

Bi-periodic Fibonacci Zeta function and related L -functions

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Abstract We introduce and investigate the bi-periodic Fibonacci zeta function; a new Dirichlet series based on the classical one and is defined using the bi-periodic Fibonacci numbers. We prove that this series has an analytic continuation over the whole complex plane $\mathbb{C}$. We also provide the values at integer numbers along with other analytic properties of this series. Similarly, based on the classical Dirichlet L -function, we define and discuss some properties of the bi-periodic Fibonacci L -function.

1 Introduction

The classical Fibonacci sequence $\{F_n\}_{n \geq 0}$ and the Lucas sequence $\{L_n\}_{n \geq 0}$ are defined as follows:

$$F_n = F_{n-1} + F_{n-2} \quad (n \geq 2), \text{ where } F_0 = 0 \text{ and } F_1 = 1,$$

$$L_n = L_{n-1} + L_{n-2} \quad (n \geq 2), \text{ where } L_0 = 2 \text{ and } L_1 = 1.$$

Many generalizations of these sequences were established. Let p be a positive integer, Falcón [8] and Falcón and Plaza [9] introduced the p -Fibonacci sequence $\{F_{p,n}\}_{n \geq 0}$ and p -Lucas sequence $\{L_{p,n}\}_{n \geq 0}$ as follows:

$$F_{p,n} = pF_{p,n-1} + F_{p,n-2} \quad (n \geq 2), \text{ where } F_{p,0} = 0 \text{ and } F_{p,1} = 1,$$

$$L_{p,n} = pL_{p,n-1} + L_{p,n-2} \quad (n \geq 2), \text{ where } L_{p,0} = 2 \text{ and } L_{p,1} = 1.$$

For $p = 1$, we obtain $\{F_{1,n}\}_{n \geq 0} = \{F_n\}_{n \geq 0}$ and $\{L_{1,n}\}_{n \geq 0} = \{L_n\}_{n \geq 0}$. For more details, see, [6, 8, 9, 11, 18].

Let a and b two nonzero real numbers. Edson and Yayenie [7] defined the bi-periodic Fibonacci sequence $\{q_n(a, b)\}_{n \geq 0} = \{q_n\}_{n \geq 0}$ as follows:

$$q_n = \begin{cases} aq_{n-1} + q_{n-2}, & \text{if } n \text{ is even,} \\ bq_{n-1} + q_{n-2}, & \text{if } n \text{ is odd,} \end{cases} \quad (n \geq 2),$$

with initial values $q_0 = 0$ and $q_1 = 1$.

The bi-periodic Lucas sequence $\{l_n(a, b)\}_{n \geq 0} = \{l_n\}_{n \geq 0}$, which is the companion sequence of $\{q_n\}_{n \geq 0}$, was defined by Bilgici [4] as follows:

$$l_n = \begin{cases} bl_{n-1} + l_{n-2}, & \text{if } n \text{ is even,} \\ al_{n-1} + l_{n-2}, & \text{if } n \text{ is odd,} \end{cases} \quad (n \geq 2).$$

with initial values $l_0 = 2$ and $l_1 = a$.

If we take $a = b = p$ for some positive integer p , we get $\{q_n\}_{n \geq 0} = \{F_{p,n}\}_{n \geq 0}$ and $\{l_n\}_{n \geq 0} = \{L_{p,n}\}_{n \geq 0}$.

The sequences $\{q_n\}_{n \geq 0}$ and $\{l_n\}_{n \geq 0}$ can be extended to negative indices as follows:

$$q_{-n} = (-1)^{n+1} q_n \quad \text{and} \quad l_{-n} = (-1)^n l_n. \quad (1.1)$$

See, [1, 3], for more details.

For any integer n , the Binet's formulas of q_n and l_n are given in [1, 4, 7] by:

$$\begin{aligned} q_n &= \left(\frac{a^{1-\xi(n)}}{(ab)^{\lfloor \frac{n}{2} \rfloor}} \right) \frac{\alpha^n - \beta^n}{\alpha - \beta}, \\ l_n &= \left(\frac{a^{\xi(n)}}{(ab)^{\lfloor \frac{n+1}{2} \rfloor}} \right) (\alpha^n + \beta^n), \end{aligned}$$

where $\alpha = \frac{ab + \sqrt{a^2b^2 + 4ab}}{2}$, $\beta = \frac{ab - \sqrt{a^2b^2 + 4ab}}{2}$ and,

$$\xi(n) = n - 2 \left\lfloor \frac{n}{2} \right\rfloor = \begin{cases} 0, & \text{if } n \text{ is even,} \\ 1, & \text{if } n \text{ is odd.} \end{cases}$$

Throughout the paper, we consider that a and b are positive real numbers and

$$\lambda := \frac{\sqrt{ab} + \sqrt{ab+4}}{2}.$$

We have clearly $\alpha - \beta = \sqrt{ab}\sqrt{ab+4}$, $\alpha = \sqrt{ab}\lambda$ and $\beta = -\sqrt{ab}\lambda^{-1}$. Therefore, the Binet's formulas of q_n and l_n can be written as:

$$q_n = \frac{1}{\sqrt{b^2 + \frac{4b}{a}}} \left(\frac{a}{b} \right)^{-\{\frac{n}{2}\}} (\lambda^n - (-\lambda)^{-n}), \quad (1.2)$$

$$l_n = \left(\frac{a}{b} \right)^{\{\frac{n}{2}\}} (\lambda^n + (-\lambda)^{-n}), \quad (1.3)$$

where $\{\frac{n}{2}\} = \frac{n}{2} - \lfloor \frac{n}{2} \rfloor$ denotes the fractional part of $\frac{n}{2}$, i.e., $\{\frac{n}{2}\} = 0$ if n is even and $\{\frac{n}{2}\} = \frac{1}{2}$ if n is odd.

Let $\{a_n\}_{n \geq 0}$ be an arbitrary sequence. We consider the Dirichlet series $\sum_{n \geq 1} a_n^{-s}$. Many authors have studied particular cases of this series. When $\{a_n\} = \{n\}$, one gets the famous Riemann zeta function $\zeta(s) = \sum_{n \geq 1} n^{-s}$, Which has been studied extensively in the literature. Navas [12] defined and studied the series $\zeta_F(s) = \sum_{n \geq 1} F_n^{-s}$ which is called Fibonacci zeta function, where $\{F_n\}_{n \geq 0}$ is the Fibonacci sequence. He gave a meromorphic continuation to the whole plane and computed the residues of this meromorphic continuation at its poles. Also, the author discussed the values at negative integers and proved that these values can be expressed in terms of the Fibonacci and Lucas numbers. His study motivated many authors to study similar series.

In 2013, Kamano [10] introduced the Lucas zeta function defined by $\zeta_{P,Q}(s) = \sum_{n \geq 1} U_n^{-s}$ where $\{U_n\} = \{U_n(P, Q)\}$ is the sequence defined by the following recurrence relation: $U_n = PU_{n-1} + QU_{n-2}$ ($n \geq 2$) with $U_0 = 0$ and $U_1 = 1$.

In 2014, Rout and Panda [13] defined and studied the Balancing zeta function $\zeta_B(s) = \sum_{n \geq 1} B_n^{-s}$, where $\{B_n\}$ is the Balancing sequence which satisfy the recurrence relation: $B_n = 6B_{n-1} - B_{n-2}$ ($n \geq 2$) with initial conditions $B_0 = 0$ and $B_1 = 1$. A similar work but using the Lucas balancing numbers was established by Behera et al. [2].

Recently, Dirichlet series defined by some recurrence sequences of higher order have been studied. In 2023, Serrano Holgado [14] introduced and studied the Tribonacci zeta function $\zeta_T(s) = \sum_{n \geq 1} T_n^{-s}$. In the same year, Serrano Holgado and Navas [15] studied the general case, namely the Dirichlet series $\sum_{n \geq 1} a_n^{-s}$, where $\{a_n\}_{n \geq 0}$ is a recurrent linear sequence with integer coefficients satisfying particular conditions.

The previous studies inspired us to introduce and study the new Dirichelt series:

$$\zeta_{q(a,b)}(s) := \sum_{n \geq 1} q_n^{-s}, \quad \Re(s) > 0,$$

where q_n is the n^{th} bi-periodic Fibonacci number. We will call this series the bi-periodic Fibonacci zeta function.

When $a = b = p$ for some positive integer p , we obtain the p -Fibonacci zeta function $\zeta_{F_p}(s) := \sum_{n \geq 1} F_{p,n}^{-s}$, where $F_{p,n}$ is the n^{th} p -Fibonacci number.

We organize our paper as follows: In Section 2, we prove that the bi-periodic Fibonacci zeta function $\zeta_{q(a,b)}(s)$ can be extended to a meromorphic function on $\mathbb{C}$. In Section 3, we discuss the values of $\zeta_{q(a,b)}(s)$ at integers. Finally, in Section 4, the bi-periodic Fibonacci L -function is introduced, and we discuss its analytic continuation.

