

S - α -generated rings and modules

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Abstract Let A be a commutative ring with identity, $S \subseteq A$ a multiplicative set and α be an infinite cardinal number. The ring A is called S - α -generated if for each ideal I of A , $sI \subseteq \langle a_\lambda, \lambda \in \Lambda \rangle \subseteq I$ for some $s \in S$ and some subset $\{a_\lambda, \lambda \in \Lambda\}$ of I with cardinality less than α . The purpose of this paper is to introduce a new class of rings " S - α -generated rings" which is closely related to the concepts α generated rings and S -Noetherian rings. We prove that the concepts " S - χ_0 -generated ring" and " S -Noetherian ring" are the same in the power series ring if S has a cardinality at most χ_0 (the smallest infinite cardinal number). But that is not true for the polynomial ring. Indeed, the ring A is S - α -generated if and only if $A[X]$ is a S - α -generated ring, provided that $(\bigcap_{n=1}^{+\infty} s_0 \cdots s_n A) \bigcap S \neq \emptyset$ for each sequence $(s_n)_{n \geq 0}$ of elements of S .

Introduction

In this paper, a ring means a commutative ring with identity and all modules considered are left side and unitary modules. Let S be a multiplicative subset of a ring A . Recall from [1] that the ring A is called S -Noetherian, if for each ideal I of A , $sI \subseteq J \subseteq I$ for some $s \in S$ and some finitely generated ideal J of A .

D. D. Anderson and T. Dumitrescu in [1] have studied different properties of this notion, for instance, the transfer of this property to the polynomial ring and the power series ring. For more results the reader can refer to [1], [2], [3], [8] and [11]. In section 1, we start by improving some well known results which are useful for this work. At the end of this section, we study the S -Noetherianity of rings of the form $A + XI[X]$, where $A \subseteq B$ is a ring extension and I is an ideal of B . In fact, we show that if S is a finite set then the ring $A + XI[X]$ is S -Noetherian if and only if the ring A is S -Noetherian, there exists $s \in S$ such that $sI \subseteq I^2$ and the A -module I is S -finite.

Let A be a ring and α an infinite cardinal number. An ideal I of A is called α -generated if it can be generated by a set of cardinality at most α . We say that A is a α -generated ring if each ideal of A is α -generated. In [10] B. G. Kang and P. T. Toan showed that A is a α -generated ring if and only if the polynomial ring $A[X]$ is α -generated and the equivalence " $A[[X]]$ is a α -generated ring if and only if A is Noetherian". It is clear that a Noetherian ring is both S -Noetherian ring and α -generated ring for every multiplicative subset S of A and every infinite cardinal number α . But the two concepts " S -Noetherian" rings and " α -generated " rings are not related to each other.

Indeed, let $A = \prod_{n=1}^{+\infty} (\mathbb{Z}/n\mathbb{Z})$ and $S = \{(\bar{1}, \bar{1}, \bar{1}, \dots), (\bar{1}, \bar{0}, \bar{0}, \bar{0}, \dots)\}$. It is easy to check that A is S -Noetherian and S is an anti-Archimedean multiplicative subset of A . By Theorem 1.1 of this paper, the ring $A[[X]]$ is S -Noetherian. On the other hand, $A[[X]]$ is not χ_0 -generated since A is not Noetherian by [10, Theorem 13]. It is easy to see that a non-Noetherian α -generated ring A is not S -Noetherian with $S = U(A)$, but we give a non-trivial example. Let $A = \mathbb{Z} + X\mathbb{Q}[X]$ and $S = \mathbb{Z} \setminus \{0\}$. Since $\mathbb{Q}$ is not an S -finite $\mathbb{Z}$ -module, then A is not S -Noetherian. But A is χ_0 -generated because its cardinality is χ_0 . The purpose of this paper is to generalize the concept α -generated rings to a new concept closely related to the two concepts above.

