

Computation of the Iwasawa invariant λ of some imaginary number fields

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Abstract Let $q \equiv 3 \pmod{8}$ and $\ell \equiv 5 \pmod{8}$ be two prime numbers such that $\left(\frac{q}{\ell}\right)_4 = 1$. The purpose of this paper is to compute the Iwasawa invariant λ of the field $\mathbb{K} := \mathbb{Q}\left(\sqrt{q\varepsilon\sqrt{\ell}}, \sqrt{-1}\right)$, where ε is the fundamental unit of $\mathbb{Q}(\sqrt{\ell})$. Furthermore, we determine the structure of the Iwasawa module of $\mathbb{K}$.

1 Introduction

Let p be a prime integer and $\mathbb{Z}_p$ be the ring of p -adic integers. A $\mathbb{Z}_p$ -extension of a number field k is an extension k_∞/k with $\text{Gal}(k_\infty/k)$ is topologically isomorphic to the additive group $\mathbb{Z}_p$. Every number field k has at least one $\mathbb{Z}_p$ -extension, namely the cyclotomic $\mathbb{Z}_p$ -extension k_∞ obtained by the compositum $k_\infty = k\mathbb{Q}_\infty$, where $\mathbb{Q}_\infty$ is the cyclotomic $\mathbb{Z}_p$ -extension of the field of rational numbers $\mathbb{Q}$. For each positive integer n , denote by k_n the unique intermediate field of k_∞/k of degree p^n over k , and by A_n the p -class group of k_n . Then the Iwasawa module A_∞ is defined as the projective limit of the family $(A_n)_n$ with respect to the norm mapping. Let λ, μ and ν be the Iwasawa invariants, then the order of A_n is $p^{\lambda n + \mu p^n + \nu}$ for n sufficiently large (cf. [16, Theorem 13.13, p. 276]). The Greenberg's conjecture (cf. [6, p. 263]) states that if k is totally real then $\lambda = \mu = 0$, i.e. A_∞ is finite. This conjecture is generally still open, except for some special cases (see for example [4, 9, 13, 15]). Many works have focused on determining the structure of A_∞ , when it is infinite, we cite for example [2, 7, 8].

Let $q \equiv 3 \pmod{8}$ and $\ell \equiv 5 \pmod{8}$ be two prime numbers such that $\left(\frac{q}{\ell}\right)_4 = 1$, and put $\mathbb{K} := \mathbb{Q}\left(\sqrt{q\varepsilon\sqrt{\ell}}, \sqrt{-1}\right)$, where ε is the fundamental unit of $\mathbb{Q}(\sqrt{\ell})$. In this paper, we first compute the Iwasawa invariant λ of the field $\mathbb{K}$ by using Kida's formula. Next, we determine the structure of the Iwasawa module A_∞ of $\mathbb{K}$.

Notations

Let k be a number field and p be a prime number. We have the following notations:

- k^+ : the maximal real subfield of a CM-field k ,
- $\mathbb{Q}_n$: the maximal real subfield of $\mathbb{Q}(\zeta_{2^{n+2}})$,
- k_n : the n th layer of the $\mathbb{Z}_2$ -extension of k ,
- $k_\infty = \bigcup_{n \geq 0} k_n$,
- A_n : the 2-class group of k_n ,
- A_n^+ : the group of strongly ambiguous classes of k_n/k_n^+ ,

- $A_n^- = A_n/A_n^+$,
- $A_\infty = \varprojlim A_n$ (resp. $A_\infty^- = \varprojlim A_n^-$),
- τ : a topological generator of $\text{Gal}(k_\infty/k)$,
- $\Lambda = \mathbb{Z}_2[[T]]$ for $T = \tau - 1$,
- $\mu(M), \lambda(M)$: the Iwasawa invariants for a Λ -torsion module M ,
- $\mu(k) = \mu(A_\infty)$,
- $\lambda(k) = \lambda(A_\infty)$,
- $\lambda^-(k) = \lambda(A_\infty^-)$,
- $\mathcal{O}_k$: the ring of integers of k ,
- $h(k)$: the class number of k ,
- $N_{L/k}$: the relative norm for an extension L/k ,
- $\left(\frac{x}{\mathfrak{p}}\right)$: the quadratic residue symbol for k ,
- $\left(\frac{a}{p}\right)$: the Legendre symbol.

2 Preliminary results

Let us first collect some results that will be useful in the sequel.