2 Analytic continuation of the Bi-periodic Fibonacci zeta function

It is well known that the zeta function $\zeta(s)$ is defined in the half plan $\Re(s) > 1$ and can be extended to meromorphic function on $\mathbb{C}$. Similarly, we can see that the series $\sum_{n \geq 1} q_n^{-s}$ is absolutely convergent when $\Re(s) > 0$. Indeed, let $s = \sigma + it$ with $\sigma > 0$, using (1.2) we get

$$\begin{aligned} \sum_{n \geq 1} |q_n^{-s}| &= \left(b^2 + \frac{4b}{a}\right)^{\frac{\sigma}{2}} \sum_{n \geq 1} \left(\frac{a}{b}\right)^{\sigma\{\frac{n}{2}\}} \left| \lambda^{-ns} \left(1 - (-\lambda^{-2})^n\right)^{-s} \right| \\ &= \left(b^2 + \frac{4b}{a}\right)^{\frac{\sigma}{2}} \sum_{n \geq 1} \left(\frac{a}{b}\right)^{\sigma\{\frac{n}{2}\}} \lambda^{-n\sigma} \left| \left(1 - (-\lambda^{-2})^n\right)^{-\sigma} \right| \\ &\leq \left(b^2 + \frac{4b}{a}\right)^{\frac{\sigma}{2}} \max \left\{ \left(\frac{a}{b}\right)^{\frac{\sigma}{2}}, 1 \right\} (1 - \lambda^{-2})^{-\sigma} \sum_{n \geq 1} \lambda^{-n\sigma}. \end{aligned}$$

The fact that $\lambda > 1$ ensures that the previous series is absolutely convergent. Hence, the bi-periodic Fibonacci zeta function $\zeta_{q(a,b)}(s)$ is analytic in the half plane $\Re(s) > 0$.

In the next theorem we will prove that the bi-periodic Fibonacci zeta function can be extended to a meromorphic function on $\mathbb{C}$.

Theorem 2.1. *The bi-periodic Fibonacci zeta function $\zeta_{q(a,b)}(s)$ can be meromorphically continued to a meromorphic function on $\mathbb{C}$ and can be expressed as:*

$$\zeta_{q(a,b)}(s) = (ab + 4)^{\frac{s}{2}} \sum_{k \geq 0} \binom{-s}{k} \frac{\lambda^{s+2k} + (-1)^k \left(\frac{b}{a}\right)^{\frac{s}{2}}}{\lambda^{2(s+2k)} - 1}. \quad (2.1)$$

It is holomorphic except at the following simple poles:

$$\begin{cases} s_1 = s_{n,k} = -2k + \frac{\pi i(2n+k)}{\log \lambda}, & \text{if } a = b, \\ s_2 = s_{k,n} = -2k + \frac{\pi i n}{\log \lambda}, & \text{if } a \neq b, \end{cases} \quad (k \geq 0, n \in \mathbb{Z}).$$

The residue of $\zeta_{q(a,b)}(s)$ at these poles are:

$$\text{Res}(\zeta_{q(a,b)}(s), w) = \begin{cases} (a^2 + 4)^{\frac{w}{2}} \binom{-w}{k} \frac{(-1)^k}{\log \lambda}, & \text{if } w = s_1, \\ (ab + 4)^{\frac{w}{2}} \binom{-w}{k} \frac{(-1)^n + (-1)^k \left(\frac{b}{a}\right)^{\frac{w}{2}}}{2 \log \lambda}, & \text{if } w = s_2. \end{cases}$$

Proof. Let s be a complex number. We have

$$\begin{aligned} q_n^{-s} &= \left(b^2 + \frac{4b}{a}\right)^{\frac{s}{2}} \left(\frac{a}{b}\right)^{s\{\frac{n}{2}\}} \lambda^{-ns} \left(1 - (-\lambda^{-2})^n\right)^{-s} \\ &= \left(b^2 + \frac{4b}{a}\right)^{\frac{s}{2}} \left(\frac{a}{b}\right)^{s\{\frac{n}{2}\}} \lambda^{-ns} \sum_{k \geq 0} \binom{-s}{k} (-1)^k (-\lambda^{-2})^{nk} \quad (\lambda^{-2} < 1) \\ &= \left(b^2 + \frac{4b}{a}\right)^{\frac{s}{2}} \left(\frac{a}{b}\right)^{s\{\frac{n}{2}\}} \sum_{k \geq 0} \binom{-s}{k} (-1)^{k(n+1)} \lambda^{-n(s+2k)}. \end{aligned} \quad (2.2)$$

Thus,

$$\zeta_{q(a,b)}(s) = \left(b^2 + \frac{4b}{a}\right)^{\frac{s}{2}} \sum_{n \geq 1} \sum_{k \geq 0} \left(\frac{a}{b}\right)^{s\{\frac{n}{2}\}} \binom{-s}{k} (-1)^{k(n+1)} \lambda^{-n(s+2k)}. \quad (2.3)$$

Let $S_{n,k} = \left(\frac{a}{b}\right)^{s\{\frac{n}{2}\}} \binom{-s}{k} (-1)^{k(n+1)} \lambda^{-n(s+2k)}$. The series $\sum_{n \geq 1} \sum_{k \geq 0} S_{n,k}$ is absolutely convergent in the half plan $\Re(s) > 0$. To do so, we need the following inequality:

$$\left| \binom{-s}{k} \right| \leq \frac{|s|(|s|+1) \dots (|s|+k-1)}{k!} = \binom{|s|+k-1}{k} = (-1)^k \binom{-|s|}{k}.$$

Let $s = \sigma + it$ with $\sigma > 0$, then we have

$$\begin{aligned} \sum_{n \geq 1} \sum_{k \geq 0} |S_{n,k}| &= \sum_{n \geq 1} \sum_{k \geq 0} \left(\frac{a}{b}\right)^{\sigma\{\frac{n}{2}\}} \left| \binom{-s}{k} \right| \lambda^{-n(\sigma+2k)} \\ &\leq \sum_{n \geq 1} \left(\frac{a}{b}\right)^{\sigma\{\frac{n}{2}\}} \lambda^{-n\sigma} \sum_{k \geq 0} (-1)^k \binom{-|s|}{k} \lambda^{-2nk} \\ &\leq \sum_{n \geq 1} \left(\frac{a}{b}\right)^{\sigma\{\frac{n}{2}\}} \lambda^{-n\sigma} (1 - \lambda^{-2n})^{-|s|} \\ &\leq \max \left\{ \left(\frac{a}{b}\right)^{\frac{\sigma}{2}}, 1 \right\} (1 - \lambda^{-2})^{-|s|} \sum_{n \geq 1} \lambda^{-n\sigma} < \infty, \quad (\lambda > 1). \end{aligned}$$

Then the series $\sum_{n \geq 1} \sum_{k \geq 0} S_{n,k}$ is absolutely convergent, thus, we can change the summation order in (2.3) to get

$$\zeta_{q(a,b)}(s) = \left(b^2 + \frac{4b}{a}\right)^{\frac{s}{2}} \sum_{k \geq 0} \binom{-s}{k} (-1)^k \sum_{n \geq 1} \left(\frac{a}{b}\right)^{s\{\frac{n}{2}\}} (-1)^{nk} \lambda^{-n(s+2k)}.$$

We consider the two cases: n is even and n is odd, we obtain

$$\begin{aligned} \zeta_{q(a,b)}(s) &= \left(b^2 + \frac{4b}{a}\right)^{\frac{s}{2}} \sum_{k \geq 0} \binom{-s}{k} (-1)^k \left[\sum_{n \geq 1} \lambda^{-2n(s+2k)} \right. \\ &\quad \left. + (-1)^k \left(\frac{a}{b}\right)^{\frac{s}{2}} \lambda^{-(s+2k)} \sum_{n \geq 0} \lambda^{-2n(s+2k)} \right] \\ &= \left(b^2 + \frac{4b}{a}\right)^{\frac{s}{2}} \sum_{k \geq 0} \binom{-s}{k} (-1)^k \left[-1 + \left(1 + (-1)^k \left(\frac{a}{b}\right)^{\frac{s}{2}} \lambda^{-(s+2k)}\right) \sum_{n \geq 0} \lambda^{-2n(s+2k)} \right] \\ &= \left(b^2 + \frac{4b}{a}\right)^{\frac{s}{2}} \sum_{k \geq 0} \binom{-s}{k} (-1)^k \left[(-1)^k \left(\frac{a}{b}\right)^{\frac{s}{2}} \frac{\lambda^{s+2k} + (-1)^k \left(\frac{b}{a}\right)^{\frac{s}{2}}}{\lambda^{2(s+2k)} - 1} \right] \\ &= (ab + 4)^{\frac{s}{2}} \sum_{k \geq 0} \binom{-s}{k} \frac{\lambda^{s+2k} + (-1)^k \left(\frac{b}{a}\right)^{\frac{s}{2}}}{\lambda^{2(s+2k)} - 1} \\ &= (ab + 4)^{\frac{s}{2}} \sum_{k \geq 0} g_k(s), \end{aligned}$$

where $g_k(s) = \binom{-s}{k} \frac{\lambda^{s+2k} + (-1)^k \left(\frac{b}{a}\right)^{\frac{s}{2}}}{\lambda^{2(s+2k)} - 1}$.

Let $\beta = \max \left\{ \frac{\log 2}{\log \lambda}, -\frac{2}{3}\sigma \right\}$. For any complex number s and any integer $k \geq \beta$, we have

$$|\lambda^{2(s+2k)} - 1| \geq \lambda^k \lambda^{2\sigma+3k} - 1 \geq 2\lambda^{2\sigma+3k} - 1 \geq \lambda^{2\sigma+3k}.$$

Therefore,

$$\begin{aligned}
\sum_{k \geq k_0} |g_k(s)| &= \sum_{k \geq k_0} \left| \binom{-s}{k} \right| \left| \frac{\lambda^{s+2k} + (-1)^k \left(\frac{b}{a}\right)^{\frac{s}{2}}}{\lambda^{2(s+2k)} - 1} \right| \\
&\leq \sum_{k \geq 0} (-1)^k \binom{-|s|}{k} \frac{\lambda^{\sigma+2k} + \left(\frac{b}{a}\right)^{\frac{\sigma}{2}}}{\lambda^{2\sigma+3k}} \\
&\leq \lambda^{-\sigma} \sum_{k \geq 0} (-1)^k \binom{-|s|}{k} \lambda^{-k} + \lambda^{-2\sigma} \left(\frac{b}{a}\right)^{\frac{\sigma}{2}} \sum_{k \geq 0} (-1)^k \binom{-|s|}{k} \lambda^{-3k} \\
&\leq \lambda^{-\sigma} (1 - \lambda^{-1})^{-|s|} + \left(\frac{b}{a\lambda^4}\right)^{\frac{\sigma}{2}} (1 - \lambda^{-3})^{-|s|} < \infty.
\end{aligned}$$

Thus the series $\sum_{k \geq 0} g_k(s)$ converges uniformly and absolutely in any compact set that does not contain any pole of $g_k(s)$ and this prove that the series $\zeta_{q(a,b)}(s)$ can be extend to a holomorphic function on $\mathbb{C}$ except for the poles of $g_k(s)$, i.e., the poles obtained from the equation $\lambda^{2s+4k} - 1 = 0$ when $a \neq b$, and $\lambda^{s+2k} + (-1)^{k-1} = 0$ when $a = b$.