Let A be a ring, $S \subseteq A$ a multiplicative set and α an infinite cardinal number. An ideal I of A is called S - α -generated if there exists $s \in S$ and $\{a_\lambda, \lambda \in \Lambda\} \subseteq I$ with cardinality at most

α such that, $sI \subseteq \langle a_\lambda, \lambda \in \Lambda \rangle$. We say that A is a $S - \alpha$ -generated ring if each ideal of A is $S - \alpha$ -generated.

The section 2 is concerned by this generalization and its consequences. It is clear that a α -generated ring is an $S - \alpha$ -generated ring for every multiplicative subset S of A . We show that the converse does not hold. Indeed, if a ring A is not Noetherian but S -Noetherian for a given multiplicative set S of A , then the power series ring $A[[X]]$ is $S - \chi_0$ -generated which is not χ_0 -generated. We prove that a ring A is $S - \alpha$ -generated if and only if the polynomial ring $A[X]$ is $S - \alpha$ -generated, but this is not true for the case of power series rings. Also, we give a sufficient condition for a sequence $\mathcal{A} = (A_n)_{n \geq 0}$ of rings in order that the ring $\mathcal{A}[X]$ is $S - \alpha$ -generated for a given infinite cardinal number α .

The Cohen's theorem type is a natural question in each concept of ideal theoretic. In fact, it is an open question in the α -generated concept. Our last goal in this paper is to give a partial answer of this question. For instance, we give an answer in the case of $S - \chi_0$ -generated rings (where χ_0 is the smallest infinite cardinal number). This result can be seen in the case of χ_0 -generated rings.

1 S -Noetherian rings of the form $A + XJ[X]$

Recall that a ring A is called S -Noetherian where $S \subseteq A$ is a given multiplicative set, if each ideal I of A is S -finite (i.e, $sI \subseteq J \subseteq I$ for some $s \in S$ and a finitely generated ideal J of A). The following theorem is known in [1]: Let S be an anti-Archimedean multiplicative subset of a ring A consisting of nonzero-divisors. If A is S -Noetherian, so is $A[[X]]$. In the introduction we say that the ring $\left(\prod_{n=1}^{+\infty} (\mathbb{Z}/n\mathbb{Z}) \right) [[X]]$ is S -Noetherian where $S = \{(\bar{1}, \bar{1}, \bar{1}, \dots), (\bar{1}, \bar{0}, \bar{0}, \bar{0}, \dots)\}$.

The problem is that in this case S does not satisfy the hypothesis of [1. Proposition 10] because it contains a zero divisor. For this reason, we give the following result.

Theorem 1.1. *Let A be a ring and S an anti-Archimedean multiplicative subset of A . If A is S -Noetherian, so is $A[[X]]$.*

Proof. By [1, Corollary 5], it suffices to show that every prime ideal of $A[[X]]$ is S -finite.

Let P be a prime ideal of $A[[X]]$ and $P' = \{f(0), f \in P\}$. Since P' is an ideal of A which is S -Noetherian, there exist $f_1, \dots, f_n \in P$ and $s \in S$ such that $sP' \subseteq \langle f_1(0), \dots, f_n(0) \rangle A$. If $X \in P$ the result has been proved by D. D. Anderson and T. Demitrescu in [1]. If $X \notin P$. Since S is

anti-Archimedean, there exists $t \in \bigcap_{k=1}^{+\infty} s^k A \cap S$. Let $f \in P$. Then $sf(0) = \sum_{i=1}^n b_{i,0} f_i(0)$ with $b_{i,0} \in A$, $1 \leq i \leq n$. Thus $sf - \sum_{i=1}^n b_{i,0} f_i = Xg_1 \in P$ (prime ideal). Since $X \notin P$, we have

