advancing our understanding of the distribution of p -class groups, where the prime p is classified as 'bad' within the framework of the Cohen-Martinet heuristic, as discussed in [2] and [3]. Furthermore, it is worth noting that the distribution of $Cl_2(K)$ is non-random, owing to its inherent algebraic structure's connection to the ramified prime numbers and the units of the quadratic subfield $k = \mathbb{Q}(\sqrt{2})$. This insight is derived from the utilization of the ambiguous class number formula, which is hidden behind arithmetic and algebraic information. This allowed us to arrange the 2-class groups according to the norm of the relative discriminant $N_{k/\mathbb{Q}}(\Delta_{K/k}) = N_{k/\mathbb{Q}}(4d\sqrt{2}) = 32d^2$.

Here, $\Delta_{K/k}$ represents the relative discriminant of K/k , and $N_{k/\mathbb{Q}}$ signifies the absolute norm of k .

2 Preliminary Results

Before delving into the assessment of density, our exploration commences by introducing a selection of tools drawn from the realm of analytic number theory. These tools will be widely used in the proof of Theorem (1.1).

Let $\pi(x)$ denote the number of primes less than x , let $\pi_s(x)$ denote the number of integers which are the product of exactly $s \geq 2$ distinct prime factors less than x , and let $\tilde{\pi}_s(x)$ denote the number of integers which are the product of exactly $s \geq 2$ odd distinct prime factors less than x . We first recall the asymptotic expansion at infinity of the counting functions $\pi(x)$ and $\pi_s(x)$, which we then prove for $\tilde{\pi}_s(x)$. This initial step necessitates the formulation of a lemma that gives the form of quartic fields having a unique subfield $\mathbb{Q}(\sqrt{2})$.

Lemma 2.1. *Let K be a real pure quartic field and $k = \mathbb{Q}(\sqrt{2})$ its unique real quadratic subfield. Then K has the form $\mathbb{Q}(\sqrt[4]{2d^2})$, where d is an odd square-free integer.*

Proof. Let m be a positive integer and consider the real pure quartic number field $K = \mathbb{Q}(\sqrt[4]{m})$. Without loss of generality, and by the result of [6], there exist d_1, d_2 , and d_3 square-free and pairwise prime integers such that $m = d_1 d_2^2 d_3^3$, where:

(i) $d_1 \neq 1$, d_3 is odd, and $d_2, d_3 > 0$;

(ii) $|d_1| \geq d_3$, if d_1 is odd.

According to [12], the unique quadratic subfield of K is $\mathbb{Q}(\sqrt{d_1 d_3})$. Since, by assumption, $k = \mathbb{Q}(\sqrt{2})$ is the unique subfield of K , there exists $t \in \mathbb{Q}^+$ such that $2 = t^2 d_1 d_3$. Clearing the denominator, we have $2a^2 = d_1 d_3 b^2$ for some $a, b \in \mathbb{Z}$ with $\gcd(a, b) = 1$. Since 2, d_1 , and d_3 are square-free, this implies $t = 1$ and $2 = d_1 d_3$. Thus, $d_1 = 2$ and $d_3 = 1$. Consequently, $m = 2d_2^2$ and $K = \mathbb{Q}(\sqrt[4]{2d^2})$ with $d = d_2$. $\square$

Theorem 2.2 (Landau, 1900 [13]). *Let x be a positive real number. Then*

$$\pi_s(x) = \frac{x(\log \log x)^{s-1}}{(s-1)! \log x} + O\left(\frac{x(\log \log x)^{s-2}}{\log x}\right).$$

Theorem 2.3. *Let x be a positive real number, and let $(\tilde{\pi}_s(x))_{s \geq 0}$ be the sequence defined by the initial values $\tilde{\pi}_0(x) = 1$, $\tilde{\pi}_1(x) = \pi(x) - 1$, and for all $s \geq 2$,*

$$\tilde{\pi}_s(x) = (-1)^s + \sum_{k=0}^{s-1} (-1)^k \pi_{s-k} \left(\frac{x}{2^k} \right).$$

Then, for $s \geq 2$, we have

$$\tilde{\pi}_s(x) = (-1)^s + \sum_{k=0}^{s-1} \frac{(-1)^k}{(s-k-1)! 2^k} \frac{x(\log \log x)^{s-k-1}}{\log(x)} + O\left(\frac{x(\log \log x)^{s-2}}{\log(x)}\right).$$

Proof. Let $p_1, \dots, p_s$ be distinct primes. Then, by breaking down the count into a series of summations, we have:

$$\tilde{\pi}_s(x) = \#\{p_1 \cdots p_s \leq x : p_i \text{ are odd primes}\} = \sum_{p_1 \cdots p_s \leq x} 1 - \sum_{\substack{p_1 \cdots p_{s-1} \leq \frac{x}{2} \\ p_i \neq 2}} 1.$$

Hence,

$$\tilde{\pi}_s(x) = \pi_s(x) - \tilde{\pi}_{s-1} \left(\frac{x}{2} \right).$$

Using induction, we deduce the general formula:

$$\tilde{\pi}_s(x) = (-1)^s + \sum_{k=0}^{s-1} (-1)^k \pi_{s-k} \left(\frac{x}{2^k} \right).$$