- For $a = b$, the poles of $g_k(s)$ are $s_1 = s_{k,n} = -2k + \frac{\pi i(k+2n)}{\log \lambda}$, ($k \geq 0, n \in \mathbb{Z}$). Using L'Hopital's rule, the residue at s_1 is

$$\text{Res}(\zeta_{q(a,b)}(s), s_1) = (a^2 + 4)^{\frac{s_1}{2}} \binom{-s_1}{k} \lim_{s \rightarrow s_1} \frac{s - s_1}{\lambda^{s+2k} + (-1)^{k-1}} = (a^2 + 4)^{\frac{s_1}{2}} \binom{-s_1}{k} \frac{(-1)^k}{\log \lambda}.$$

- For $a \neq b$, the poles of $g_k(s)$ are $s_2 = s_{k,n} = -2k + \frac{\pi i n}{\log \lambda}$, ($k \geq 0, n \in \mathbb{Z}$). Using L'Hopital's rule, the residue at s_2 is easily seen to be

$$\begin{aligned}
\text{Res}(\zeta_{q(a,b)}(s), s_2) &= (ab + 4)^{\frac{s_2}{2}} \binom{-s_2}{k} \left(\lambda^{s_2+2k} + (-1)^k \left(\frac{b}{a}\right)^{\frac{s_2}{2}} \right) \lim_{s \rightarrow s_2} \frac{s - s_2}{\lambda^{2(s+2k)} - 1} \\
&= (ab + 4)^{\frac{s_2}{2}} \binom{-s_2}{k} \frac{(-1)^n + (-1)^k \left(\frac{b}{a}\right)^{\frac{s_2}{2}}}{2 \log \lambda}.
\end{aligned}$$

Thus the proof is completed. □

For $a = b = p$, we get

Corollary 2.2. *The p -Fibonacci zeta function $\zeta_{F_p}(s)$ can be meromorphically continued to the whole complex plane and can be expressed as:*

$$\zeta_{F_p}(s) = (p^2 + 4)^{\frac{s}{2}} \sum_{k \geq 0} \binom{-s}{k} \frac{1}{\lambda^{s+2k} + (-1)^{k+1}}, \quad \text{where } \lambda = \frac{p + \sqrt{p^2 + 4}}{2}.$$

It is holomorphic except at the following simple poles: $s_1 = s_{n,k} = -2k + \frac{\pi i(2n+k)}{\log \lambda}$, ($k \geq 0, n \in \mathbb{Z}$). Moreover, the residue of $\zeta_{F_p}(s)$ at $s = s_1$ is as follows:

$$\text{Res}(\zeta_{F_p}(s), s_1) = (p^2 + 4)^{\frac{s_1}{2}} \binom{-s_1}{k} \frac{(-1)^k}{\log \lambda}.$$

3 Values of $\zeta_{q(a,b)}(s)$ at integers

In this section, we discuss the values of $\zeta_{q(a,b)}(s)$ at integers.

3.1 Values of $\zeta_{q(a,b)}(s)$ at negative integers

In this part, we discuss the values of $\zeta_{q(a,b)}(s)$ at negative integers. Theorem 2.1 yields the following lemma.

Lemma 3.1. *Let ℓ be a non negative integer. Then $\zeta_{q(a,b)}(s)$ possesses pole at $s = -\ell$ if and only*

$$\text{if } \begin{cases} \ell \equiv 0 \pmod{4}, & \text{if } a = b, \\ \ell \equiv 0 \pmod{2}, & \text{if } a \neq b. \end{cases}$$

Now, we will be interested in finding the constant terms in the Laurent series expansion of the function $\zeta_{q(a,b)}(s)$ at the negative integers that are poles of $\zeta_{q(a,b)}(s)$.

Proposition 3.2. *In each of the two cases listed in the above lemma, the constant term in the Laurent series expansion of $\zeta_{q(a,b)}(s)$ at $s = -\ell$ is given by:*

$$\begin{aligned} CT_{s=-\ell}\zeta_{q(a,b)}(s) = & (ab+4)^{-\frac{\ell}{2}} \binom{\ell}{\frac{\ell}{2}} (-1)^{\frac{\ell}{2}} \left(\frac{b}{a}\right)^{-\frac{\ell}{2}} \left[-\frac{1}{2} + \frac{\log(ab+4) + \log\left(\frac{b}{a}\right)}{4\log(\lambda)} - \frac{H_\ell - H_{\frac{\ell}{2}}}{2\log(\lambda)} \right] \\ & + (ab+4)^{-\frac{\ell}{2}} \binom{\ell}{\frac{\ell}{2}} \left[\frac{\log(ab+4)}{4\log(\lambda)} - \frac{H_\ell - H_{\frac{\ell}{2}}}{2\log(\lambda)} \right] \\ & + (ab+4)^{-\frac{\ell}{2}} \sum_{k=0, k \neq \frac{\ell}{2}}^{\ell} \binom{\ell}{k} \frac{\lambda^{-\ell+2k} + (-1)^k \left(\frac{b}{a}\right)^{-\frac{\ell}{2}}}{\lambda^{2(-\ell+2k)} - 1}, \end{aligned}$$

where $H_n = \sum_{j=1}^n \frac{1}{j}$ denotes the n -th harmonic number and $H_0 = 0$.

Proof. From Theorem 2.1, we have

$$\begin{aligned} \zeta_{q(a,b)}(s) = & (ab+4)^{\frac{s}{2}} \sum_{k \geq 0} \binom{-s}{k} \frac{\lambda^{s+2k} + (-1)^k \left(\frac{b}{a}\right)^{\frac{s}{2}}}{\lambda^{2(s+2k)} - 1} \\ = & (ab+4)^{\frac{s}{2}} \binom{-s}{\frac{\ell}{2}} \frac{\lambda^{s+\ell} + (-1)^{\frac{\ell}{2}} \left(\frac{b}{a}\right)^{\frac{s}{2}}}{\lambda^{2(s+\ell)} - 1} + (ab+4)^{\frac{s}{2}} \sum_{k \geq 0, k \neq \frac{\ell}{2}} \binom{-s}{k} \frac{\lambda^{s+2k} + (-1)^k \left(\frac{b}{a}\right)^{\frac{s}{2}}}{\lambda^{2(s+2k)} - 1} \\ = & T(s) + K(s). \end{aligned}$$

Since $K(s)$ is holomorphic at $s = -\ell$ and $\binom{\ell}{k} = 0$ for $k > \ell$, we obtain

$$CT_{s=-\ell}K(s) = K(-\ell) = (ab+4)^{-\frac{\ell}{2}} \sum_{k=0, k \neq \frac{\ell}{2}}^{\ell} \binom{\ell}{k} \frac{\lambda^{-\ell+2k} + (-1)^k \left(\frac{b}{a}\right)^{-\frac{\ell}{2}}}{\lambda^{2(-\ell+2k)} - 1}. \quad (3.1)$$

Now, as $s \rightarrow -\ell$, we have

$$(ab+4)^{\frac{s}{2}} = (ab+4)^{-\frac{\ell}{2}} + \frac{1}{2}(ab+4)^{-\frac{\ell}{2}} \log(ab+4)(s+\ell) + o(s+\ell), \quad (3.2)$$

$$\begin{aligned} \lambda^{s+\ell} + (-1)^{\frac{\ell}{2}} \left(\frac{b}{a}\right)^{\frac{s}{2}} &= \lambda^{s+\ell} + (-1)^{\frac{\ell}{2}} \left(\frac{b}{a}\right)^{-\frac{\ell}{2}} \left(\frac{b}{a}\right)^{\frac{s+\ell}{2}} \\ &= 1 + (-1)^{\frac{\ell}{2}} \left(\frac{b}{a}\right)^{-\frac{\ell}{2}} \\ &\quad + \left(\log(\lambda) + \frac{1}{2}(-1)^{\frac{\ell}{2}} \left(\frac{b}{a}\right)^{-\frac{\ell}{2}} \log\left(\frac{b}{a}\right) \right) (s+\ell) + o(s+\ell) \end{aligned} \quad (3.3)$$

$$\begin{aligned}
\frac{1}{\lambda^{2(s+\ell)} - 1} &= \frac{1}{e^{2(s+\ell)\log(\lambda)} - 1} \\
&= \frac{1}{2(s+\ell)\log(\lambda) + 2(s+\ell)^2\log^2(\lambda) + \frac{4}{3}(s+\ell)^3\log^3(\lambda) + o(s+\ell)^3} \\
&= \frac{1}{2(s+\ell)\log(\lambda)} \times \frac{1}{1 + (s+\ell)\log(\lambda) + \frac{2}{3}(s+\ell)^2\log^2(\lambda) + o(s+\ell)^2} \\
&= \frac{1}{2(s+\ell)\log(\lambda)} \left(1 - (s+\ell)\log(\lambda) + \frac{5}{3}(s+\ell)^2\log^2(\lambda) + o(s+\ell)^2 \right) \\
&= \frac{1}{2(s+\ell)\log(\lambda)} - \frac{1}{2} + \frac{5}{6}(s+\ell)\log^2(\lambda) + o(s+\ell), \tag{3.4}
\end{aligned}$$