$g_1 \in P$. Doing the same process with $g_1 \in P$ instead of f , we obtain $sg_1 - \sum_{i=1}^n b_{i,1} f_i = Xg_2$ with $b_{1,1}, \dots, b_{n,1} \in A$ and $g_2 \in P$. Thus

$$s^2 f - \sum_{i=1}^n (sb_{i,0} + b_{i,1}X) f_i = X^2 g_2.$$

By induction, for each $k \in \mathbb{N}^*$,

$$s^{k+1} f - \sum_{i=1}^n (s^k b_{i,0} + s^{k-1} b_{i,1}X + \dots + b_{i,k}X^k) f_i = X^{k+1} g_{k+1}.$$

with $g_{k+1} \in P$. For every $j \in \mathbb{N}^*$, there exists $c_j \in A$ such that, $t = s^j c_j$. Let

$$h_k = \sum_{i=1}^n (s^k b_{i,0} + s^{k-1} b_{i,1}X + \dots + b_{i,k}X^k) f_i.$$

We have $t^2f - tc_{k+1}h_k = tc_{k+1}(s^{k+1}f - \sum_{i=1}^n(s^k b_{i,0} + s^{k-1}b_{i,1}X + \cdots + b_{i,k}X^k)f_i) = tc_{k+1}X^{k+1}g_{k+1} \in X^{k+1}A[[X]]$. Thus t^2f is the limit of the sequence $(tc_{k+1}h_k)_{k \geq 1}$ for the X -adic topology. Since

$$\begin{aligned} tc_{k+1}h_k &= tc_{k+1} \sum_{i=1}^n (s^k b_{i,0} + s^{k-1}b_{i,1}X + \cdots + b_{i,k}X^k)f_i = \\ &\sum_{i=1}^n (tc_{k+1}s^k b_{i,0} + tc_{k+1}s^{k-1}b_{i,1}X + \cdots + tc_{k+1}b_{i,k}X^k)f_i = \\ &\sum_{i=1}^n (c_1 s c_{k+1} s^k b_{i,0} + c_2 s^2 c_{k+1} s^{k-1}b_{i,1}X + \cdots + c_{k+1} t b_{i,k} X^k)f_i = \\ &\sum_{i=1}^n (c_1 t b_{i,0} + c_2 t b_{i,1}X + \cdots + c_{k+1} t b_{i,k} X^k)f_i = \\ &t \sum_{i=1}^n (c_1 b_{i,0} + c_2 b_{i,1}X + \cdots + c_{k+1} b_{i,k} X^k)f_i, \end{aligned}$$

then $t^2f = \lim_{k \rightarrow +\infty} tc_{k+1}h_k = \sum_{i=1}^n (t \sum_{k=0}^{+\infty} c_{k+1} b_{i,k} X^k)f_i \in \langle f_1, \dots, f_n \rangle A[[X]]$. Thus $t^2P \subseteq \langle f_1, \dots, f_n \rangle A[[X]] \subseteq P$. Hence P is S -finite. $\square$

By induction we have immediately the following corollary.

Corollary 1.2. *Let A be a ring and S an anti-Archimedean multiplicative subset of A . If A is S -Noetherian, so is $A[[X_1, X_2, \dots, X_n]]$.*

Lemma 1.3. *Let A be a ring, T a multiplicative subset of A consisting of nonzero-divisors and S an anti-Archimedean multiplicative subset of A . If A is S -Noetherian, so is A_T .*

Proof. T does not contain zero-divisor, then $A \subseteq A_T$. Thus a prime ideal of A_T is of the form P_T with P a prime ideal of A disjointed of T . Since A is S -Noetherian, the ideal P of A is S -finite. There exist $s \in S$ and a finitely generated ideal I of A such that, $sP \subseteq I \subseteq P$. Thus $sP_T \subseteq I_T \subseteq P_T$. Hence A_T is S -Noetherian. $\square$

Proposition 1.4. *Let A be a ring and S a anti-Archimedean multiplicative subset of A . If A is S -Noetherian, so is $A[[X_1, \dots, X_n]][X_1^{-1}, \dots, X_n^{-1}]$.*

Proof. By Corollary 1.2, $A[[X_1, \dots, X_n]]$ is S -Noetherian. We have $A[[X_1, \dots, X_n]][X_1^{-1}, \dots, X_n^{-1}] = A[[X_1, \dots, X_n]]_T$ where $T = \{X_1^{i_1} \cdots X_n^{i_n}, i_j \in \mathbb{N}\}$ is a multiplicative subset of $A[[X_1, \dots, X_n]]$ consists of nonzero-divisors. By Lemma 1.3, $A[[X_1, \dots, X_n]]_T$ is S -Noetherian. $\square$

A. Faisol, B. Surodjo and S. Wahyuni in [6, Theorem 5] have shown that if A is a S -Noetherian ring where S is a given anti-Archimedean multiplicative subset of A consisting of nonzero-divisors, then for each finitely generated A -module M , the $A[[X]]$ -module $M[[X]]$ is S -Noetherian. We improve this result as follows.