For each term $\pi_{s-k}\left(\frac{x}{2^k}\right)$, we use its asymptotic expression:

$$\pi_{s-k}\left(\frac{x}{2^k}\right) = \frac{\frac{x}{2^k}(\log \log(x) - \log \log(2^k))^{s-k-1}}{(s-k-1)! \log\left(\frac{x}{2^k}\right)} + O\left(\frac{\frac{x}{2^k}(\log \log(x) - \log \log(2^k))^{s-k-2}}{\log\left(\frac{x}{2^k}\right)}\right).$$

Observing that $\frac{x}{2^s} \leq \frac{x}{2^k} \leq x$ for $0 \leq k \leq s$, we derive the bounds:

$$\frac{x}{\log(x)} \leq \frac{x}{\log\left(\frac{x}{2^k}\right)} \leq \frac{x}{\log\left(\frac{x}{2^s}\right)},$$

and

$$(\log \log(x) - \log \log(2^s))^{s-k-1} \leq (\log \log(x) - \log \log(2^k))^{s-k-1} \leq (\log \log(x))^{s-k-1}.$$

Using standard series expansions for $\frac{1}{1-x}$, we find:

$$\frac{x}{\log\left(\frac{x}{2^s}\right)} = \frac{x}{\log(x) - s \log(2)} = \frac{x}{\log(x)} + O\left(\frac{x}{(\log(x))^2}\right).$$

Similarly, for $(\log \log\left(\frac{x}{2^s}\right))^{s-k-1}$, expanding $\log(1-x)$ and $(1-x)^\alpha$ gives:

$$(\log \log(x) - \log \log(2^s))^{s-k-1} = (\log \log(x))^{s-k-1} + O((\log \log(x))^{s-k-2}).$$

Substituting these approximations back, we calculate:

$$\sum_{k=0}^{s-1} \frac{(-1)^k}{(s-k-1)! 2^k} \frac{x(\log \log(x))^{s-k-1}}{\log(x)} + O\left(\frac{x(\log \log(x))^{s-2}}{\log(x)}\right).$$

Combining terms, we conclude:

$$\tilde{\pi}_s(x) = (-1)^s + \sum_{k=0}^{s-1} \frac{(-1)^k}{(s-k-1)! 2^k} \frac{x(\log \log(x))^{s-k-1}}{\log(x)} + O\left(\frac{x(\log \log(x))^{s-2}}{\log(x)}\right).$$

□

The following corollary will be used in the proof of our main result.

Corollary 2.4. *Let x be a positive real number. Then*

$$\tilde{\pi}_s\left(\sqrt{\frac{x}{32}}\right) = (-1)^s + \sum_{k=0}^{s-1} \frac{(-1)^k}{(s-k-1)! 2^{k+1}} \frac{\sqrt{x}(\log \log x)^{s-k-1}}{\sqrt{2} \log x} + O\left(\frac{\sqrt{x}(\log \log x)^{s-2}}{\log x}\right).$$

Proof. We begin by considering the expression for $\tilde{\pi}_s\left(\sqrt{\frac{x}{32}}\right)$. This can be written as:

$$\begin{aligned} \tilde{\pi}_s\left(\sqrt{\frac{x}{32}}\right) &= (-1)^s + \sum_{k=0}^{s-1} \frac{(-1)^k}{(s-k-1)! 2^k} \left(\frac{\sqrt{\frac{x}{32}} (\log \log^{s-k-1}\left(\sqrt{\frac{x}{32}}\right))}{\log\left(\sqrt{\frac{x}{32}}\right)} \right) \\ &\quad + O\left(\frac{\sqrt{\frac{x}{32}} \log \log^{s-2}\left(\sqrt{\frac{x}{32}}\right)}{\log\left(\sqrt{\frac{x}{32}}\right)}\right). \end{aligned}$$

Next, we simplify the term $\frac{\sqrt{\frac{x}{32}}}{\log\left(\sqrt{\frac{x}{32}}\right)}$:

$$\frac{\sqrt{\frac{x}{32}}}{\log\left(\sqrt{\frac{x}{32}}\right)} = \frac{2\sqrt{x}}{4\sqrt{2}(\log x - 5 \log 2)} = \frac{\sqrt{x}}{2\sqrt{2} \log x \left(1 - \frac{5 \log 2}{\log x}\right)} = \frac{\sqrt{x}}{2\sqrt{2} \log x} + O\left(\frac{\sqrt{x}}{(\log x)^2}\right).$$

Furthermore, using the asymptotic expansions for the logarithmic terms:

$$\begin{aligned} \left(\log \log \left(\sqrt{\frac{x}{32}} \right) \right)^{s-1} &= (\log \log x)^{s-1} + O \left(\log \log^{s-2}(x) \right), \\ \left(\log \log \left(\sqrt{\frac{x}{32}} \right) \right)^{s-k-1} &= (\log \log x)^{s-k-1} + O \left(\log \log^{s-k-2}(x) \right). \end{aligned}$$

