

A New Generalization of the Euler Totient Function

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Abstract In this paper, for an arithmetic function f , we define the related arithmetic function φ_f which can be considered as a generalization of the classical Euler's totient function φ , and we study some of its properties, including multiplicativity, the Dirichlet inverse, and Dirichlet series.

1 Introduction

An arithmetic function is a complex-valued function whose domain is the set of positive integers. For example, the Euler totient function, denoted by $\varphi(n)$, is an arithmetic function that counts the positive integers up to a given integer n that are coprime with n ; thus,

$$\varphi(n) = \text{card}\{x \mid 1 \leq x < n, \gcd(x, n) = 1\}.$$

It is known that (see, e.g. [1, 2]):

$$\varphi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right),$$

where the product is taken over all distinct prime factors p of n .

Euler's totient function is a versatile and essential tool that plays a crucial role in various branches of mathematics and its applications extend beyond number theory into diverse areas of mathematical research and computer science. For these reasons, it has garnered significant attention from mathematicians, and one form of this interest is to find more than ten known generalizations and analogues of it (see [4, Chapter 3.7]). For example, the Schemmel totient function S_k ($k \in \mathbb{N}$) is defined as:

$$S_k(n) = \text{card}\{(x, \dots, x+k-1) \mid 1 \leq x+i < n, \gcd(x+i, n) = 1 \ (0 \leq i \leq k-1)\},$$

and the Jordan totient function J_k ($k \in \mathbb{N}$) is defined as:

$$J_k(n) = n^k \prod_{p|n} \left(1 - \frac{1}{p^k}\right).$$

Clearly, we have $S_1(n) = J_1(n) = \varphi(n)$.

In the present paper, we introduce a new generalization associated with an arithmetic function f of the Euler totient function. Furthermore, we will discuss some of its properties, including multiplicativity, the Dirichlet inverse, and Dirichlet series. The following arithmetic functions are also well-known and may be used throughout this paper:

- μ, λ are the Möbius and the Liouville functions, respectively.
- $\mathbf{u}$ is the unit function: $\mathbf{u}(n) = 1$ for all n .
- N_α ($\alpha \in \mathbb{C}$) is the α -th power function: $N_\alpha(n) = n^\alpha$, ($N_1 = N$).
- ψ is the Dedekind function: $\psi(n) = n \prod_{p|n} \left(1 + \frac{1}{p}\right)$.

2 Main Results

Let f be an arithmetic function. Then we define φ_f to be the arithmetic function such that:

$$\varphi_f(n) = n \prod_{p|n} \left(1 - \frac{f(p)}{p}\right), \quad \varphi_f(1) = 1.$$

Note that if $f(p) = 1$ for all prime numbers p , then $\varphi_f = \varphi$. For example:

$$\varphi_u = \varphi, \quad \varphi_\mu = \varphi_\lambda = \psi, \quad \text{and} \quad \varphi_\varphi = -\varphi_\psi.$$

Let n and m be positive integers and let $d = \gcd(n, m)$. Then

$$\begin{aligned} \varphi_f(mn) &= mn \prod_{p|mn} \left(1 - \frac{f(p)}{p}\right) \\ &= \frac{m \prod_{p|m} \left(1 - \frac{f(p)}{p}\right) n \prod_{p|n} \left(1 - \frac{f(p)}{p}\right)}{\prod_{p|d} \left(1 - \frac{f(p)}{p}\right)} \\ &= \varphi_f(m) \varphi_f(n) \frac{d}{\varphi_f(d)}. \end{aligned}$$

In particular, if $d = 1$ then:

$$\varphi_f(mn) = \varphi_f(m) \varphi_f(n)$$

which means that φ_f is a multiplicative function. Hence

$$\varphi_f(n) = \varphi_f \left(\prod_{i=1}^r p_i^{e_i} \right) = \prod_{i=1}^r p_i^{e_i-1} (p_i - f(p_i)) \quad (n > 1),$$

where the product is taken over all prime factors of n .

It is well-known (see, e.g. [1, p. 29]) that the Euler totient function φ satisfies the following relation:

$$\varphi = \mu * N,$$

where $*$ denotes the Dirichlet product (recall that $(f * g)(n) = \sum_{ab=n} f(a)g(b)$). The next theorem establishes that φ_f satisfies an analogous identity.