Proposition 1.5. *Let A be a ring, $S \subseteq A$ an anti-Archimedean multiplicative set and M a A -module. If A is S -Noetherian and M is a S -finite A -module, then the $A[[X]]$ -module $M[[X]]$ is S -Noetherian.*

Proof. By Theorem 1.1, the ring $A[[X]]$ is S -Noetherian. The A -module M is S -finite, then there exist $s \in S$ and an A -submodule N of M such that,

$$sM \subseteq N.$$

Consequently,

$$sM[[X]] \subseteq N[[X]] \subseteq M[[X]].$$

Thus $M[[X]]$ is a S -finite $A[[X]]$ -module. Hence $M[[X]]$ is a S -Noetherian $A[[X]]$ -module. $\square$

Example 1.6. Let A be a ring, $S \subseteq A$ an anti-Archimedean multiplicative subset additively closed and M a A -module. If A is S -Noetherian and M is an A -module S -finite, then the $A[[X]]$ -module $M[[X]]$ is $S[[X]]$ -Noetherian. Indeed, since S is additively closed, By [5, Lemma 7], $S[[X]]$ is a multiplicative subset of $A[[X]]$. By Proposition 1.5, the $A[[X]]$ -module $M[[X]]$ is S -Noetherian. Since $S \subseteq S[[X]]$, the $A[[X]]$ -module $M[[X]]$ is $S[[X]]$ -Noetherian.

Our next goal is to study the S -Noetherian property of rings of the form $A + XJ[X]$.

Proposition 1.7. Let $A \subseteq B$ be a ring extension, $S \subseteq A$ an anti-Archimedean multiplicative subset, J an ideal of B and $X = \{X_1, \dots, X_n\}$ a finite number of indeterminates over B . If the ring $A + XJ[X]$ is S -Noetherian, then the ring A is S -Noetherian. There exists $s \in S$ such that $sJ \subseteq J^2$ and the A -module J is S -finite.

Proof. (*) Let I be an ideal of A . Then the ideal $I(A + XJ[X])$ of $A + XJ[X]$ is S -finite. Hence there exist $s \in S$ and $f_1, \dots, f_k \in I(A + XJ[X])$ such that $sI(A + XJ[X]) \subseteq \langle f_1, \dots, f_k \rangle(A + XJ[X])$. Thus $sI \subseteq \langle f_1(0), \dots, f_k(0) \rangle A$ with $f_1(0), \dots, f_k(0) \in I$. It yields that I is S -finite. (***) The ideal I generated by $\{JX_1, JX_1^2, \dots\}$ in $A + XJ[X]$ is S -finite. Then there exist $s \in S$, $m \in \mathbb{N}^*$ and $a_{i,j} \in J$, $1 \leq i \leq m$, $1 \leq j \leq k_i$ such that

$$sI \subseteq \langle a_{i,j}X_1^i, 1 \leq i \leq m, 1 \leq j \leq k_i \rangle(A + XJ[X]).$$

Let $a \in J$. Write $saX_1^{m+1} = \sum_{i=1}^m \sum_{j=1}^{k_i} a_{i,j}X_1^i f_{i,j}$ with $f_{i,j} \in A + XJ[X]$. We pose $f_{i,j} = b_{i,j} + g_{i,j}$ with $b_{i,j} \in A$ and $g_{i,j} \in XJ[X]$, $1 \leq i \leq m$, $1 \leq j \leq k_i$. Hence

$$saX_1^{m+1} = \sum_{i=1}^m \sum_{j=1}^{k_i} b_{i,j}a_{i,j}X_1^i + \sum_{i=1}^m \sum_{j=1}^{k_i} a_{i,j}X_1^i g_{i,j}.$$

By identification of the coefficient of X_1^{m+1} , we obtain $sa = \sum_i \sum_j a_{i,j}c_{i,j}$ where $c_{i,j} \in J$ for each i, j . Thus $sa \in J^2$. Consequently, $sJ \subseteq J^2$.