Multiplying the asymptotic expansions, we obtain:

$$\frac{\sqrt{x}(\log \log x)^{s-k-1}}{2\sqrt{2} \log x} + O \left(\frac{\sqrt{x}(\log \log x)^{s-k-2}}{\log x} \right).$$

Additionally, for the higher-order terms, we have:

$$O \left(\frac{\sqrt{\frac{x}{32}} \log \log^{s-1} \left(\sqrt{\frac{x}{32}} \right)}{\log^2 \left(\sqrt{\frac{x}{32}} \right)} \right) = O \left(\frac{\sqrt{x}(\log \log x)^{s-2}}{\log x} \right).$$

Thus, we conclude that:

$$\tilde{\pi}_s \left(\sqrt{\frac{x}{32}} \right) = (-1)^s + \sum_{k=0}^{s-1} \frac{(-1)^k}{(s-k-1)!2^{k+1}} \frac{\sqrt{x}(\log \log x)^{s-k-1}}{\sqrt{2} \log x} + O \left(\frac{\sqrt{x}(\log \log x)^{s-2}}{\log x} \right).$$

□

Remark 2.5. For $s \in \{1, 2, 3\}$, we have the following asymptotic expansions:

$$\begin{aligned} (i) \quad & \tilde{\pi}_1 \left(\sqrt{\frac{x}{32}} \right) = \frac{\sqrt{x}}{2\sqrt{2} \log(x)} - 1 + O \left(\frac{\sqrt{x}}{(\log(x))^2} \right). \\ (ii) \quad & \tilde{\pi}_2 \left(\sqrt{\frac{x}{32}} \right) = \frac{\sqrt{x} \log \log(x)}{2\sqrt{2} \log(x)} + 1 + O \left(\frac{\sqrt{x}}{\log(x)} \right). \\ (iii) \quad & \tilde{\pi}_3 \left(\sqrt{\frac{x}{32}} \right) = \frac{\sqrt{x}(\log \log(x))^2}{4\sqrt{2} \log(x)} + \frac{\sqrt{x}}{8\sqrt{2} \log(x)} - 1 + O \left(\frac{\sqrt{x} \log \log(x)}{\log(x)} \right). \end{aligned}$$

In the following, the notation $f \sim g$ means that for given real-valued functions $f(x)$ and $g(x)$, we have

$$\lim_{x \rightarrow +\infty} \frac{f(x)}{g(x)} = 1.$$

Theorem 2.6 (See Chapter 4, Section IV, page 274 of [16]). *Let p be a prime number. For positive integers a and n with $\gcd(a, n) = 1$, and for x a positive real number, the following asymptotic equivalence holds:*

$$\#\{p \leq x : p \equiv a \pmod{n}\} \sim \frac{1}{\varphi(n)} \pi(x),$$

where φ represents Euler's totient function.

Proposition 2.7. *Let $p_1, \dots, p_s$ be distinct prime numbers and $k \in \mathbb{N}$. For positive integers a and n with $\gcd(a, n) = 1$, and for x a positive real number, the following asymptotic equivalence holds:*

$$\#\{p_1 \cdots p_s \leq x : p_i \equiv a \pmod{n_i}\} \sim \frac{1}{\prod_{i=1}^s \varphi(n_i)} \pi_s(x).$$

Moreover, if $p_1, \dots, p_s$ are distinct odd prime numbers, then

$$\#\{p_1 \cdots p_s \leq x : p_i \equiv a \pmod{n_i}\} \sim \frac{1}{\prod_{i=1}^s \varphi(n_i)} \tilde{\pi}_s(x),$$

where φ represents Euler's totient function.

Proof. The proof of this proposition is loosely based on a combination of Landau's Theorem (2.2) and Theorem (2.6) on primes in arithmetic progressions. □

3 Asymptotic Expansions

We use the results from the previous section to derive asymptotic expansions for the sets $\#\mathcal{A}_{u,x}$ and $\#\mathcal{A}_{u,x}^r$. Recall that

$$\mathcal{A}_{u,x} = \{K \in \mathcal{A}_u : 32d^2 \leq x\} \quad \text{and} \quad \mathcal{A}_{u,x}^r = \{K \in \mathcal{A}_{u,x} : r_2(K) = r\},$$

where r takes the values 0, 1, or 2.

Proposition 3.1. *We have the following asymptotic expansions:*

(i) For $u = 0$, $\#\mathcal{A}_{0,x} = 1$.

