

The 2-Iwasawa module over certain octic elementary fields

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Abstract The aim of this paper is to construct families of number fields whose 2-Iwasawa module is isomorphic to $\mathbb{Z}_2^3$, where $\mathbb{Z}_2$ denotes the ring of 2-adic integers.

1 Introduction

Let ℓ denote a prime number. Let $\mathbb{Z}_\ell$ be the additive group of ℓ -adic integers, $\Lambda = \mathbb{Z}_\ell[[T]]$ and M a finitely generated Λ -module. It is known, by the structure theorem, that there exist $r, s, t, n_i, m_i \in \mathbb{Z}$, and f_i distinguished and irreducible polynomials of $\mathbb{Z}_\ell[T]$ such that

$$M \sim \Lambda^r \oplus \left(\bigoplus_{i=1}^s \Lambda/(\ell^{n_i}) \right) \oplus \left(\bigoplus_{j=1}^t \Lambda/(f_j(T)^{m_j}) \right).$$

Denote by k_∞/k the cyclotomic $\mathbb{Z}_\ell$ -extension of a number field k . For any integer $n \geq 1$, it is known that k_∞ contains a unique cyclic subfield k_n of degree ℓ^n over k . The field k_n is called the n -th layer of the cyclotomic $\mathbb{Z}_\ell$ -extension of k . Let $A_n(k)$ denote the ℓ -class group of k_n and consider the Iwasawa module $A_\infty(k) = \varinjlim A_n(k)$ which is a finitely generated Λ -module.

The determination of the structure of $A_\infty(k)$, for a given number field k , is a difficult and yet interesting problem in Iwasawa theory. In this paper we shall deal with the problem (in the case $\ell = 2$) for some infinite families of number fields of the form $F = \mathbb{Q}(\sqrt{p}, \sqrt{q}, i)$, where p and q are two primes satisfying one of the following conditions:

$$q \equiv 7 \pmod{8}, p \equiv 5 \text{ or } 3 \pmod{8} \text{ and } \left(\frac{p}{q}\right) = 1. \quad (1.1)$$

$$q \equiv 7 \pmod{8}, p \equiv 5 \text{ or } 3 \pmod{8} \text{ and } \left(\frac{p}{q}\right) = -1. \quad (1.2)$$

Let us adopt the following notations: $h_2(d)$ the 2-class number of $\mathbb{Q}(\sqrt{d})$, $h(k)$ (with respect $h_2(k)$) the class number (resp. 2-class number) of a number field k , k^+ the maximal real subfield of k , $\text{Cl}(k)$ the class group of k , $q(k)$ the unit index of a multiquadratic number field k and $N_{k'/k}$ the norm map of an extension k'/k .

2 The 2-Iwasawa module

Let us recall some results that will be useful as follows.

Lemma 2.1 ([2]). *Let $m \geq 1$ and p a prime such that $p \equiv 3 \text{ or } 5 \pmod{8}$. Then p decomposes into the product of 2 prime ideals of $K_n = \mathbb{Q}(\zeta_{2^{n+2}})$ while it is inert in K_n^+ .*

Theorem 2.2 ([7]). *Let F and K be CM-fields and K/F a finite 2-extension. Assume that $\mu^-(F) = 0$. Then $\mu^-(K) = 0$ and*

$$\lambda^-(K) - \delta(K) = [K_\infty : F_\infty] (\lambda^-(F) - \delta(F)) + \sum (e_\beta - 1) - \sum (e_{\beta^+} - 1),$$

where $\delta(k)$ takes the values 1 or 0 in accordance with whether F_∞ contains the fourth roots of unity or not. The e_β is the ramification index of a prime β in K_∞/F_∞ coprime to 2 and e_{β^+} is the ramification index for a prime and coprime to 2 in K_∞^+/F_∞^+ .

Let $\text{rank}_\ell(G)$ denote the ℓ -rank of an abelian group G and $\mu(k)$, $\lambda(k)$ denote the Iwasawa invariants of k . We have:

Theorem 2.3. *Let K_∞/k be a $\mathbb{Z}_\ell$ -extension of a number field k and assume that any prime which is ramified in K_∞/k is totally ramified. If $\mu(k) = 0$ and $\varprojlim A_n$ is an elementary Λ -module, then $\text{rank}_\ell(A_n) = \lambda(k)$ for all $n \geq \lambda(k)$.*

Proof. We have $\text{rank}_\ell(A_n) = \lambda(k)$, for n large enough and we have $\text{rank}_\ell(A_n) \leq \lambda(k)$, for all n . Assume that $\text{rank}_\ell(A_{m_0}) \leq \lambda(k) - 1$ for some $m_0 \geq \lambda(k)$. This implies that necessarily there exists r such that $r \leq \lambda(k)$ and $\text{rank}_\ell(A_r) = \text{rank}_\ell(A_{r+1})$ and therefore by [6, Theorem 1], $\text{rank}_\ell(A_n) = \text{rank}_\ell(A_{k+1}) \leq \lambda(k) - 1$, for all $n \geq r$. Which is absurd. $\square$

Lemma 2.4. *Let p and q be two primes satisfying conditions (1.1) or (1.2). Set $\pi_1 = 2$, $\pi_2 = 2 + \sqrt{2}, \dots$, $\pi_n = 2 + \sqrt{\pi_{n-1}}$. Then, the 2-class group of $\mathbb{Q}(\sqrt{p}, \sqrt{q}, \sqrt{\pi_n})$ is trivial for all $n \geq 1$.*