(***) The ideal I of $A + XJ[X]$ generated by $\{bX_1, b \in J\}$ is S -finite, then there exists $s \in S$ and $b_1, \dots, b_r \in J$ such that

$$sI \subseteq \langle b_1, \dots, b_r \rangle(A + XJ[X]).$$

Let $b \in J$. There exist $g_1, \dots, g_r \in A + XJ[X]$ such that

$$sbX_1 = \sum_{i=1}^r g_i b_i X_1.$$

Then $(sb - \sum_{i=1}^r g_i b_i)X_1 = 0$. Since X_1 is a regular element, $sb = \sum_{i=1}^r g_i b_i$. Hence, $sb = \sum_{i=1}^r g_i(0)b_i$ with $g_i(0) \in A$. Thus $sJ \subseteq \langle b_1, \dots, b_r \rangle A$. $\square$

Lemma 1.8. Let A be a ring, S a multiplicative subset of A and I an S -finite ideal of A . If there exists $s \in S$ such that, $sI \subseteq I^2$, then there exist $b \in I$ and $t \in S$ such that, $tI \subseteq bA$ and $tb = b^2$.

Proof. Write $wI \subseteq \langle a_1, \dots, a_n \rangle$ with $w \in S$ and $a_1, \dots, a_n \in I$. We have

$$wsI \subseteq wI^2 \subseteq I\langle a_1, \dots, a_n \rangle = Ia_1 + \dots + Ia_n.$$

Then for each $1 \leq k \leq n$, $wsa_k = \sum_{i=1}^n \alpha_{k,i}a_i$ with $\alpha_{k,i} \in I$. Hence for each $1 \leq k \leq n$,

$\sum_{i=1}^n \delta_{k,i}wsa_i = \sum_{i=1}^n \alpha_{k,i}a_i$ where $\delta_{k,i}$ is Kronecker's symbole. Thus $\sum_{i=1}^n (\delta_{k,i}ws - \alpha_{k,i})a_i = 0$, $1 \leq k \leq n$. Let d be the determinant of the matrix $(\delta_{k,i}ws - \alpha_{k,i})_{1 \leq i,k \leq n}$. Then $d = (ws)^n - b$ with $b \in I$. By Cramer's law, $da_i = 0$ for each $1 \leq i \leq n$. Then $(ws)^n a_i = ba_i$ for each $1 \leq i \leq n$. Thus $(ws)^n I \subseteq bA \subseteq I$ with $(ws)^n \in S$. Since $dI = \{0\}$, $db = 0$ consequently $(ws)^n b = b^2$. $\square$

Proposition 1.9. *Let $A \subseteq B$ be a rings extension, $S \subseteq A$ an anti-Archimedean multiplicative subset, J an ideal of B and $X = \{X_1, \dots, X_n\}$ a finite number of indeterminates over B . If the ring A is S -Noetherian, there exists $s \in S$ such that $sJ \subseteq J^2$ and the A -module J is S -finite, then the ring $(A + XJ[X])_S$ is Noetherian.*

Proof. By hypothesis, the A -module J is S -finite. Then J is an S -finite ideal of B . Since there exists $s \in S$ such that $sJ \subseteq J^2$, by Lemma 1.8, there exist $t \in S$ and $x \in J$ such that $tJ \subseteq xB$ and $tx = x^2$. Let $b_1, \dots, b_r \in J$ and $w \in S$ satisfying $sJ \subseteq \langle b_1, \dots, b_r \rangle A$. Add a family of indeterminates $Y = \{Y_{i,j}, 1 \leq i \leq n, 1 \leq j \leq r\}$ over B . We define the A_S -homomorphism of rings $\phi : A[X, Y]_S = A_S[X, Y] \rightarrow B[X]_S = B_S[X]$ by $\phi(X_i) = xX_i$ and $\phi(Y_{i,j}) = b_jX_i$. It is clear that $\phi(A[X, Y]_S) \subseteq (A + XJ[X])_S$.