(ii) For $u = 1$,

$$\#\mathcal{A}_{1,x} = \frac{\sqrt{x}}{2\sqrt{2}\log(x)} - 1 + O\left(\frac{\sqrt{x}}{(\log(x))^2}\right).$$

(iii) For $u = 2$,

$$\#\mathcal{A}_{2,x} = \frac{\sqrt{x}\log\log(x)}{2\sqrt{2}\log(x)} + 1 + O\left(\frac{\sqrt{x}}{\log(x)}\right).$$

(iv) For $u = 3$,

$$\#\mathcal{A}_{3,x} = \frac{\sqrt{x}(\log\log(x))^2}{4\sqrt{2}\log(x)} + \frac{\sqrt{x}}{8\sqrt{2}\log(x)} - 1 + O\left(\frac{\sqrt{x}\log\log(x)}{\log(x)}\right).$$

Proof. • For $u = 0$, it is straightforward to verify that $\mathcal{A}_{0,x} = \{\mathbb{Q}(\sqrt[4]{2})\}$, hence $\#\mathcal{A}_{0,x} = 1$.

• For $u = 1$, we compute:

$$\#\mathcal{A}_{1,x} = \sum_{\substack{32q^2 \leq x \\ q \neq 2}} 1 = \tilde{\pi}_1\left(\sqrt{\frac{x}{32}}\right) = \frac{\sqrt{x}}{2\sqrt{2}\log(x)} - 1 + O\left(\frac{\sqrt{x}}{(\log(x))^2}\right).$$

• For $u \geq 2$, we express $\#\mathcal{A}_{u,x} = \tilde{\pi}_u\left(\sqrt{\frac{x}{32}}\right)$. According to Corollary (2.4), this expands to:

$$\begin{aligned} \#\mathcal{A}_{u,x} &= (-1)^u + \sum_{k=0}^{u-1} \frac{(-1)^k}{(u-k-1)! \cdot 2^{k+1}} \cdot \frac{\sqrt{x}(\log\log(x))^{u-k-1}}{\sqrt{2}\log(x)} \\ &\quad + O\left(\frac{\sqrt{x}(\log\log(x))^{u-2}}{\log(x)}\right). \end{aligned}$$

Specifically, for $u = 2$:

$$\#\mathcal{A}_{2,x} = \tilde{\pi}_2\left(\sqrt{\frac{x}{32}}\right) = \frac{\sqrt{x}\log\log(x)}{2\sqrt{2}\log(x)} + 1 + O\left(\frac{\sqrt{x}}{\log(x)}\right).$$

For $u = 3$:

$$\begin{aligned} \#\mathcal{A}_{3,x} &= \tilde{\pi}_3\left(\sqrt{\frac{x}{32}}\right) \\ &= -1 + \sum_{k=0}^2 \frac{(-1)^k}{(2-k)! \cdot 2^{k+1}} \cdot \frac{\sqrt{x}(\log\log(x))^{2-k}}{\sqrt{2}\log(x)} + O\left(\frac{\sqrt{x}\log\log(x)}{\log(x)}\right) \\ &= \frac{\sqrt{x}(\log\log(x))^2}{4\sqrt{2}\log(x)} + \frac{\sqrt{x}}{8\sqrt{2}\log(x)} - 1 + O\left(\frac{\sqrt{x}\log\log(x)}{\log(x)}\right). \end{aligned}$$

These computations establish the results for all u .

□

Remark 3.2. (i) Consider all prime numbers q satisfying the congruence $q \equiv 1 \pmod{8}$ and $q \leq \sqrt{\frac{x}{32}}$. For these primes, we have:

$$\begin{aligned} \left| \frac{(\log(x))^2}{\sqrt{x}} \sum_{\substack{q \equiv 1 \pmod{8} \\ q \leq \sqrt{\frac{x}{32}}}} \left(\frac{2}{q} \right)_4 (-1)^{\frac{q-1}{8}} \right| &\leq \frac{(\log(x))^2}{\sqrt{x}} \sum_{\substack{q \equiv 1 \pmod{8} \\ q \leq \sqrt{\frac{x}{32}}}} \left| \left(\frac{2}{q} \right)_4 (-1)^{\frac{q-1}{8}} \right| \\ &= \frac{(\log(x))^2}{\sqrt{x}} \sum_{\substack{q \equiv 1 \pmod{8} \\ q \leq \sqrt{\frac{x}{32}}}} 1 = \frac{1}{8\sqrt{2}}. \end{aligned}$$

This leads to the conclusion:

$$\sum_{\substack{q \equiv 1 \pmod{8} \\ q \leq \sqrt{\frac{x}{32}}}} \left(\frac{2}{q} \right)_4 (-1)^{\frac{q-1}{8}} = O\left(\frac{\sqrt{x}}{(\log(x))^2}\right).$$

(ii) Consider all prime pairs (q_1, q_2) such that $(q_1, q_2) \equiv (1, 5) \pmod{8}$ and $q_1 q_2 \leq \sqrt{\frac{x}{32}}$. For these primes, we find:

$$\begin{aligned} \left| \frac{\log(x)}{\sqrt{x}} \sum_{\substack{(q_1, q_2) \equiv (1, 5) \pmod{8} \\ q_1 q_2 \leq \sqrt{\frac{x}{32}}}} \left(\frac{2}{q_1} \right)_4 (-1)^{\frac{q_1-1}{8}} \right| &\leq \frac{\log(x)}{\sqrt{x}} \sum_{\substack{(q_1, q_2) \equiv (1, 5) \pmod{8} \\ q_1 q_2 \leq \sqrt{\frac{x}{32}}}} \left| \left(\frac{2}{q_1} \right)_4 (-1)^{\frac{q_1-1}{8}} \right| \\ &= \frac{\log(x)}{\sqrt{x}} \sum_{\substack{(q_1, q_2) \equiv (1, 5) \pmod{8} \\ q_1 q_2 \leq \sqrt{\frac{x}{32}}}} 1 = \frac{1}{32\sqrt{2}}. \end{aligned}$$