Lemma 2.1. *If p is a prime number such that $p \equiv \pm 3 \pmod{8}$, then p splits into the product of 2 prime ideals of $\mathbb{Q}(\zeta_{2^{n+2}})$ while it is inert in $\mathbb{Q}_n$, for all $n \geq 1$.*

Proof. It is well known that $\mathbb{Q}_n = \mathbb{Q}(\sqrt{\pi_{n+1}})$, where $\pi_2 = 2$ and $\pi_{n+1} = 2 + \sqrt{\pi_n}$ for all $n \geq 2$. First we prove by induction that p is inert in $\mathbb{Q}_n$. For $n = 1$, p is inert in $\mathbb{Q}_1 = \mathbb{Q}(\sqrt{2})$ because $\left(\frac{2}{p}\right) = -1$. Assume that p is inert in $\mathbb{Q}_n$ and prove that it is inert in $\mathbb{Q}_{n+1} = \mathbb{Q}(\sqrt{\pi_{n+2}})$. Let $\mathfrak{p}_m$ be the unique prime ideal of $\mathbb{Q}_m$ lying above p , for $1 \leq m \leq n$, then, by successively applying the norm in $\mathbb{Q}_m/\mathbb{Q}_{m-1}$ and using the properties of quadratic residue symbol (see [12, Proposition 4.2]), we have

$$\begin{aligned} \left(\frac{\pi_{n+2}}{\mathfrak{p}_n}\right) &= \left(\frac{N_{\mathbb{Q}_n/\mathbb{Q}_{n-1}}(\pi_{n+2})}{\mathfrak{p}_{n-1}}\right) = \left(\frac{N_{\mathbb{Q}_n/\mathbb{Q}_{n-1}}(2 + \sqrt{\pi_{n+1}})}{\mathfrak{p}_{n-1}}\right) = \left(\frac{4 - \pi_{n+1}}{\mathfrak{p}_{n-1}}\right) \\ &= \left(\frac{2 - \sqrt{\pi_n}}{\mathfrak{p}_{n-1}}\right) = \dots = \left(\frac{2}{p}\right) = -1 \end{aligned}$$

We claim that p is inert in $\mathbb{Q}_{n+1}$.

Now we prove p splits into the product of 2 prime ideals of $\mathbb{Q}(\zeta_{2^{n+2}})$. Note that $\mathbb{Q}(\zeta_{2^{n+2}}) = \mathbb{Q}_n(\sqrt{-1})$ which coincides with $\mathbb{Q}_n(\sqrt{-2})$ and taking into account that p is inert in $\mathbb{Q}_n$.

- If $p \equiv 3 \pmod{8}$, then p splits in $\mathbb{Q}(\sqrt{-2})$ because $\left(\frac{-2}{p}\right) = 1$, so it splits necessarily into the product of 2 prime ideals of $\mathbb{Q}(\zeta_{2^{n+2}})$.
- If $p \equiv -3 \pmod{8}$, then p splits in $\mathbb{Q}(\sqrt{-1})$ because $\left(\frac{-1}{p}\right) = 1$, so it splits necessarily into the product of 2 prime ideals of $\mathbb{Q}(\zeta_{2^{n+2}})$.

□

Lemma 2.2 ([3], Theorem 20.3). *Let $\ell \equiv 5 \pmod{8}$ be a prime number, then the class number of $\mathbb{F} = \mathbb{Q}(\sqrt{\ell}, \sqrt{-1})$ is odd.*

Theorem 2.3 ([14], Theorem 2.5, p. 374). *Let k be a CM-field containing the fourth roots of unity. Then there is no finite Λ -submodule in A_{∞}^- .*

Theorem 2.4 ([10], Theorem 3, p. 341). *Let L/F a finite 2-extension of abelian CM-fields. Then we have*

$$\lambda^-(L) - \delta(L) = [L_{\infty} : F_{\infty}] \cdot (\lambda^-(F) - \delta(F)) + \sum_{\beta \nmid 2} (e_{\beta} - 1) - \sum_{\beta^+ \nmid 2} (e_{\beta^+} - 1) \quad (2.1)$$

where $\delta(k)$ takes the values 1 or 0 according to whether k_{∞} contains the fourth roots of unity or not, and e_{β} (resp. e_{β^+}) is the ramification index in L_{∞}/F_{∞} (resp. $L_{\infty}^+/F_{\infty}^+$) of a finite prime β of L_{∞} (resp. β^+ of L_{∞}^+).