Conversely, let $f \in (A + XJ[X])_S$, $f = \frac{1}{d}(a + \sum_{i=1}^n (\sum_{\alpha} b_{\alpha,i} X^\alpha) X_i)$

with $d \in S$, $a \in A$, $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, $b_{\alpha,i} \in J$ and $X^\alpha = X_1^{\alpha_1} \dots X_n^{\alpha_n}$.

Let $f_1 = a + \sum_{i=1}^n (\sum_{\alpha} b_{\alpha,i} X^\alpha) X_i$. Write $sb_{\alpha,i} = \sum_{j=1}^r a_{\alpha,i,j} b_j$ with $a_{\alpha,i,j} \in A$. Then

$$sf_1 = sa + \sum_{i=1}^n (\sum_{\alpha} (\sum_{j=1}^r a_{\alpha,i,j} b_j) X^\alpha) X_i = sa + \sum_{i=1}^n \sum_{j=1}^r (\sum_{\alpha} a_{\alpha,i,j} X^\alpha) b_j X_i.$$

Since for each $1 \leq j \leq r$, $tb_j \in tJ \subseteq xB$ and $tx = x^2$, $tb_j = txb_j$, $1 \leq j \leq r$. Then

$$tsf_1 = tsa + \sum_{i=1}^n \sum_{j=1}^r (\sum_{\alpha} a_{\alpha,i,j} x X^\alpha) b_j X_i = tsa + \sum_{i=1}^n \sum_{j=1}^r (\sum_{\alpha} a_{\alpha,i,j} (xX)^\alpha) b_j X_i = \phi(g).$$

with $(xX)^\alpha = (xX)^{\alpha_1} \dots (xX)^{\alpha_n}$ and $g = tsa + \sum_{i=1}^n \sum_{j=1}^r (\sum_{\alpha} a_{\alpha,i,j} (X)^\alpha) Y_{i,j} \in A[X, Y]$. Thus

$tsf = \frac{ts}{d} f_1 = \frac{1}{d} \phi(g)$. Since $\frac{1}{tsd} g \in (A + XJ[X])_S$, $f = \phi(\frac{1}{tsd} g)$. Hence ϕ is surjective. It yields that $(A + XJ[X])_S \simeq A[X, Y]_S / \ker(\phi)$. The ring A is S -Noetherian and S is anti-Archimedean, then $A[X, Y]$ is S -Noetherian. By Lemma 1.3, $(A[X, Y])_S$ is Noetherian. Hence $(A + XJ[X])_S \simeq (A[X, Y])_S / \ker(\phi)$ is Noetherian. $\square$

Let A be a ring and S be a multiplicative subset of A . Recall that for an ideal I of A the set $S(I) = IA_S \cap A$ is called the saturation of I .

Lemma 1.10. *Let A be a ring and S be a finite multiplicative subset of A . Then the ring A is S -Noetherian if and only if the ring A_S is Noetherian.*

Proof. Let $S = \{s_1, \dots, s_k\}$, $s = s_1 s_2 \dots s_k$ and I an ideal of A . For $x \in I : s$, we have $sx \in I$. Then $x = \frac{sx}{s} \in S(I)$. Conversely, let $x \in S(I)$. Write $x = \frac{a}{s}$ with $a \in I$. Thus $sx = a \in I$. Hence $x \in I : s$. It follows that $S(I) = I : s$. By [1, Proposition 2], we have the result. $\square$

Corollary 1.11. *Let $A \subseteq B$ be a ring extension, J an ideal of B , S a finite multiplicative subset of A and $X = \{X_1, \dots, X_n\}$ a finite number of indeterminates over B . The following statements are equivalent :*