Theorem 2.1. *Let f be a multiplicative function. Then*

$$\varphi_f = \mu f * N, \tag{2.1}$$

where $(\mu f)(n) = \mu(n)f(n)$.

Proof. We note that the equality holds when $n = 1$. So, we assume that $n > 1$ and let p_i ($1 \leq i \leq r$) be the distinct prime factors of n . Thus

$$\begin{aligned} \frac{\varphi_f(n)}{n} &= \prod_{i=1}^r \left(1 - \frac{f(p_i)}{p_i}\right) \\ &= 1 - \sum \frac{f(p_i)}{p_i} + \sum \frac{f(p_i)f(p_j)}{p_i p_j} + \dots + (-1)^r \frac{f(p_1) \cdots f(p_r)}{p_1 \cdots p_r} \\ &= 1 - \sum \frac{f(p_i)}{p_i} + \sum \frac{f(p_i p_j)}{p_i p_j} + \dots + (-1)^r \frac{f(p_1 \cdots p_r)}{p_1 \cdots p_r} \quad (f \text{ is multiplicative}) \\ &= \mu(1) + \mu(p_i) \sum \frac{f(p_i)}{p_i} + \mu(p_i p_j) \sum \frac{f(p_i p_j)}{p_i p_j} + \dots + \mu(p_1 \cdots p_r) \frac{f(p_1 \cdots p_r)}{p_1 \cdots p_r} \\ &= \sum_{d|n} \frac{\mu f(d)}{d}, \end{aligned}$$

since $\mu(d) = 0$ if d is not a square-free integer. Note that, in a term such as $\sum \frac{f(p_i p_j)}{p_i p_j}$, it is implied that we are considering all possible products $p_i p_j$ involving distinct prime factors of n . Hence we can rewrite the above equalities as:

$$\varphi_f(n) = \sum_{d|n} \mu f(d) \frac{n}{d} = (\mu f * \mathbf{N})(n),$$

as claimed. The proof is complete. $\square$

Corollary 2.2. *Let f be a completely multiplicative function. Then*

$$\varphi_f = f^{-1} * \mathbf{N}, \quad (2.2)$$

where f^{-1} is the Dirichlet inverse of f .

Proof. The result follows at once from (2.1) and the fact that $f^{-1} = \mu f$ (see, e.g. [1, Theorem 2.17]). $\square$

The Dirichlet inverse of the Euler totient function is given by (see, e.g. [1, p. 37]):

$$\varphi^{-1} = \mathbf{u} * \mu \mathbf{N}.$$

The following theorem provides the Dirichlet inverse of φ_f .

Theorem 2.3. *If f is multiplicative, then*

$$\varphi_f^{-1} = (\mu f)^{-1} * \mu \mathbf{N}.$$

Proof. It is well-known that the identity $(g * h)^{-1} = g^{-1} * h^{-1}$ holds true for all arithmetic functions g and h (see, e.g. [1, 3]). Furthermore, since $\mathbf{N}$ is completely multiplicative, we have $\mathbf{N}^{-1} = \mu \mathbf{N}$. It follows by using these facts and Theorem 2.1 that:

$$\begin{aligned} \varphi_f = \mu f * \mathbf{N} &\Rightarrow \varphi_f^{-1} = (\mu f)^{-1} * \mathbf{N}^{-1} \\ &\Rightarrow \varphi_f^{-1} = (\mu f)^{-1} * \mu \mathbf{N}. \end{aligned}$$

As required, the proof is finished. $\square$

Corollary 2.4. *Let f be a completely multiplicative function and let $\prod_{i=1}^r p_i^{e_i}$ be the prime factorization of the positive integer $n > 1$. Then*

$$\varphi_f^{-1}(n) = (f * \mu \mathbf{N})(n) = \prod_{i=1}^r f(p_i)^{e_i-1} (f(p_i) - p_i).$$

Proof. Because f is completely multiplicative we have $f^{-1} = \mu f$, so

$$\begin{aligned} \varphi_f^{-1}(n) &= ((f^{-1})^{-1} * \mu \mathbf{N})(n) = (f * \mu \mathbf{N})(n) \\ &= \sum_{d|n} d \mu(d) f\left(\frac{n}{d}\right). \end{aligned}$$