and

$$\begin{aligned}
\binom{-s}{\frac{\ell}{2}} &= \frac{\Gamma(-(s+\ell) + \ell + 1)}{(\frac{\ell}{2})!\Gamma(-(s+\ell) + \frac{\ell}{2} + 1)} \\
&= \binom{\ell}{\frac{\ell}{2}} \left[1 + (s+\ell) \left(\psi\left(\frac{\ell}{2} + 1\right) - \psi(\ell + 1) \right) \right] + o(s+\ell), \tag{3.5}
\end{aligned}$$

where $\psi(s)$ denotes the digamma function defined by $\psi(s) = \frac{\Gamma'(s)}{\Gamma(s)}$. By multiplying (3.2), (3.3), (3.4) and (3.5), and using the functional equation $\psi(s+1) = \psi(s) + \frac{1}{s}$ for the digamma function, we deduce that the constant term in the Laurent series expansion of $T(s)$ at $s = -\ell$ equals

$$\begin{aligned}
CT_{s=-\ell}T(s) &= (ab+4)^{-\frac{\ell}{2}} \binom{\ell}{\frac{\ell}{2}} (-1)^{\frac{\ell}{2}} \left(\frac{b}{a} \right)^{\frac{-\ell}{2}} \left[-\frac{1}{2} + \frac{\log(ab+4) + \log\left(\frac{b}{a}\right)}{4\log(\lambda)} - \frac{H_\ell - H_{\frac{\ell}{2}}}{2\log(\lambda)} \right] \\
&+ (ab+4)^{-\frac{\ell}{2}} \binom{\ell}{\frac{\ell}{2}} \left[\frac{\log(ab+4)}{4\log(\lambda)} - \frac{H_\ell - H_{\frac{\ell}{2}}}{2\log(\lambda)} \right]. \tag{3.6}
\end{aligned}$$

Thus, the proof is complete by summing (3.1) and (3.6). $\square$

Corollary 3.3. 1) If $a = b = p$, we obtain

$$\begin{aligned}
CT_{s=-\ell}\zeta_{F_p}(s) &= (p^2+4)^{-\frac{\ell}{2}} \binom{\ell}{\frac{\ell}{2}} \left[-\frac{1}{2} + \frac{\log(p^2+4)}{2\log(\lambda)} - \frac{H_\ell - H_{\frac{\ell}{2}}}{\log(\lambda)} \right] \\
&+ (p^2+4)^{-\frac{\ell}{2}} \sum_{k=0, k \neq \frac{\ell}{2}}^{\ell} \binom{\ell}{k} \frac{1}{\lambda^{-\ell+2k} + (-1)^{k+1}}.
\end{aligned}$$

1) If $\ell = 0$, we obtain

$$CT_{s=0}\zeta_{q(a,b)}(s) = -\frac{1}{2} + \frac{\log(ab+4)}{2\log(\lambda)} + \frac{\log\left(\frac{b}{a}\right)}{4\log(\lambda)}.$$

From Lemma 3.1, we have that $0, -4, -8, \dots$ are simple poles of $\zeta_{q(a,b)}(s)$ when $a = b$, and $0, -2, -4, \dots$ are simple poles of $\zeta_{q(a,b)}(s)$ when $a \neq b$. The following theorem gives the values of $\zeta_{q(a,b)}(s)$ at the negative integers which are not poles.

Theorem 3.4. Let m be a positive integer. Then we have

$$\zeta_{q(a,b)}(-m) = \begin{cases} 0, & \text{if } m \text{ is even,} \\ \frac{a}{b} \left(b^2 + \frac{4b}{a} \right)^{\frac{1-m}{2}} \sum_{k=0}^{\frac{m-1}{2}} \binom{m}{k} (-1)^{k+1} \frac{q_{m-2k}}{l_{m-2k}}, & \text{if } m \text{ is odd,} \end{cases} \tag{3.7}$$

except for singularities.

Proof. Let m be a positive integer. From (2.1), we obtain

$$\begin{aligned}\zeta_{q(a,b)}(-m) &= (ab+4)^{-\frac{m}{2}} \sum_{k=0}^m \binom{m}{k} \frac{\lambda^{2k-m} + (-1)^k \left(\frac{b}{a}\right)^{-\frac{m}{2}}}{\lambda^{2(2k-m)} - 1} \\ &= \frac{1}{2}(ab+4)^{-\frac{m}{2}} \sum_{k=0}^m (\theta_k + \theta_{m-k}),\end{aligned}$$

where $\theta_k := \binom{m}{k} \frac{\lambda^{2k-m} + (-1)^k \left(\frac{b}{a}\right)^{-\frac{m}{2}}}{\lambda^{2(2k-m)} - 1}$. We have

$$\begin{aligned}\theta_k + \theta_{m-k} &= \binom{m}{k} \left[\frac{\lambda^{2k-m} + (-1)^k \left(\frac{b}{a}\right)^{-\frac{m}{2}}}{\lambda^{2(2k-m)} - 1} + \frac{\lambda^{m-2k} + (-1)^{m-k} \left(\frac{b}{a}\right)^{-\frac{m}{2}}}{\lambda^{2(m-2k)} - 1} \right] \\ &= \left(\frac{b}{a}\right)^{-\frac{m}{2}} \binom{m}{k} (-1)^k \left[\frac{(-1)^k \left(\frac{b}{a}\right)^{\frac{m}{2}} \lambda^{2k-m} + 1}{\lambda^{2(2k-m)} - 1} + \frac{(-1)^k \left(\frac{b}{a}\right)^{\frac{m}{2}} \lambda^{m-2k} + (-1)^m}{\lambda^{2(m-2k)} - 1} \right] \\ &= \left(\frac{b}{a}\right)^{-\frac{m}{2}} \binom{m}{k} (-1)^{k+1} \frac{\lambda^{2(m-2k)} + (-1)^m \lambda^{2(2k-m)} - 1 + (-1)^{m+1}}{(\lambda^{m-2k} - \lambda^{2k-m})^2}.\end{aligned}$$

Hence

$$\begin{aligned}\zeta_{q(a,b)}(-m) &= \\ \frac{1}{2} \left(b^2 + \frac{4b}{a}\right)^{-\frac{m}{2}} \sum_{k=0}^m \binom{m}{k} (-1)^{k+1} \frac{\lambda^{2(m-2k)} + (-1)^m \lambda^{2(2k-m)} - 1 + (-1)^{m+1}}{(\lambda^{m-2k} - \lambda^{2k-m})^2}.\end{aligned}$$

- If m is even, we have

$$\zeta_{q(a,b)}(-m) = \frac{1}{2} \left(b^2 + \frac{4b}{a}\right)^{-\frac{m}{2}} \sum_{k=0}^m \binom{m}{k} (-1)^{k+1} = 0.$$

- If m is odd, we have

$$\begin{aligned}\zeta_{q(a,b)}(-m) &= \frac{1}{2} \left(b^2 + \frac{4b}{a}\right)^{-\frac{m}{2}} \sum_{k=0}^m \binom{m}{k} (-1)^{k+1} \left[\frac{(\lambda^{m-2k} - \lambda^{2k-m})(\lambda^{m-2k} + \lambda^{2k-m})}{(\lambda^{m-2k} - \lambda^{2k-m})^2} \right] \\ &= \frac{1}{2} \left(b^2 + \frac{4b}{a}\right)^{-\frac{m}{2}} \sum_{k=0}^m \binom{m}{k} (-1)^{k+1} \left[\frac{\lambda^{m-2k} + \lambda^{2k-m}}{\lambda^{m-2k} - \lambda^{2k-m}} \right].\end{aligned}\tag{3.8}$$

From (1.2) and (1.3), we obtain $\frac{q_n}{l_n} = \frac{\left(\frac{a}{b}\right)^{-\xi(n)}}{\left(b^2 + \frac{4b}{a}\right)^{\frac{1}{2}}} \frac{\lambda^n - (-\lambda)^{-n}}{\lambda^n + (-\lambda)^{-n}}$. Hence

$$\frac{\lambda^n - (-\lambda)^{-n}}{\lambda^n + (-\lambda)^{-n}} = \left(\frac{a}{b}\right)^{\xi(n)} \left(b^2 + \frac{4b}{a}\right)^{\frac{1}{2}} \frac{q_n}{l_n}.$$

Since $|m - 2k|$ is an odd number,

$$\frac{\lambda^{m-2k} + \lambda^{2k-m}}{\lambda^{m-2k} - \lambda^{2k-m}} = \frac{a}{b} \left(b^2 + \frac{4b}{a}\right)^{\frac{1}{2}} \frac{q_{m-2k}}{l_{m-2k}}.$$

Thus, Identity (3.8) becomes as follows:

$$\zeta_{q(a,b)}(-m) = \frac{a}{2b} \left(b^2 + \frac{4b}{a}\right)^{\frac{1-m}{2}} \sum_{k=0}^m \binom{m}{k} (-1)^{k+1} \frac{q_{m-2k}}{l_{m-2k}}.$$

Using (1.1), we get the desired.