- (i) *The ring $A + XJ[X]$ is S -Noetherian.*
- (ii) *The ring A is S -Noetherian, and there exists $s \in S$ such that $sJ \subseteq J^2$ and the A -module J is S -finite.*

Proof. "1) $\implies$ 2)" By Proposition 1.7. "2) $\implies$ 1)" Since S is finite, one can check that S is anti-Archimedean set. By Proposition 1.17, the ring $(A + XJ[X])_S$ is Noetherian. By Lemma 1.10, the ring $A + XJ[X]$ is S -Noetherian. $\square$

2 S - α -generated rings

We start this section by the following definition.

Definition 2.1. Let A be a ring, S a multiplicative subset of A , I an ideal of A and α an infinite cardinal number.

- (i) We say that I is a S - α -generated ideal, if there exist $s \in S$ and $\{x_\lambda, \lambda \in \Lambda\} \subseteq I$ such that, $|\Lambda| \leq \alpha$ and

$$sI \subseteq \langle x_\lambda, \lambda \in \Lambda \rangle.$$

- (ii) We say that A is a S - α -generated ring, if each ideal of A is S - α -generated ideal.

- (iii) We say that A is a α -generated ring, if A is S - α -generated with $S = \{1\}$.

It is natural to ask when the ring $A[X]$ is S - α -generated? The following Proposition give an affirmative answer of this question.

Proposition 2.2. Let A be a ring, $S \subseteq A$ multiplicative subset such that, for every sequence $(s_n)_{n \geq 0}$ of elements of S , $(\bigcap_{n=0}^{+\infty} s_0 \cdots s_n A) \cap S \neq \emptyset$ and α an infinite cardinal number. Then A is S - α -generated ring, if and only if, the polynomial ring $A[X]$ is S - α -generated.

Proof. " $\implies$ " Let J be an ideal of $A[X]$. For each positive integer n , we define

$$I_n = \{a \in A \mid a = 0 \text{ or } a \text{ is the leader coefficient of a polynomial } f \in J \text{ of degree } n\}.$$

It is clear that I_n is an ideal of A , thus it is S - α -generated ideal. There exist $s_n \in S$ and a set of cardinality less than α , $\{r_{n,\lambda}, \lambda \in \Lambda\} \subseteq I_n$, such that, $s_n I_n \subseteq \langle r_{n,\lambda}, \lambda \in \Lambda \rangle$.

Let $w \in (\bigcap_{n=0}^{+\infty} s_0 \cdots s_n A) \cap S$. For $n \geq 1$, $w = s_0 \cdots s_n c_n$ with $c_n \in A$. Then $w I_n = (s_0 \cdots s_{n-1} c_n) s_n I_n \subseteq \langle r_{n,\lambda}, \lambda \in \Lambda \rangle$. It is clear that S is anti-Archimedien (it suffices to take

the constant sequence $s_n = s, n \in \mathbb{N}$ where $s \in S$), let $t \in (\bigcap_{n=1}^{+\infty} w^n A) \cap S$. For every $n \geq 1$, Let

$t = w^n a_n$ with $a_n \in A$. By the axiom of choice, we choose a set $\{f_{n,\lambda}, n \geq 0, \lambda \in \Lambda\} \subseteq J$ such that, for every $n \geq 0$ and every $\lambda \in \Lambda$, $r_{n,\lambda}$ is the leader coefficient of $f_{n,\lambda}$. Let $f = \alpha_0 + \alpha_1 X + \cdots + \alpha_k X^k \in J$. We have $w \alpha_k = \sum_{i=1}^{n_k} \beta_{k,i} r_{k,\lambda_i}$ with $\beta_{k,i} \in A$. Then $\deg(wf - \sum_{i=1}^{n_k} \beta_{k,i} f_{k,\lambda_i}) \leq$