We know that φ_f^{-1} is multiplicative, which means that it suffices to compute $\varphi_f^{-1}(p_i^{e_i})$ to deter-

mine $\varphi_f^{-1}(n)$. Thus

$$\begin{aligned}
 \varphi_f^{-1}(n) &= \prod_{i=1}^r \varphi_f^{-1}(p_i^{e_i}) \\
 &= \prod_{i=1}^r \sum_{d|p_i^{e_i}} d\mu(d) f\left(\frac{p_i^{e_i}}{d}\right) \\
 &= \prod_{i=1}^r \sum_{k=0}^{e_i} p_i^k \mu(p_i^k) f\left(\frac{p_i^{e_i}}{p_i^k}\right) \\
 &= \prod_{i=1}^r f(p_i^{e_i}) - p_i f(p_i^{e_i-1}) \\
 &= \prod_{i=1}^r f(p_i)^{e_i-1} (f(p_i) - p_i),
 \end{aligned}$$

since f is completely multiplicative. This completes the proof. $\square$

Example 2.5. (i) If $f = \mathbf{u}$, then

$$\varphi_f^{-1}(n) = \varphi^{-1}(n) = \prod_{i=1}^r (1 - p_i).$$

(ii) If $f = \lambda$, then

$$\varphi_f^{-1}(n) = \psi^{-1}(n) = \prod_{i=1}^r (-1)^{e_i} (1 + p_i).$$

(iii) If $f = N_\alpha$, then

$$\varphi_f^{-1}(n) = n^\alpha \prod_{i=1}^r \left(1 - \frac{1}{p_i^{\alpha-1}}\right).$$

The Dirichlet series associated with an arithmetic function f is an infinite series of the form [1, 3]:

$$D(f, s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s} \quad (s \in \mathbb{C}).$$

The most famous example is the Riemann ζ function, given by $\zeta(s) = D(\mathbf{u}, s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ ($\Re(s) > 1$). Since $\varphi_f = \mu f * N$ (for multiplicative function f), we have according to [1, Theorem 11.5]:

$$D(\varphi_f, s) = D(\mu f, s) D(N, s), \quad (2.3)$$

which holds in the half-plane where both series on the right-hand side of (2.3) converge absolutely. In particular, if f is completely multiplicative, then we have the following theorem:

Theorem 2.6. *Let $D(f, s)$ be the Dirichlet series associated with a completely multiplicative function f . Suppose that $D(f, s)$ converges absolutely in the half-plane $\Re(s) > a$, then:*

$$D(\varphi_f, s) = \frac{\zeta(s-1)}{D(f, s)} \quad (\Re(s) > \max\{2, a\}).$$

Proof. This follows at once from (2.2), (2.3), and [1, Examples 2,3 p. 229]. $\square$

Example 2.7. (i) If $f = \mathbf{u}$, then

$$D(\varphi_f, s) = D(\varphi, s) = \sum_{n=1}^{\infty} \frac{\varphi(n)}{n^s} = \frac{\zeta(s-1)}{\zeta(s)} \quad (\Re(s) > 2).$$

(ii) If $f = \lambda$, then

$$D(\varphi_f, s) = D(\psi, s) = \sum_{n=1}^{\infty} \frac{\psi(n)}{n^s} = \frac{\zeta(s)\zeta(s-1)}{\zeta(2s)} \quad (\Re(s) > 2).$$

(iii) If $f = N_\alpha$, then $\varphi_f(n) = n \prod_{p|n} (1 - p^{\alpha-1})$ and

$$D(\varphi_f, s) = \sum_{n=1}^{\infty} \frac{\varphi_f(n)}{n^s} = \frac{\zeta(s-1)}{\zeta(s-\alpha)} \quad (\Re(s) > \max\{2, 1 + \Re(\alpha)\}).$$

(iv) If $f = \chi$ is Dirichlet character, then

$$D(\varphi_f, s) = \sum_{n=1}^{\infty} \frac{\varphi_f(n)}{n^s} = \frac{\zeta(s-1)}{L(s, \chi)} \quad (\Re(s) > 2),$$

where $L(s, \chi)$ is the well-known Dirichlet L -function associated with χ . Note that this case connects φ_f with the Euler product for $L(s, \chi)$.

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