□

If we take $a = b = p$ for some positive integer p , we obtain:

Corollary 3.5. *Let m be a positive integer. Then we have*

$$\zeta_{F_p}(-m) = \begin{cases} 0, & \text{if } m \equiv 2 \pmod{4}, \\ (p^2 + 4)^{\frac{1-m}{2}} \sum_{k=0}^{\frac{m-1}{2}} \binom{m}{k} (-1)^{k+1} \frac{F_{p,m-2k}}{L_{p,m-2k}}, & \text{if } m \text{ is odd.} \end{cases} \quad (3.9)$$

3.2 Values of $\zeta_{q(a,b)}(s)$ at positive integers

In this part, we discuss the values of $\zeta_{q(a,b)}(s)$ at positive integers.

The values of bi-periodic Fibonacci zeta function at positive integers are given by:

Theorem 3.6. *Let m be a positive integer. We have*

$$\zeta_{q(a,b)}(m) = \left(b^2 + \frac{4b}{a}\right)^{\frac{m}{2}} \sum_{\ell \geq 1} a_{m,\ell} \lambda^{-\ell},$$

$$\text{where } a_{m,\ell} = \sum_{\substack{d \in S_m^+ \\ d|\ell}} \binom{\frac{m+d-2}{2}}{m-1} \left(\frac{a}{b}\right)^{m\{\frac{\ell}{2d}\}} + \sum_{\substack{d \in S_m^- \\ d|\ell}} (-1)^{\frac{\ell}{d}} \binom{\frac{m+d-2}{2}}{m-1} \left(\frac{a}{b}\right)^{m\{\frac{\ell}{2d}\}},$$

$$S_m^+ := \{d \geq m : d \equiv m \pmod{4}\}, \text{ and } S_m^- := \{d \geq m : d \equiv m+2 \pmod{4}\}.$$

Proof. From Identity (2.2), we obtain

$$q_n^{-m} = \left(b^2 + \frac{4b}{a}\right)^{\frac{m}{2}} \left(\frac{a}{b}\right)^{m\{\frac{n}{2}\}} \sum_{k \geq 0} \binom{-m}{k} (-1)^{k(n+1)} \lambda^{-n(m+2k)}.$$

Using the fact that $(-1)^k \binom{-m}{k} = \binom{m+k-1}{m-1} = \binom{m+k-1}{m-1}$,

$$q_n^{-m} = \left(b^2 + \frac{4b}{a}\right)^{\frac{m}{2}} \left(\frac{a}{b}\right)^{m\{\frac{n}{2}\}} \sum_{k \geq 0} \binom{m+k-1}{m-1} (-1)^{nk} \lambda^{-n(m+2k)}.$$

If we set $d = m + 2k$, we get

$$q_n^{-m} = \left(b^2 + \frac{4b}{a}\right)^{\frac{m}{2}} \left(\frac{a}{b}\right)^{m\{\frac{n}{2}\}} \sum_{d \in S_m} \binom{\frac{m+d-2}{2}}{m-1} (-1)^{n\frac{d-m}{2}} \lambda^{-nd},$$

where $S_m = \{d \geq m : d \equiv m \pmod{2}\}$.

Let $S_m^+ = \{d \geq m : d \equiv m \pmod{4}\}$ and $S_m^- = \{d \geq m : d \equiv m+2 \pmod{4}\}$. We have then

$$q_n^{-m} = \left(b^2 + \frac{4b}{a}\right)^{\frac{m}{2}} \left(\frac{a}{b}\right)^{m\{\frac{n}{2}\}} \left[\sum_{\substack{d \in S_m^+ \\ d|\ell}} \binom{\frac{m+d-2}{2}}{m-1} \lambda^{-nd} + \sum_{\substack{d \in S_m^- \\ d|\ell}} \binom{\frac{m+d-2}{2}}{m-1} (-1)^n \lambda^{-nd} \right].$$

To take sum over n , we will collect like powers $\ell = nd$, so that ℓ runs over all natural numbers and we restrict to $d|\ell$. We thus have

$$\zeta_{q(a,b)}(m) = \left(b^2 + \frac{4b}{a}\right)^{\frac{m}{2}} \sum_{\ell \geq 1} \left[\sum_{\substack{d \in S_m^+ \\ d|\ell}} \binom{\frac{m+d-2}{2}}{m-1} \left(\frac{a}{b}\right)^{m\{\frac{\ell}{2d}\}} + \sum_{\substack{d \in S_m^- \\ d|\ell}} (-1)^{\frac{\ell}{d}} \binom{\frac{m+d-2}{2}}{m-1} \left(\frac{a}{b}\right)^{m\{\frac{\ell}{2d}\}} \right] \lambda^{-\ell}.$$

This ends the proof. □

When $a = b = p$, we obtain:

Corollary 3.7. *Let m be a positive integer. We have*

$$\zeta_{F_p}(m) = (p^2 + 4)^{\frac{m}{2}} \sum_{\ell \geq 1} a_{m,\ell} \lambda^{-\ell},$$

$$\text{where } a_{m,\ell} = \sum_{\substack{d \in S_m^+ \\ d|\ell}} \binom{\frac{m+d-2}{2}}{m-1} + \sum_{\substack{d \in S_m^- \\ d|\ell}} (-1)^{\frac{\ell}{d}} \binom{\frac{m+d-2}{2}}{m-1}.$$

At the end of this section, we deal with the case $m = 1$ in Theorem 3.6. We have

Corollary 3.8.

$$\sum_{n \geq 1} q_{2n}^{-1} = (b^2 + \frac{4b}{a})^{\frac{1}{2}} \sum_{\ell \equiv 0 \pmod{2}} (d_1(\ell) + d_3(\ell)) \lambda^{-\ell}, \quad (3.10)$$

$$\sum_{n \geq 0} q_{2n+1}^{-1} = (ab + 4)^{\frac{1}{2}} \sum_{\ell \equiv 1 \pmod{2}} (d_1(\ell) - d_3(\ell)) \lambda^{-\ell}, \quad (3.11)$$

$$\text{where } d_i(\ell) = \sum_{\substack{d \equiv i \pmod{4} \\ d|\ell}} 1.$$

Proof. From Theorem 3.6 we have

$$\zeta_{q(a,b)}(1) = \sum_{n \geq 1} q_n^{-1} = (b^2 + \frac{4b}{a})^{\frac{1}{2}} \sum_{\ell \geq 1} \left[\sum_{\substack{d \in S_1^+ \\ d|\ell}} \left(\frac{a}{b}\right)^{\{\frac{\ell}{2d}\}} + \sum_{\substack{d \in S_1^- \\ d|\ell}} (-1)^{\frac{\ell}{d}} \left(\frac{a}{b}\right)^{\{\frac{\ell}{2d}\}} \right] \lambda^{-\ell},$$

where $S_1^+ = \{1, 5, 9, \dots\}$ and $S_1^- = \{3, 7, 11, \dots\}$. We therefore have the following identities:

$$\sum_{n \geq 1} q_{2n}^{-1} = (b^2 + \frac{4b}{a})^{\frac{1}{2}} \sum_{\ell \equiv 0 \pmod{2}} \left[\sum_{\substack{d \in S_1^+ \\ d|\ell}} \left(\frac{a}{b}\right)^{\{\frac{\ell}{2d}\}} + \sum_{\substack{d \in S_1^- \\ d|\ell}} (-1)^{\frac{\ell}{d}} \left(\frac{a}{b}\right)^{\{\frac{\ell}{2d}\}} \right] \lambda^{-\ell}.$$

$$\sum_{n \geq 0} q_{2n+1}^{-1} = (b^2 + \frac{4b}{a})^{\frac{1}{2}} \sum_{\ell \equiv 1 \pmod{2}} \left[\sum_{\substack{d \in S_1^+ \\ d|\ell}} \left(\frac{a}{b}\right)^{\{\frac{\ell}{2d}\}} + \sum_{\substack{d \in S_1^- \\ d|\ell}} (-1)^{\frac{\ell}{d}} \left(\frac{a}{b}\right)^{\{\frac{\ell}{2d}\}} \right] \lambda^{-\ell}.$$

Since $d|\ell$ and d is an odd number, $\{\frac{\ell}{2d}\} = 0$ when ℓ is even and $\{\frac{\ell}{2d}\} = \frac{1}{2}$ when ℓ is an odd number. Therefore,

$$\sum_{n \geq 1} q_{2n}^{-1} = (b^2 + \frac{4b}{a})^{\frac{1}{2}} \sum_{\ell \equiv 0 \pmod{2}} (d_1(\ell) + d_3(\ell)) \lambda^{-\ell},$$

and,

$$\sum_{n \geq 0} q_{2n+1}^{-1} = (ab + 4)^{\frac{1}{2}} \sum_{\ell \equiv 1 \pmod{2}} (d_1(\ell) - d_3(\ell)) \lambda^{-\ell},$$

$$\text{where } d_i(\ell) = \sum_{\substack{d \equiv i \pmod{4} \\ d|\ell}} 1. \quad \square$$

Identity (3.11) can be expressed using the Jacobi's theta function $\Theta(s) = \sum_{n \in \mathbb{Z}} e^{i\pi n^2 s}$ as follows:

Theorem 3.9. *We have*

$$\sum_{\ell \geq 0} q_{2n+1}^{-1} = \frac{(ab + 4)^{\frac{1}{2}}}{4} \left[\Theta^2\left(\frac{i \log \lambda}{\pi}\right) - \Theta^2\left(\frac{2i \log \lambda}{\pi}\right) \right]. \quad (3.12)$$

Proof. Here, we write $\Theta(q) = \sum_{n \in \mathbb{Z}} q^{n^2}$, for $q = e^{i\pi s}$ with $\text{Im}(s) > 0$. It is well known that $\Theta^2(q) = \sum_{\ell \geq 0} r_2(\ell) q^\ell$, where $r_2(\ell)$ is the number of representation of ℓ as sum of two integer squares, also $r_2(\ell) = 4(d_1(\ell) - d_3(\ell))$ and $r_2(2\ell) = r_2(\ell)$. We have then

$$\Theta^2(q^2) = \sum_{\ell \geq 0} r_2(\ell) q^{2\ell} = \sum_{\ell \geq 0} r_2(2\ell) q^{2\ell}.$$

Hence

$$\sum_{\ell \equiv 1 \pmod{2}} r_2(\ell) q^\ell = \Theta^2(q) - \Theta^2(q^2).$$

Thus, $\sum_{\ell \equiv 1 \pmod{2}} (d_1(\ell) - d_3(\ell)) q^\ell = \frac{1}{4}(\Theta^2(q) - \Theta^2(q^2))$. Using the fact that $q = \lambda^{-1}$ when $s = \frac{i \log \lambda}{\pi}$ with Identity (3.11), we obtain the disered identity. $\square$

For $a = b = p$, we obtain:

Corollary 3.10. *We have*

$$\sum_{\ell \geq 0} F_{p, 2n+1}^{-1} = \frac{(p^2 + 4)^{\frac{1}{2}}}{4} \left[\Theta^2 \left(\frac{i \log \lambda}{\pi} \right) - \Theta^2 \left(\frac{2i \log \lambda}{\pi} \right) \right], \quad (3.13)$$

where $\lambda = \frac{p + \sqrt{p^2 + 4}}{2}$.