Theorem 2.5 ([2], Theorem 3.3, p. 8). *Let k_{∞} be a $\mathbb{Z}_2$ -extension of a number field k and assume that any prime of k lying above 2 is totally ramified in k_{∞}/k . If $\mu(k) = 0$ and A_{∞} is an elementary Λ -module, then $\text{rank}_2(A_n) = \lambda(k)$ for all $n \geq \lambda(k)$.*

Proposition 2.6 ([16], Proposition 13.22, p. 284). *Let k_{∞} be a $\mathbb{Z}_2$ -extension of a number field k and assume that there exists only one prime of k lying above 2 and that this prime is totally ramified in k_{∞}/k . Then*

$$2 \nmid h(k) \iff 2 \nmid h(k_n), \text{ for all } n \geq 0.$$

3 Main theorem

Let $q \equiv 3 \pmod{8}$ and $\ell \equiv 5 \pmod{8}$ be two prime numbers such that $\left(\frac{q}{\ell}\right)_4 = 1$, and put $\mathbb{K} := \mathbb{Q}\left(\sqrt{q\varepsilon\sqrt{\ell}}, \sqrt{-1}\right)$, where ε is the fundamental unit of $\mathbb{Q}(\sqrt{\ell})$. The main theorem of this paper is stated as follows:

Theorem 3.1. *Let q and ℓ be as above, and let A_n denote the 2-class group of the n th layer of the cyclotomic $\mathbb{Z}_2$ -extension of $\mathbb{K}$. Then*

(i) *The structure of the Iwasawa module A_{∞} is given by:*

$$A_{\infty} \simeq \mathbb{Z}_2^2$$

(ii) *The 2-rank of A_n is given by:*

$$\text{rank}_2(A_n) = 2, \quad \text{for all } n \geq 0.$$

Proof. We begin by computing the value of $\lambda^-(\mathbb{K})$ using Kida's formula (2.1). For this, consider the figure below:

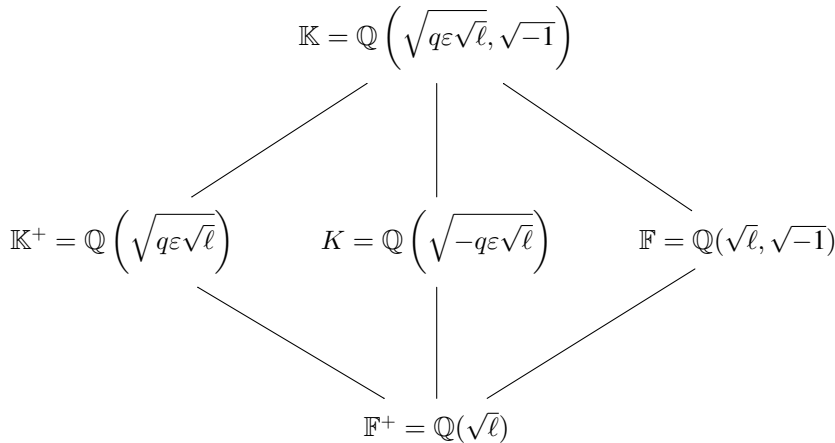


Figure 1. Subextensions of $\mathbb{K}/\mathbb{F}^+$

By Lemma 2.2, the class number of $\mathbb{F}$ is odd. Moreover, there is only one prime of $\mathbb{F}$ lying above 2, because 2 stay inert in $\mathbb{F}^+$ and it ramifies in $\mathbb{Q}(\sqrt{-1})$, and this prime is totally ramified in $\mathbb{F}_\infty/\mathbb{F}$. Thus the class numbers of the layers of the cyclotomic $\mathbb{Z}_2$ -extension of $\mathbb{F}$ are odd by Proposition 2.6, which is equivalent to $\lambda(\mathbb{F}) = 0$. Therefore $\lambda^-(\mathbb{F}) = \lambda(\mathbb{F}) - \lambda^+(\mathbb{F}) = 0$, because $\lambda^+(\mathbb{F}) = \lambda(\mathbb{F}^+) = 0$ (one can prove as above, by using Proposition 2.6, that the class numbers of the layers of the cyclotomic $\mathbb{Z}_2$ -extension of $\mathbb{F}^+$ are odd, this is equivalent to $\lambda(\mathbb{F}^+) = 0$). On the other hand, we have q splits into two prime ideals of $\mathbb{F}^+$ because $\left(\frac{\ell}{q}\right) = 1$ and ℓ ramifies in $\mathbb{F}^+$.