This leads to the conclusion:

$$\sum_{\substack{(q_1, q_2) \equiv (1, 5) \pmod{8} \\ q_1 q_2 \leq \sqrt{\frac{x}{32}}}} \left(\frac{2}{q_1} \right)_4 (-1)^{\frac{q_1-1}{8}} = O\left(\frac{\sqrt{x}}{\log(x)}\right).$$

These results provide insights into the behavior of certain sums involving prime numbers and their residues in specific arithmetic progressions, helping to establish their growth rates as x becomes larger.

We now consider the sets $\mathcal{A}_{u,x}^r$ for each value of $r = 0, 1$, or 2 .

Proposition 3.3. *We have the following asymptotic expansions:*

(i) For $u = 0$, $\#\mathcal{A}_{0,x}^0 = 1$.

(ii) For $u = 1$, $\#\mathcal{A}_{1,x}^0 = \frac{\sqrt{x}}{8\sqrt{2}\log(x)} - 1 + O\left(\frac{\sqrt{x}}{(\log(x))^2}\right)$.

Proof. Suppose that $Cl_2(K)$ is trivial, as asserted in (i) of Theorem (1.1). This leads to the following consequences:

- (i) When $u = 0$, it directly follows that $\#\mathcal{A}_{0,x}^0 = 1$.
(ii) When $u = 1$, we proceed as follows:

$$\#\mathcal{A}_{1,x}^0 = \left(\sum_{\substack{q \equiv 3 \pmod{8} \\ 32q^2 \leq x}} 1 \right) - 1 = \frac{1}{4} \tilde{\pi} \left(\sqrt{\frac{x}{32}} \right) - 1 = \frac{\sqrt{x}}{8\sqrt{2} \log(x)} - 1 + O \left(\frac{\sqrt{x}}{(\log(x))^2} \right).$$

□

Proposition 3.4. *We have the following asymptotic expansions:*

- (i) For $u = 1$,

$$\#\mathcal{A}_{1,x}^1 = \frac{5\sqrt{x}}{16\sqrt{2} \log(x)} - 1 + O \left(\frac{\sqrt{x}}{(\log(x))^2} \right).$$

- (ii) For $u = 2$,

$$\#\mathcal{A}_{2,x}^1 = \frac{\sqrt{x} \log \log(x)}{16\sqrt{2} \log(x)} + 1 + O \left(\frac{\sqrt{x}}{\log(x)} \right).$$

Proof. When the class group $Cl_2(K)$ is cyclic, from the assertion (ii) in Theorem (1.1), we analyze the cases based on the value of u :

- (i) **Case $u = 1$:** Using the definitions (1.3) and Theorem (2.6), we have:

$$\#\mathcal{A}_{1,x}^1 = \sum_{\substack{q \equiv 5 \text{ or } 7 \pmod{8} \\ 32q^2 \leq x}} 1 + \frac{1}{2} \sum_{\substack{q \equiv 1 \pmod{8} \\ 32q^2 \leq x}} \left(1 - \left(\frac{2}{q} \right)_4 (-1)^{\frac{q-1}{8}} \right).$$

From Remark (3.2) and since $q \equiv 1 \pmod{8}$, it follows that:

$$\begin{aligned} \#\mathcal{A}_{1,x}^1 &= \sum_{\substack{q \equiv 5 \text{ or } 7 \pmod{8} \\ q^2 \leq \frac{x}{32}}} 1 + \frac{1}{2} \sum_{\substack{q \equiv 1 \pmod{8} \\ q^2 \leq \frac{x}{32}}} 1 \\ &= \frac{1}{2} \tilde{\pi} \left(\sqrt{\frac{x}{32}} \right) + \frac{1}{8} \tilde{\pi} \left(\sqrt{\frac{x}{32}} \right) \\ &= \frac{5}{8} \tilde{\pi} \left(\sqrt{\frac{x}{32}} \right) \\ &= \frac{5\sqrt{x}}{16\sqrt{2} \log(x)} - 1 + O \left(\frac{\sqrt{x}}{(\log(x))^2} \right). \end{aligned}$$

- (ii) **Case $u = 2$:** Using Proposition (2.7), we have:

$$\#\mathcal{A}_{2,x}^1 = \sum_{\substack{(q_1, q_2) \equiv (3, 5) \pmod{8} \\ 32q_1^2 q_2^2 \leq x}} 1 + \sum_{\substack{(q_1, q_2) \equiv (3, 3) \pmod{8} \\ 32q_1^2 q_2^2 \leq x}} 1.$$