4 The bi-periodic Fibonacci L -Function

Let χ be a Dirichlet character modulo d . The Dirichlet L -function associated to χ is defined as $L(s, \chi) = \sum_{n \geq 1} \frac{\chi(n)}{n^s}$, ($\Re(s) > 1$). Similarly, we define the bi-periodic Fibonacci L -function as follows:

$$L_{q(a,b)}(s, \chi) := \sum_{n \geq 1} \frac{\chi(n)}{q_n^s} \quad (\Re(s) > 0).$$

In order to study the function $L_{q(a,b)}(s, \chi)$, we introduce the partial bi-periodic Fibonacci zeta function $\zeta_{q(a,b)}(s, (r, d))$ as follows:

$$\zeta_{q(a,b)}(s, (r, d)) := \sum_{\substack{n \geq 1 \\ n \equiv r \pmod{d}}} \frac{1}{q_n^s} \quad (\Re(s) > 0).$$

where d and r are positive integers with $1 \leq r \leq d$.

It is clear that

$$\begin{aligned} L_{q(a,b)}(s, \chi) &= \sum_{n \geq 1} \frac{\chi(n)}{q_n^s} \\ &= \sum_{r=1}^d \chi(r) \zeta_{q(a,b)}(s, (r, d)). \end{aligned} \quad (4.1)$$

Theorem 4.1. *The function $\zeta_{q(a,b)}(s, (r, d))$ can be meromorphically continued to a meromorphic function on $\mathbb{C}$ and can be expressed as:*

$$\begin{aligned} \zeta_{q(a,b)}(s, (r, d)) &= \\ \left(b^2 + \frac{4b}{a} \right)^{\frac{s}{2}} \sum_{k \geq 0} \binom{-s}{k} (-1)^{k(r+1)} \lambda^{(d-r)(s+2k)} \frac{\left(\frac{a}{b} \right)^{s \left\{ \frac{r}{2} \right\}} \lambda^{d(s+2k)} + (-1)^{kd} \left(\frac{a}{b} \right)^{s \left\{ \frac{d+r}{2} \right\}}}{\lambda^{2d(s+2k)} - 1}. \end{aligned} \quad (4.2)$$

It is holomorphic except at the following simple poles:

$$\begin{cases} s_1 = s_{n,k} = -2k + \left(k + \frac{2n}{d}\right) \frac{\pi i}{\log \lambda}, & \text{if } a = b, \\ s_2 = s_{k,n} = -2k + \frac{2ni\pi}{d \log \lambda}, & \text{if } a \neq b \text{ and } d \text{ is even, } (k \geq 0, n \in \mathbb{Z}). \\ s_3 = s_{k,n} = -2k + \frac{in\pi}{d \log \lambda}, & \text{if } a \neq b \text{ and } d \text{ is odd,} \end{cases}$$

The residue of $\zeta_{q(a,b)}(s)$ at these poles are:

$$\begin{aligned} \text{Res}(\zeta_{q(a,b)}(s, (d, r)); w) = \\ (b^2 + \frac{4b}{a})^{\frac{w}{2}} (-w) \frac{(-1)^k \eta_d^{-nr}}{d \log \lambda} \times \begin{cases} 1, & \text{if } w = s_1, \\ (-1)^{kr} \left(\frac{a}{b}\right)^{w\{\frac{r}{2}\}}, & \text{if } w = s_2, \\ \frac{1}{2}(-1)^{kr} \eta_d^{\frac{nr}{2}} \left(\left(\frac{a}{b}\right)^{w\{\frac{r}{2}\}} + (-1)^{k+n} \left(\frac{a}{b}\right)^{w\{\frac{r+1}{2}\}}\right), & \text{if } w = s_3, \end{cases} \end{aligned}$$

where $\eta_d := \exp(2i\pi/d)$.

Proof. From (2.2), we have

$$\begin{aligned} \zeta_{q(a,b)}(s, (r, d)) \\ = \left(b^2 + \frac{4b}{a}\right)^{\frac{s}{2}} \sum_{n \geq 0} \sum_{k \geq 0} \left(\frac{a}{b}\right)^{s\{\frac{dn+r}{2}\}} \binom{-s}{k} (-1)^{k(dn+r+1)} \lambda^{-(dn+r)(s+2k)} \\ = \left(b^2 + \frac{4b}{a}\right)^{\frac{s}{2}} \sum_{k \geq 0} \binom{-s}{k} (-1)^{k(r+1)} \lambda^{-r(s+2k)} \sum_{n \geq 0} \left(\frac{a}{b}\right)^{s\{\frac{dn+r}{2}\}} (-1)^{dnk} \lambda^{-dn(s+2k)}. \end{aligned}$$

Considering the n even and n odd cases, we get

$$\begin{aligned} & \sum_{n \geq 0} \left(\frac{a}{b}\right)^{s\{\frac{dn+r}{2}\}} (-1)^{dnk} \lambda^{-dn(s+2k)} \\ &= \left(\left(\frac{a}{b}\right)^{s\{\frac{r}{2}\}} + (-1)^{kd} \left(\frac{a}{b}\right)^{s\{\frac{d+r}{2}\}} \lambda^{-d(s+2k)}\right) \sum_{n \geq 0} \lambda^{-2nd(s+2k)} \\ &= \left(\left(\frac{a}{b}\right)^{s\{\frac{r}{2}\}} + (-1)^{kd} \left(\frac{a}{b}\right)^{s\{\frac{d+r}{2}\}} \lambda^{-d(s+2k)}\right) \frac{\lambda^{2d(s+2k)}}{\lambda^{2d(s+2k)} - 1} \\ &= \lambda^{d(s+2k)} \frac{\left(\frac{a}{b}\right)^{s\{\frac{r}{2}\}} \lambda^{d(s+2k)} + (-1)^{kd} \left(\frac{a}{b}\right)^{s\{\frac{d+r}{2}\}}}{\lambda^{2d(s+2k)} - 1}. \end{aligned}$$

Therefore,

$$\begin{aligned} \zeta_{q(a,b)}(s, (r, d)) = \\ \left(b^2 + \frac{4b}{a}\right)^{\frac{s}{2}} \sum_{k \geq 0} \binom{-s}{k} (-1)^{k(r+1)} \lambda^{(d-r)(s+2k)} \frac{\left(\frac{a}{b}\right)^{s\{\frac{r}{2}\}} \lambda^{d(s+2k)} + (-1)^{kd} \left(\frac{a}{b}\right)^{s\{\frac{d+r}{2}\}}}{\lambda^{2d(s+2k)} - 1}. \end{aligned}$$

Let $\eta_d = \exp(2i\pi/d)$, we have then,

(i) If $a = b$, we get

$$\zeta_{q(a,a)}(s, (r, d)) = (a^2 + 4)^{\frac{s}{2}} \sum_{k \geq 0} \binom{-s}{k} (-1)^{k(r+1)} \frac{\lambda^{(d-r)(s+2k)}}{\lambda^{d(s+2k)} - (-1)^{kd}}.$$

Thus, the poles of $\zeta_{q(a,a)}(s, (r, d))$ are $s_1 = s_{n,k} = -2k + \left(k + \frac{2n}{d}\right) \frac{i\pi}{\log \lambda}$ ($k \geq 0, n \in \mathbb{Z}$). Using L'Hopital's rule, the residue at s_1 can be easily seen:

$$\begin{aligned} \text{Res}(\zeta_{q(a,a)}(s, (d, r)); s_1) &= (a^2 + 4)^{\frac{s_1}{2}} \binom{-s_1}{k} (-1)^{k(d+1)} \eta_d^{-nr} \lim_{s \rightarrow s_1} \frac{s - s_1}{\lambda^{d(s+2k)} - (-1)^{kd}} \\ &= (a^2 + 4)^{\frac{s_1}{2}} \binom{-s_1}{k} (-1)^k \frac{\eta_d^{-nr}}{d \log \lambda}. \end{aligned}$$

(ii) If $a \neq b$, we distinguish two cases:

- Case d is an even number: we have then $\{\frac{d+r}{2}\} = \{\frac{r}{2}\}$, hence

$$\zeta_{q(a,b)}(s, (r, d)) = \quad (4.3)$$

$$\left(b^2 + \frac{4b}{a}\right)^{\frac{s}{2}} \sum_{k \geq 0} \binom{-s}{k} (-1)^{k(r+1)} \lambda^{(d-r)(s+2k)} \left(\frac{a}{b}\right)^{s\{\frac{r}{2}\}} \frac{1}{\lambda^{d(s+2k)} - 1}. \quad (4.4)$$