Proof. Assume that $p \equiv 5 \pmod{8}$. Put $k = \mathbb{Q}(\sqrt{p}, \sqrt{q})$. If $\left(\frac{p}{q}\right) = -1$, then by [4, Lemma 3.2] we easily deduce that $q(k) = 2$ (Similarly we show that we have the same equality for other cases). Note that by class number formula (cf. [9]) and the fact that we have $h_2(p) = h_2(q) = 1$ and $h_2(pq) = 2$ (cf. [5, Corollaries 18.4, 19.7 and 19.8]). we have

$$h_2(k) = \frac{1}{4} q(k) h_2(p) h_2(q) h_2(pq) = \frac{1}{4} \cdot 2 \cdot 1 \cdot 1 \cdot 2 = 1.$$

By [4, Theorem 4.4] and [1, Theorem 3.14] the class number of $k_1 = \mathbb{Q}(\sqrt{p}, \sqrt{q}, \sqrt{2})$ is odd. Therefore the result follows by [6, Theorem 1]. We prove similarly the result for the case $p \equiv 3 \pmod{8}$. $\square$

Let p be a prime and K be an arbitrary CM number field containing the p -th root of unity (the 4-th roots of unity if $p = 2$). Consider the cyclotomic $\mathbb{Z}_p$ -extension of K , denoted by K_∞ . The complex conjugation of K , denoted by j , acts on the p part of the class group A_n of the intermediate fields K_n as well as on the projective limit $\varprojlim A_n$. In general one defines the minus part of the class group as $\widehat{A}_n^- = \{a \in A_n \mid ja = -a\}$ and the plus part as $\widehat{A}_n^+ = \{a \in A_n \mid ja = a\}$. For $p \neq 2$ this yields a direct decomposition of A_n . Further, it is well known that there is no capitulation on the minus part for $p \neq 2$. For $p = 2$ this is in general not true. To avoid this problem Müller (cf. [8]) defined A_n^+ as the group of strongly ambiguous classes with respect to the group K_n/K_n^+ and $A_n^- = A_n/A_n^+$. Note that $A^+ = \widehat{A}^+$ and $A^- \cong \widehat{A}^-$ for $p \neq 2$. The projective limit $A_\infty^- = \varprojlim_{n \leftarrow \infty} A_n^-$ is a finitely generated Λ -torsion module. In the following we will for every λ -module M denote its λ -invariant by $\lambda(M)$.

Note that Kida proved in his result (i.e. Theorem) for $\lambda(\widehat{A^-})$. But due to the next lemma this λ -invariant equals the λ -invariant of A^- .

Lemma 2.5 ([3], Lemma 2). *Assume that $\mu(A_\infty^-) = 0$. Then $\lambda(A_\infty^-) = \lambda(\widehat{A_\infty^-})$.*

Now we can prove the main result.

Theorem 2.6. *Let p and q be two primes satisfying conditions (1.1) or (1.2). Put $F = \mathbb{Q}(\sqrt{p}, \sqrt{q}, i)$. Let $A_n(F)$ denote the 2-class group of the n -th layer of the cyclotomic $\mathbb{Z}_2$ -extension of F . Then the Iwasawa module $A_\infty(F) = \varprojlim A_n(F)$ of F is a Λ -module without finite part. If moreover $q \equiv 7 \pmod{16}$, then*

- We have $A_\infty(F) \simeq \mathbb{Z}_2^3$.
- $\text{rank}_2(A_n(F)) = 3$, for all $n \geq 3$.

Proof. Assume that $p \equiv 5 \pmod{8}$. Let F_n denote the n -th ($n \geq 1$) layer of the cyclotomic $\mathbb{Z}_2$ -extension of F and $A_n(F)$ denote the 2-class group of F_n . Let us define A_n^+ as the group of strongly ambiguous classes with respect to the extension F_n/F_n^+ and $A_n^- = A_n/A_n^+$. Since the class number of F_n^+ is odd (Lemma 2.4), then A_n^+ is generated by the ramified primes above 2. Since F_1/F_1^+ is unramified, then F_n/F_n^+ is also unramified. Therefore A_n^+ is trivial. So $A_n = A_n^-$. By [8, Theorem 2.5] there is no finite submodule in A_∞^- . Hence $A_\infty(F) = A_\infty^-$ is a Λ -module without finite part.

- Assume now that $q \equiv 7 \pmod{16}$.

Set $K_n = \mathbb{Q}(\zeta_{2^{n+2}})$ and $K = \mathbb{Q}(\sqrt{q}, i)$. Note that p splits into 2 prime ideals in $\mathbb{Q}(\sqrt{q})$ or $\mathbb{Q}(\sqrt{2q})$. Since by Lemma 2.1, p splits into 2 primes of K_n and inert in K_n^+ , for n large enough, then p splits into 4 primes in F_n while it splits into 2 primes in F_n^+ . By [3, Theorem 4] we have $\lambda^-(K) = 1$. Since $[F_\infty : K_\infty] = [F_\infty^+ : K_\infty^+] = 2$, then by Kida's formula (Theorem 2.2) applied on F/K we have

$$\lambda^-(F) - 1 = 2(1 - 1) + 4 - 2.$$

Thus by Lemma 2.5, $\lambda(F) = \lambda^-(F) = 3$. It follows that

$$A_\infty(F) \sim \bigoplus \Lambda/(f_i(T)),$$

for some distinguished polynomials f_i such that $\sum \deg(f_i) = 3$. Since by the division algorithm we have $\Lambda/(f_i(T)) \simeq \mathbb{Z}_2^{\deg(f_i)}$, then $A_\infty(F) \simeq \mathbb{Z}_2^3$. Therefore, Theorem 2.3 completes the proof of the result for $p \equiv 5 \pmod{8}$. We proceed similarly for the other cases. $\square$

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