By symmetry and applying the asymptotics for $\tilde{\pi}_2$, we find:

$$\#\mathcal{A}_{2,x}^1 = \frac{1}{16} \tilde{\pi}_2 \left(\sqrt{\frac{x}{32}} \right) + \frac{1}{16} \tilde{\pi}_2 \left(\sqrt{\frac{x}{32}} \right) = \frac{\sqrt{x} \log \log(x)}{16\sqrt{2} \log(x)} + 1 + O \left(\frac{\sqrt{x}}{\log(x)} \right).$$

□

Proposition 3.5. Let $\mathcal{A}_{u,x}^2$ be the set defined as in the introduction. Then, the asymptotic expansion of $\#\mathcal{A}_{u,x}^2$ as $x \rightarrow +\infty$ is given by:

(i) For $u = 1$,

$$\#\mathcal{A}_{1,x}^2 = \frac{\sqrt{x}}{16\sqrt{2}\log(x)} - 1 + O\left(\frac{\sqrt{x}}{(\log(x))^2}\right).$$

(ii) For $u = 2$,

$$\#\mathcal{A}_{2,x}^2 = \frac{9\sqrt{x}\log\log(x)}{64\sqrt{2}\log(x)} + 1 + O\left(\frac{\sqrt{x}}{\log(x)}\right).$$

(iii) For $u = 3$,

$$\#\mathcal{A}_{3,x}^2 = \frac{3\sqrt{x}(\log\log(x))^2}{256\sqrt{2}\log(x)} + \frac{3\sqrt{x}}{512\sqrt{2}\log(x)} - 1 + O\left(\frac{\sqrt{x}\log\log(x)}{\log(x)}\right).$$

Proof. Suppose $Cl_2(K) \cong \mathbb{Z}/2^{n_1} \times \mathbb{Z}/2^{n_2}$. We analyze the cases based on the value of u using the assertion (iii) in Theorem (1.1):

(i) ****Case $u = 1$:**** Using Theorem (2.7) and Remark (3.2), we have:

$$\#\mathcal{A}_{1,x}^2 = \frac{1}{2} \sum_{\substack{q \equiv 1 \pmod{8} \\ 32q^2 \leq x}} \left(\left(\frac{2}{q} \right)_4 (-1)^{\frac{q-1}{8}} + 1 \right).$$

From these, it follows that:

$$\#\mathcal{A}_{1,x}^2 = \frac{1}{8} \tilde{\pi} \left(\sqrt{\frac{x}{32}} \right) = \frac{\sqrt{x}}{16\sqrt{2}\log(x)} - 1 + O\left(\frac{\sqrt{x}}{(\log(x))^2}\right).$$

(ii) ****Case $u = 2$:**** Using Proposition (2.7) and Remark (3.2), we write:

$$\begin{aligned} \#\mathcal{A}_{2,x}^2 = & \sum_{\substack{(q_1, q_2) \equiv (7, 3) \pmod{8} \\ 32q_1^2 q_2^2 \leq x}} 1 + \sum_{\substack{(q_1, q_2) \equiv (7, 5) \pmod{8} \\ 32q_1^2 q_2^2 \leq x}} 1 + \sum_{\substack{(q_1, q_2) \equiv (1, 3) \pmod{8} \\ 32q_1^2 q_2^2 \leq x}} 1 \\ & + \sum_{\substack{(q_1, q_2) \equiv (5, 5) \pmod{8} \\ 32q_1^2 q_2^2 \leq x}} 1 + \frac{1}{2} \sum_{\substack{(q_1, q_2) \equiv (1, 5) \pmod{8} \\ 32q_1^2 q_2^2 \leq x}} \left(1 - \left(\frac{2}{q_1} \right)_4 (-1)^{\frac{q_1-1}{8}} \right). \end{aligned}$$

By simplifying, we find:

$$\#\mathcal{A}_{2,x}^2 = \frac{9}{32} \tilde{\pi}_2 \left(\sqrt{\frac{x}{32}} \right) = \frac{9\sqrt{x}\log\log(x)}{64\sqrt{2}\log(x)} + 1 + O\left(\frac{\sqrt{x}}{\log(x)}\right).$$

(iii) ****Case $u = 3$:**** Using Proposition (2.7), we compute:

$$\#\mathcal{A}_{3,x}^2 = \sum_{\substack{(q_1, q_2, q_3) \equiv (3, 3, 3) \pmod{8} \\ 32q_1^2 q_2^2 q_3^2 \leq x}} 1 + \sum_{\substack{(q_1, q_2, q_3) \equiv (3, 3, 5) \pmod{8} \\ 32q_1^2 q_2^2 q_3^2 \leq x}} 1 + \sum_{\substack{(q_1, q_2, q_3) \equiv (3, 5, 5) \pmod{8} \\ 32q_1^2 q_2^2 q_3^2 \leq x}} 1.$$