So, the poles of $\zeta_{q(a,b)}(s, (r, d))$ are $s_2 = s_{n,k} = -2k + \frac{2ni\pi}{d \log \lambda}$ ($k \geq 0, n \in \mathbb{Z}$). Using L'Hopital's rule, the residue at s_1 can be easily seen to be:

$$\begin{aligned} \text{Res}(\zeta_{q(a,b)}(s, (d, r)); s_2) &= \\ \left(b^2 + \frac{4b}{a}\right)^{\frac{s_2}{2}} \binom{-s_2}{k} (-1)^{k(r+1)} \eta_d^{-nr} \left(\frac{a}{b}\right)^{s_2\{\frac{r}{2}\}} \lim_{s \rightarrow s_2} \frac{s - s_1}{\lambda^{d(s+2k)} - 1} \\ &= \left(b^2 + \frac{4b}{a}\right)^{\frac{s_2}{2}} \binom{-s_2}{k} (-1)^{k(r+1)} \frac{\eta_d^{-nr} \left(\frac{a}{b}\right)^{s_2\{\frac{r}{2}\}}}{d \log \lambda}. \end{aligned}$$

- Case d is an odd number: the poles of $\zeta_{q(a,b)}(s, (r, d))$ are $s_3 = s_{k,n} = -2k + \frac{in\pi}{d \log \lambda}$ ($k \geq 0, n \in \mathbb{Z}$). Using L'Hopital's rule, the residue at s_3 can be easily seen.

$$\begin{aligned} \text{Res}(\zeta_{q(a,b)}(s, (d, r)); s_3) &= \\ \left(b^2 + \frac{4b}{a}\right)^{\frac{s_3}{2}} \binom{-s_3}{k} (-1)^{k(r+1)} (-1)^n \eta_d^{-\frac{nr}{2}} \\ &\quad \times \left((-1)^n \left(\frac{a}{b}\right)^{s_3\{\frac{r}{2}\}} + (-1)^k \left(\frac{a}{b}\right)^{s_3\{\frac{r+1}{2}\}} \right) \lim_{s \rightarrow s_3} \frac{s - s_3}{\lambda^{2d(s+2k)} - 1} \\ &= \left(b^2 + \frac{4b}{a}\right)^{\frac{s_3}{2}} \binom{-s_3}{k} (-1)^{k(r+1)} \eta_d^{-\frac{nr}{2}} \frac{\left(\frac{a}{b}\right)^{s_3\{\frac{r}{2}\}} + (-1)^{k+n} \left(\frac{a}{b}\right)^{s_3\{\frac{r+1}{2}\}}}{2d \log \lambda}. \end{aligned}$$

□

Let χ be a Dirichlet character modulo d . The Gaussian sum associated to χ is defined by: $G_n(d, \chi) := \sum_{r=1}^d \chi(r) \eta_d^{nr}$ where $\eta_d = \exp(2i\pi/d)$.

We define the following partial Gaussian sums according to the definition found in [5]:

$$G_n(\ell, \chi) = \sum_{r=1}^{\ell} \chi(r) \eta_d^{nr},$$

where ℓ is a positive integer and n is a fractional number.

Now, we return to the bi-periodic Fibonacci L -function $L_{q(a,b)}(s, \chi)$. The following theorem gives that the function $L_{q(a,b)}(s, \chi)$ can be extended to a meromorphic function on $\mathbb{C}$.

Theorem 4.2. *The bi-periodic Fibonacci L -function $L_{q(a,b)}(s, \chi)$ can be meromorphically continued to the whole complex plane except at the following simple poles:*

$$\begin{cases} s_1 = s_{n,k} = -2k + \left(k + \frac{2n}{d}\right) \frac{\pi i}{\log \lambda}, & \text{if } a = b, \\ s_2 = s_{k,n} = -2k + \frac{2ni\pi}{d \log \lambda}, & \text{if } a \neq b \text{ and } d \text{ is even,} \\ s_3 = s_{k,n} = -2k + \frac{in\pi}{d \log \lambda}, & \text{if } a \neq b \text{ and } d \text{ is odd,} \end{cases} \quad (k \geq 0, n \in \mathbb{Z}).$$

The residue of $L_{q(a,b)}(s, \chi)$ at these poles are:

$$\begin{aligned} \text{Res}(L_{q(a,b)}(s, (d, r)); w) &= \binom{-w}{k} \frac{(-1)^k}{d \log \lambda} \times \\ &\begin{cases} (a^2 + 4)^{\frac{w}{2}} \chi(-1) G_n(d, \chi) & \text{if } w = s_1, \\ (ab + 4)^{\frac{w}{2}} (-1)^k \chi(-1) G_n(d, \chi), & \text{if } w = s_2, \\ \frac{1}{2} \left(b^2 + \frac{4b}{a}\right)^{\frac{w}{2}} \left(1 + (-1)^{k+n} \left(\frac{a}{b}\right)^{\frac{w}{2}}\right) \left(\chi(2)(1 - (-1)^n) G_{-n}\left(\frac{d-1}{2}, \chi\right) + (-1)^n G_{-\frac{n}{2}}(d, \chi)\right), & \text{if } w = s_3, \end{cases} \end{aligned}$$

Proof. From Theorem 4.1 and (4.1), the poles of $L_{q(a,b)}(s, \chi)$ are

- (i) If $a = b$: $s_1 = s_{n,k} = -2k + \left(k + \frac{2n}{d}\right) \frac{\pi i}{\log \lambda}$ ($k \geq 0, n \in \mathbb{Z}$), and the residue at $s = s_1$ is

$$\begin{aligned} \text{Res}(L_{q(a,a)}(s, \chi); s_1) &= \sum_{r=1}^d \chi(r) \text{Res}(\zeta_{q(a,a)}(s, (d, r)); s_1) \\ &= (a^2 + 4)^{s_1/2} \binom{-s_1}{k} \frac{(-1)^k}{d \log \lambda} \sum_{r=1}^d \chi(r) \eta_d^{-nr} \\ &= (a^2 + 4)^{s_1/2} \binom{-s_1}{k} \frac{(-1)^k \chi(-1)}{d \log \lambda} G_n(d, \chi). \end{aligned}$$

- (ii) If $a \neq b$, we distinguish two cases:

- Case d is an even number: $s_2 = s_{k,n} = -2k + \frac{2ni\pi}{d \log \lambda}$ ($k \geq 0, n \in \mathbb{Z}$), in this case

$$\begin{aligned} \text{Res}(L_{q(a,b)}(s, \chi); s_2) &= \sum_{r=1}^d \chi(r) \text{Res}(\zeta_{q(a,b)}(s, (d, r)); s_2) \\ &= \left(b^2 + \frac{4b}{a}\right)^{\frac{s_2}{2}} \binom{-s_2}{k} \frac{(-1)^k}{d \log \lambda} \sum_{r=1}^d \chi(r) (-1)^{kr} \eta_d^{-nr} \left(\frac{a}{b}\right)^{s_2 \{ \frac{r}{2} \}}. \end{aligned}$$

Since d is an even number, $\chi(r) = 0$ for all even number r . Therefore,

$$\begin{aligned} \sum_{r=1}^d \chi(r) (-1)^{kr} \eta_d^{-nr} \left(\frac{a}{b}\right)^{s_2 \{ \frac{r}{2} \}} &= \left(\frac{a}{b}\right)^{\frac{s_2}{2}} (-1)^k \sum_{r=1}^d \chi(r) \eta_d^{-nr} \\ &= \left(\frac{a}{b}\right)^{\frac{s_2}{2}} (-1)^k \chi(-1) G_n(d, \chi). \end{aligned}$$

So

$$\text{Res}(L_{q(a,b)}(s, \chi); s_2) = (ab + 4)^{\frac{s_2}{2}} \binom{-s_2}{k} \frac{\chi(-1)}{d \log \lambda} G_n(d, \chi).$$

- Case d is an odd number: $s_3 = s_{k,n} = -2k + \frac{in\pi}{d \log \lambda}$ ($k \geq 0, n \in \mathbb{Z}$), in this case, the residue at $s = s_2$ is:

$$\begin{aligned} \text{Res}(L_{q(a,b)}(s, \chi); s_3) &= \sum_{r=1}^d \chi(r) \text{Res}(\zeta_{q(a,b)}(s, (r, d)); s_3) \\ &= \left(b^2 + \frac{4b}{a}\right)^{\frac{s_3}{2}} \binom{-s_3}{k} \frac{(-1)^k}{2d \log \lambda} \sum_{r=1}^d (-1)^{kr} \chi(r) \eta_d^{-\frac{nr}{2}} \left[\left(\frac{a}{b}\right)^{s_3 \{ \frac{r}{2} \}} + (-1)^{k+n} \left(\frac{a}{b}\right)^{s_3 \{ \frac{r+1}{2} \}} \right] \\ &= \left(b^2 + \frac{4b}{a}\right)^{\frac{s_3}{2}} \binom{-s_3}{k} \frac{(-1)^k}{2d \log \lambda} \left[\left(1 + (-1)^{k+n} \left(\frac{a}{b}\right)^{\frac{s_3}{2}}\right) \sum_{r=1}^{\frac{d-1}{2}} \chi(2r) \eta_d^{-nr} \right. \\ &\quad \left. + \left(\left(\frac{a}{b}\right)^{\frac{s_3}{2}} + (-1)^{k+n}\right) (-1)^k \eta_d^{-\frac{n}{2}} \sum_{r=0}^{\frac{d-1}{2}} \chi(2r+1) \eta_d^{-nr} \right] \\ &= \left(b^2 + \frac{4b}{a}\right)^{\frac{s_3}{2}} \binom{-s_3}{k} \frac{(-1)^k}{2d \log \lambda} \left(1 + (-1)^{k+n} \left(\frac{a}{b}\right)^{\frac{s_3}{2}}\right) \\ &\quad \times \left(\sum_{r=1}^{\frac{d-1}{2}} \chi(2r) \eta_d^{-nr} + (-1)^n \eta_d^{-\frac{n}{2}} \sum_{r=0}^{\frac{d-1}{2}} \chi(2r+1) \eta_d^{-nr} \right) \\ &= \left(b^2 + \frac{4b}{a}\right)^{\frac{s_3}{2}} \binom{-s_3}{k} \frac{(-1)^k}{2d \log \lambda} \left(1 + (-1)^{k+n} \left(\frac{a}{b}\right)^{\frac{s_3}{2}}\right) \\ &\quad \times \left(\chi(2)(1 - (-1)^n) G_{-n} \left(\frac{d-1}{2}, \chi\right) + (-1)^n G_{-\frac{n}{2}}(d, \chi) \right). \end{aligned}$$

□

Let χ be a Dirichlet character modulo d . It is known that if $\chi(-1) = 1$, then $L(-2m, \chi) = 0$, for any positive integer m . If $\chi(-1) = -1$, then $L(-2m+1, \chi) = 0$, for any positive integer m . Similarly, in the case of $L_{q(a,b)}(s)$, we obtain the following results:

From Theorem 4.2, we obtain a sufficient condition such that $L_{q(a,b)}(s)$ is holomorphic at negative integers.