As $q \equiv 3 \pmod{8}$ and $\ell \equiv 5 \pmod{8}$, by Lemma 2.1, we have q (resp. ℓ) splits into 2 primes of $\mathbb{Q}(\zeta_{2^{n+2}})$ and it is inert in $\mathbb{Q}_n$, with $n \geq 1$, then q (resp. ℓ) splits into the product of 4 (resp. 2) primes in $\mathbb{F}_n = \mathbb{F}^+(\zeta_{2^{n+2}})$ while it splits into 2 primes (resp. one prime) in $\mathbb{F}_n^+ = \mathbb{Q}_n(\sqrt{\ell})$. Hence there exist 6 prime ideals ramified in $\mathbb{K}_\infty/\mathbb{F}_\infty$ and 3 prime ideals ramified in $\mathbb{K}_\infty^+/\mathbb{F}_\infty^+$ (which are above q and ℓ).

Note that $[\mathbb{K}_\infty : \mathbb{F}_\infty] = [\mathbb{K}_\infty^+ : \mathbb{F}_\infty^+] = 2$ and $e_\beta = e_{\beta^+} = 2$, then by Theorem 2.4, we have:

$$\lambda^-(\mathbb{K}) - 1 = 2 \cdot (0 - 1) + 6 - 3 \iff \lambda^-(\mathbb{K}) = 2$$

By definition, we recall that $\lambda^+(\mathbb{K}) = \lambda(\mathbb{K}^+)$, then from [8, Proposition 5.2] we deduce that $\lambda(\mathbb{K}^+) = 0$. Therefore,

$$\lambda(\mathbb{K}) = \lambda^+(\mathbb{K}) + \lambda^-(\mathbb{K}) = 2$$

Now we determine the structure of the module A_∞ . By the definition of the part plus A_n^+ , it is clear that A_n^+ is generated by the classes of the ramified primes and the inert primes in $\mathbb{K}_n/\mathbb{K}_n^+$; since the extension $\mathbb{K}_n/\mathbb{K}_n^+$ is ramified and $h(\mathbb{K}_n^+)$ is odd, then A_n^+ is generated by the classes of the ramified primes which are above 2. Let $\mathcal{P}_{\mathbb{F}_n}$ (resp. $\mathcal{P}_{\mathbb{K}_n}$) be the unique prime ideal of $\mathbb{F}_n$ (resp. $\mathbb{K}_n$) lying above 2, then we have $2\mathcal{O}_{\mathbb{F}_n} = \mathcal{P}_{\mathbb{F}_n}^{2^{n+1}}$ and $2\mathcal{O}_{\mathbb{K}_n} = \mathcal{P}_{\mathbb{K}_n}^{2^{n+1}}$; as $h(\mathbb{F}_n)$ is odd and $\mathcal{P}_{\mathbb{F}_n}^{2^{n+1}} \sim 1$ in $\mathbb{F}_n$ then $\mathcal{P}_{\mathbb{F}_n} \sim 1$, so $\mathcal{P}_{\mathbb{K}_n} \sim 1$ in $\mathbb{K}_n$ because $\mathcal{P}_{\mathbb{F}_n}\mathcal{O}_{\mathbb{K}_n} = \mathcal{P}_{\mathbb{K}_n}$. Therefore $A_n^+ = \langle [\mathcal{P}_{\mathbb{K}_n}] \rangle = \{[1]\}$ (i.e. it is trivial). Thus $A_n^- = A_n$. In the projective limit we obtain $A_\infty^- = A_\infty$. On the other hand, by Theorem 2.3 there is no finite Λ -submodule in A_∞^- , hence A_∞ is a Λ -module without finite part. So,

$$A_\infty \simeq \mathbb{Z}_2^2$$

Finally, we have

$$A_\infty \simeq \mathbb{Z}_2^{\lambda(\mathbb{K})} \simeq \bigoplus_j \Lambda/(f_j(T))$$

where each f_j is distinguished and $\sum_j \deg f_j = \lambda(\mathbb{K})$; and we have $\mathbb{K}/\mathbb{Q}$ is an abelian extension, this implies that $\mu(\mathbb{K}) = 0$ by [5, Theorem, p. 377]. Then, by Theorem 2.5,

$$\text{rank}_2(A_n) = 2, \quad \text{for all } n \geq 2.$$

Moreover, we have A_0 is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ (see [1, Theorem 2.4]), which means that $\text{rank}_2(A_0) = 2$ and $|A_0| = 4$. Then $\text{rank}_2(A_1) = 2$. This completes the proof of our theorem. $\square$

Example 3.2. In Table 1, we give some numerical examples that illustrate Theorem 3.1 by using PARI/GP.

Table 1. Examples for Theorem 3.1

q	ℓ	$A(\mathbb{K})$	$A(\mathbb{K}_1)$	$A(\mathbb{K}_2)$
11	5	[2, 2]	[8, 4]	[16, 8]
131	5	[2, 2]	[8, 4]	[16, 8]
211	5	[2, 2]	[8, 4]	[16, 8]

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