This simplifies to:

$$\#\mathcal{A}_{3,x}^2 = \frac{3}{64} \tilde{\pi}_3 \left(\sqrt{\frac{x}{32}} \right),$$

which gives:

$$\#\mathcal{A}_{3,x}^2 = \frac{3\sqrt{x}(\log\log(x))^2}{256\sqrt{2}\log(x)} + \frac{3\sqrt{x}}{512\sqrt{2}\log(x)} - 1 + O\left(\frac{\sqrt{x}\log\log(x)}{\log(x)}\right).$$

These computations cover all cases for the values of u and the corresponding asymptotics of $\#\mathcal{A}_{u,x}^2$. $\square$

4 Proofs of the main theorem

In this section, we utilize the previously established results to furnish a proof of our principal outcome. Recall that for given values of u and r , we express:

$$d_{u,r} = \lim_{x \rightarrow +\infty} \frac{\#\mathcal{A}_{u,x}^r}{\#\mathcal{A}_{u,x}}.$$

We proceed by considering the cases when the 2-class group $Cl_2(K)$ of the quartic fields K takes on different structures.

Case 1: The 2-class group is trivial

Using Propositions (3.1) and (3.3), we determine the following limits:

(i) **For $d = 1$:** We have

$$d_{0,0} = \lim_{x \rightarrow +\infty} \frac{\#\mathcal{A}_{0,x}^0}{\#\mathcal{A}_{0,x}} = 1.$$

(ii) **For $d = q$:** From the asymptotics,

$$d_{1,0} = \lim_{x \rightarrow +\infty} \frac{\#\mathcal{A}_{1,x}^0}{\#\mathcal{A}_{1,x}} = \lim_{x \rightarrow +\infty} \frac{\frac{\sqrt{x}}{8\sqrt{2}\log(x)} - 1 + O\left(\frac{\sqrt{x}}{(\log(x))^2}\right)}{\frac{\sqrt{x}}{2\sqrt{2}\log(x)} - 1 + O\left(\frac{\sqrt{x}}{(\log(x))^2}\right)} = \frac{1}{4}.$$

Thus, the quartic fields with trivial class group $Cl_2(K)$ contribute with a density of $\frac{1}{4}$ in the set of quartic fields of the form $\mathbb{Q}(\sqrt[4]{2q^2})$.

Case 2: The 2-class group is cyclic

Here, we use Propositions (3.1) and (3.4) to compute densities:

(i) **For $d = q$:** We compute

$$d_{1,1} = \lim_{x \rightarrow +\infty} \frac{\#\mathcal{A}_{1,x}^1}{\#\mathcal{A}_{1,x}} = \lim_{x \rightarrow +\infty} \frac{\frac{5\sqrt{x}}{16\sqrt{2}\log(x)} - 1 + O\left(\frac{\sqrt{x}}{(\log(x))^2}\right)}{\frac{\sqrt{x}}{2\sqrt{2}\log(x)} - 1 + O\left(\frac{\sqrt{x}}{(\log(x))^2}\right)} = \frac{5}{8}.$$

(ii) **For $d = q_1q_2$:** Using the results,

$$d_{2,1} = \lim_{x \rightarrow +\infty} \frac{\#\mathcal{A}_{2,x}^1}{\#\mathcal{A}_{2,x}} = \lim_{x \rightarrow +\infty} \frac{\frac{\sqrt{x}\log\log(x)}{16\sqrt{2}\log(x)} + 1 + O\left(\frac{\sqrt{x}}{\log(x)}\right)}{\frac{\sqrt{x}\log\log(x)}{2\sqrt{2}\log(x)} + 1 + O\left(\frac{\sqrt{x}}{\log(x)}\right)} = \frac{1}{8}.$$

Thus, cyclic $Cl_2(K)$ appears with densities $\frac{5}{8}$ and $\frac{1}{8}$ in the sets of quartic fields of the forms $\mathbb{Q}(\sqrt[4]{2q^2})$ and $\mathbb{Q}(\sqrt[4]{2(q_1q_2)^2})$, respectively.

Case 3: The 2-rank of 2-class group is 2

Using Propositions (3.1) and (3.5), we compute the following limits:

(i) **For $d = q$:** We find

$$d_{1,2} = \lim_{x \rightarrow +\infty} \frac{\#\mathcal{A}_{1,x}^2}{\#\mathcal{A}_{1,x}} = \lim_{x \rightarrow +\infty} \frac{\frac{\sqrt{x}}{16\sqrt{2}\log(x)} - 1 + O\left(\frac{\sqrt{x}}{(\log(x))^2}\right)}{\frac{\sqrt{x}}{2\sqrt{2}\log(x)} - 1 + O\left(\frac{\sqrt{x}}{(\log(x))^2}\right)} = \frac{1}{8}.$$

(ii) **For** $d = q_1 q_2$: We compute

$$d_{2,2} = \lim_{x \rightarrow +\infty} \frac{\#\mathcal{A}_{2,x}^2}{\#\mathcal{A}_{2,x}} = \lim_{x \rightarrow +\infty} \frac{\frac{9\sqrt{x} \log \log(x)}{64\sqrt{2} \log(x)} + 1 + O\left(\frac{\sqrt{x}}{\log(x)}\right)}{\frac{\sqrt{x} \log \log(x)}{2\sqrt{2} \log(x)} + 1 + O\left(\frac{\sqrt{x}}{\log(x)}\right)} = \frac{9}{32}.$$