Lemma 4.3. *Let ℓ be a positive integer. Then $L_{q(a,b)}(s)$ possesses pole at $s = -\ell$ if and only if*

$$\begin{cases} \ell \equiv 0 \pmod{2}, & \text{if } a = b \text{ with } d \text{ is even,} \\ \ell \equiv 0 \pmod{4}, & \text{if } a = b \text{ with } d \text{ is odd,} \\ \ell \equiv 0 \pmod{2}, & \text{if } a \neq b. \end{cases}$$

Kamano [10] (Corollary 12) proves that $L_{q(a,a)}(-m, \chi) = 0$ for any positive integer m (except for singularities) and any Dirichlet character χ satisfies $\chi(-1) = 1$.

We continue this investigation in the case $a \neq b$:

Theorem 4.4. *Let χ be a Dirichlet character modulo even number d such that $\chi(-1) = 1$, then for any positive odd integer m , we have*

$$L_{q(a,b)}(-m, \chi) = 0.$$

Proof. From Identities (4.1) and (4.3), we get

$$L_{q(a,b)}(-m, \chi) = \frac{1}{2} \left(b^2 + \frac{4b}{a} \right)^{-\frac{m}{2}} \sum_{r=1}^d \chi(r) \sum_{k=0}^m (\sigma_k + \sigma_{m-k}),$$

where

$$\begin{aligned} \sigma_k &:= \binom{m}{k} (-1)^{k(r+1)} \lambda^{(d-r)(2k-m)} \left(\frac{a}{b} \right)^{-m\{\frac{r}{2}\}} \frac{1}{\lambda^{d(2k-m)} - 1} \\ &= \binom{m}{k} (-1)^k \left(\frac{a}{b} \right)^{-m\{\frac{r}{2}\}} \frac{\lambda^{-d(m-2k)} ((-1)^k \lambda^{m-2k})^r}{\lambda^{-d(m-2k)} - 1}. \end{aligned}$$

Since m is an odd number, we have then

$$\sigma_{m-k} = \binom{m}{k} (-1)^k \left(\frac{a}{b} \right)^{-m\{\frac{r}{2}\}} (-1)^{r+1} \frac{\lambda^{d(m-2k)} ((-1)^k \lambda^{m-2k})^{-r}}{\lambda^{d(m-2k)} - 1}.$$

Let $X = (-1)^k \lambda^{m-2k}$. Thus $\sigma_k + \sigma_{m-k} = \binom{m}{k} (-1)^k \left(\frac{a}{b} \right)^{-m\{\frac{r}{2}\}} \left[\frac{X^{r-d}}{X^{-d}-1} + (-1)^{r+1} \frac{X^{d-r}}{X^d-1} \right]$. Therefore,

$$\begin{aligned} L_{q(a,b)}(-m, \chi) &= \\ &= \frac{1}{2} \left(b^2 + \frac{4b}{a} \right)^{-\frac{m}{2}} \sum_{k=0}^m \binom{m}{k} (-1)^k \sum_{r=1}^d \chi(r) \left(\frac{a}{b} \right)^{-m\{\frac{r}{2}\}} \left[\frac{X^{r-d}}{X^{-d}-1} + (-1)^{r+1} \frac{X^{d-r}}{X^d-1} \right] \\ &= \frac{1}{2} \left(b^2 + \frac{4b}{a} \right)^{-\frac{m}{2}} \sum_{k=0}^m \binom{m}{k} \frac{(-1)^k}{1-X^d} \sum_{r=1}^d \chi(r) \left(\frac{a}{b} \right)^{-m\{\frac{r}{2}\}} [X^r + (-1)^r X^{d-r}]. \end{aligned}$$

Since d is an even number, $\chi(r) = 0$ for all even integer r . Then we have

$$L_{q(a,b)}(-m, \chi) = \frac{1}{2} (ab + 4)^{-\frac{m}{2}} \sum_{k=0}^m \binom{m}{k} \frac{(-1)^k}{1-X^d} \sum_{r=1}^d \chi(r) [X^r - X^{d-r}].$$

We have clearly,

$$\sum_{r=1}^d \chi(r) X^{d-r} = \sum_{r=0}^{d-1} \chi(d-r) X^r = \sum_{r=1}^d \chi(r) X^r \quad (\chi(-1) = 1)$$

and this gives that $L_{q(a,b)}(-m, \chi) = 0$. □

5 Concluding remarks

In this work, we have introduced the bi-periodic Fibonacci zeta function as a natural extension of the classical Fibonacci zeta function, constructed using the bi-periodic Fibonacci numbers. Through our investigation, we have demonstrated that this novel Dirichlet series admits an analytic continuation across the entire complex plane $\mathbb{C}$. Moreover, we have established values for the series at integer points and examined other analytical properties.

Building on this study, we also proposed and studied the bi-periodic Fibonacci L -function.

This work follows a series of previous generalizations and opens new perspectives for studying zeta and L -functions associated with generalized Fibonacci sequences; Namely, possible applications in dynamical systems or cryptography.

References

- [1] J. P. Ascano, E. N. Gueco, *The bi-periodic Fibonacci-Horadam matrix*, *Integers*, **21**, (2021), Paper A29, 21p.
- [2] D. Behera, U. K. Dutta, P. K. Ray, *On Lucas-balancing zeta function*, *Acta Comment. Univ. Tartu. Math.*, **21**(1), 65–74, (2018).
- [3] H. Belbachir, I. E. Djellas, F. Krim, *Permanental Representations of Negatively Subscripted Generalized Order- k Fibonacci Numbers*, *J. Integer Seq.*, **25**(2), (2022), 3.
- [4] G. Bilgici, *Two generalizations of Lucas sequence*, *Appl. Math. Comput.*, **245**, 526–538, (2014).
- [5] D. A. Burgess, *Partial Gaussian sums*, *Bull. Lond. Math. Soc.*, **21**(6), 589–592, (1988).
- [6] I. E. Djellas, A. Taane, H. Belbachir, *Some Identities for the Chebyshev Polynomials Using Graph Matchings*, *Mediterr. J. Math.*, **22**(5), (2025), 108.
- [7] M. Edson, O. Yayenie, *A new generalization of Fibonacci sequence & extended Binet's formula*, *Integers*, **9**, 639–654, (2009), Paper A48.
- [8] S. Falcón, *On the k -Lucas numbers*, *Int. J. Contemp. Math. Sci.*, **6**(21), 1039–1050, (2011).
- [9] S. Falcón, Á. Plaza, *On the Fibonacci k -numbers*, *Chaos Solitons Fractals*, **32**(5), 1615–1624, (2007).
- [10] K. Kamano, *Analytic continuation of the Lucas zeta and L -functions*, *Indag. Math.*, **24**(3), 637–646, (2013).
- [11] M. Mekkaoui, A. Taane, S. Badidja, *Some new identities involving p -Fibonacci and p -Lucas numbers with applications*, *Asian-Eur. J. Math.*, **17**(2), (2024), 2450001, 14 p.
- [12] L. Navas, *Analytic continuation of the Fibonacci Dirichlet series*, *Fibonacci Q.*, **39**(5), 409–418, (2001).
- [13] S. Rout, G. K. Panda, *Balancing Dirichlet series and related L -functions*, *Indian J. Pure Appl. Math.*, **45**(6), 943–952, (2014).
- [14] Á. Serrano Holgado, *The Tribonacci Dirichlet series*, *Acta Math. Hung.* **170**(1), 102–109, (2023).
- [15] Á. Serrano Holgado, N. Vicente, L. Manuel, *The zeta function of a recurrence sequence of arbitrary degree*, *Mediterr. J. Math.*, **20**(4), (2023), Paper No 224, 18p.
- [16] L. Smajlović, Z. Šabanac, L. Šćeta, *On the generalized Tribonacci zeta function*, *Fibonacci Q.*, **60**(5), 344–354, (2022), Spec. Iss.
- [17] L. Smajlović, Z. Šabanac, L. Šćeta, *On the Hurwitz-type zeta function associated to the Lucas sequence*, *Fibonacci Q.*, **60**(5), 355–371, (2022), Spec. Iss.
- [18] A. Taane, I. E. Djellas, M. Mekkaoui, *On some nested sums involving q -Fibonacci numbers*, *Palest. J. Math.*, **12**(2), 183–193, (2023).

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