(iii) **For** $d = q_1 q_2 q_3$: We find

$$d_{3,2} = \lim_{x \rightarrow +\infty} \frac{\#\mathcal{A}_{3,x}^2}{\#\mathcal{A}_{3,x}} = \lim_{x \rightarrow +\infty} \frac{\frac{3\sqrt{x}(\log \log(x))^2}{256\sqrt{2} \log(x)} + \frac{3\sqrt{x}}{512\sqrt{2} \log(x)} - 1 + O\left(\frac{\sqrt{x} \log \log(x)}{\log(x)}\right)}{\frac{\sqrt{x}(\log \log(x))^2}{4\sqrt{2} \log(x)} + \frac{\sqrt{x}}{8\sqrt{2} \log(x)} - 1 + O\left(\frac{\sqrt{x} \log \log(x)}{\log(x)}\right)} = \frac{3}{64}.$$

Hence, the cases where $Cl_2(K) \cong \mathbb{Z}/2^{n_1} \times \mathbb{Z}/2^{n_2}$ appear with densities $\frac{1}{8}$, $\frac{9}{32}$, and $\frac{3}{64}$, respectively, in the sets of quartic fields of the forms $\mathbb{Q}(\sqrt[4]{2q^2})$, $\mathbb{Q}(\sqrt[4]{2(q_1 q_2)^2})$, and $\mathbb{Q}(\sqrt[4]{2(q_1 q_2 q_3)^2})$. This concludes the proof of Theorem (1.1).

5 Some Remarks

(1) For $K = \mathbb{Q}(\sqrt[4]{2d^2})$ where d is an odd square-free integer with exactly u finite distinct odd prime factors, we have the following results by Theorem (1.1):

- **Case** $u = 1$: When $K = \mathbb{Q}(\sqrt[4]{2q^2})$ with q a prime, we find that $d_{1,0} + d_{1,1} + d_{1,2} = \frac{1}{4} + \frac{5}{8} + \frac{1}{8} = 1$. This implies that the 2-class group $Cl_2(K)$ is either trivial, cyclic, or of type $(2^{n_1}, 2^{n_2})$.
- **Case** $u = 2$: When $K = \mathbb{Q}(\sqrt[4]{2(q_1 q_2)^2})$ with q_1 and q_2 distinct primes, we find that $d_{2,0} + d_{2,1} + d_{2,2} = 0 + \frac{1}{8} + \frac{9}{32} < 1$. This indicates that $r_2(K)$ can take additional values greater than 3 for this type of number field.
- **Case** $u = 3$: When $K = \mathbb{Q}(\sqrt[4]{2(q_1 q_2 q_3)^2})$ with q_1 , q_2 , and q_3 distinct primes, we find that $d_{3,0} + d_{3,1} + d_{3,2} = \frac{3}{64} < 1$. This result similarly indicates that $r_2(K)$ can take additional values greater than 3 for this type of number field.

(2) Using the Pari/GP calculator (version 2-11-3), [15], we numerically evaluate the density $\delta_{u,r}$ for the 2-class group types of $Cl_2(K)$ in all pure quartic number fields of the form $K = \mathbb{Q}(\sqrt[4]{2d^2})$. The results are as follows:

- For odd primes $3 \leq q < 6 \cdot 10^5$, the densities corresponding to fields of the form $\mathbb{Q}(\sqrt[4]{2q^2})$ are: $\delta_{1,0} \simeq 0.1916206$, $\delta_{1,1} = 0.6823023$, and $\delta_{1,2} \simeq 0.1259343$.
- For odd primes $3 \leq q_1 < 50$ and $3 \leq q_2 < 10^4$, the densities corresponding to fields of the form $\mathbb{Q}(\sqrt[4]{2(q_1 q_2)^2})$ are: $\delta_{2,1} \simeq 0.2173710$ and $\delta_{2,2} \simeq 0.5199674$.
- For odd primes $3 \leq q_1 < 50$, $3 \leq q_2 < 50$, and $3 \leq q_3 < 10^3$, the densities corresponding to fields of the form $\mathbb{Q}(\sqrt[4]{2(q_1 q_2 q_3)^2})$ are: $\delta_{3,2} \simeq 0.1418581$.

According to Theorem (1.1), the theoretical values for the densities are: $d_{1,0} = \frac{1}{4} \simeq 0.25$, $d_{1,1} = \frac{5}{8} \simeq 0.625$, $d_{1,2} = \frac{1}{8} \simeq 0.125$, $d_{2,1} = \frac{1}{8} = 0.125$, $d_{2,2} = \frac{9}{32} \simeq 0.281$, and $d_{3,2} = \frac{3}{64} \simeq 0.0468$.

As a result, we observe that the numerical density values closely align with the theoretical results presented in Theorem (1.1). The disparities between the theoretical and numerical outcomes remain within a narrow margin, approximately $d_{u,r} - \delta_{u,r} \simeq 0.09$.

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