$k - 1$. Let $g_1 = wf - \sum_{i=1}^{n_k} \beta_{k,\lambda_i} f_{k,\lambda_i} = \sum_{j=0}^d \alpha_{1,j} X^j$. If $\deg(g_1) \leq k - 2$, let $\beta_{k-1,i} = 0$ else we

have $\alpha_{1,k-1} \in I_{k-1}$. Thus $w \alpha_{1,k-1} = \sum_{j=1}^{n_{k-1}} \beta_{k-1,j} r_{k-1,\lambda_j}$. Hence, $\deg(w^2 f - w \sum_{i=1}^{n_k} \beta_{k,i} f_{k,\lambda_i} -$

$\sum_{i=1}^{n_{k-1}} \beta_{k-1,i} f_{k-1,\lambda_i}) \leq k - 2$. By induction, we find $w^k f = \sum_{j=0}^k (\sum_{i=1}^{n_{k-j}} w^{k-j-1} \beta_{k-j,i} f_{k-j,\lambda_i})$. It

yields that $t f = a_k w^k f = \sum_{j=0}^k (\sum_{i=1}^{n_{k-j}} a_k w^{k-j-1} \beta_{k-j,i} f_{k-j,\lambda_i}) \in \langle f_{n,\lambda}, n \geq 0, \lambda \in \Lambda \rangle$. There-

fore, $tJ \subseteq \langle f_{n,\lambda}, n \geq 0, \lambda \in \Lambda \rangle$ with $t \in S$ and $|\bigcup_{n=0}^{+\infty} \Lambda_n| \leq \sum_{n=0}^{+\infty} |\Lambda_n| \leq \max(\chi_0, |\Lambda_n|, n \geq 0) \leq \alpha$. Thus $A[X]$ is S - α -generated. $\square$

Our next target is to show that the equivalence in the Proposition 2.2 is not true in the case of power series rings. We start by recalling the fathomless set constructed in [8].

Let U be the collection of all subsets of $\mathbb{N}$ of the form $\{n, n+1, n+2, \dots\}$ where $n \geq 1$. Let $s = (s_n)_{n \geq 1}$ and $t = (t_n)_{n \geq 1}$ be two strictly increasing sequences of nonzero positive integers.

We denote $s \gg t$ (or $t \ll s$) if for every $k \geq 1$, there exists $O \in U$ such that, for every $n \in O$, $s_n > kt_n$.

Let Ω be the collection of all A having the following properties :

(1) A is a nonempty collection of strictly increasing sequences of nonzero positive integers.

(2) If $s \in A$, then $s \gg b$ where $b = (n)_{n \geq 1}$.

(3) If s and t are two elements of A , then $s = t$ or $s \ll t$ or $t \ll s$.

The sequence $s = (n^2)_{n \geq 1}$ is strictly increasing. Let $k \in \mathbb{N}^*$, for every $n \in \{k, k+1, k+2, \dots\} \in U$, we have $n^2 > kn$, thus Ω is not empty. We order Ω by set-theoretic inclusion. By Zorn's lemma, there exists a maximal element of Ω . Let $\mathcal{A}$ be a maximal element of Ω . This choice of $\mathcal{A}$ will be fixed till the end of the next proposition. For $s, t \in \mathcal{A}$, we define $s \leq t$, if and only if, $s = t$ or $s \ll t$. Then $(\mathcal{A}, \leq)$ becomes a totally ordered set.

Definition 2.3. A set A totally ordered is called fathomless if for every countable subset B of A , there exists $y \in A$ such that, $y \ll c$, for every $c \in B$.

Theorem 2.4. [9, Theorem 5] The set $(\mathcal{A}, \leq)$ is fathomless.

It is clear that a fathomless set is not countably.

Lemma 2.5. Let A be a ring and $S = \{s_k, k \in \mathbb{N}^*\}$ a multiplicative subset of A . If A is not S -Noetherian, then there exists a sequence $(a_n)_{n \geq 0}$ of elements of A such that, for every $k \geq 1$, $s_1 \cdots s_k a_k \notin \langle a_0, a_1, \dots, a_{k-1} \rangle$.

In particular, for each $k \geq 1$, $s_k a_k \notin \langle a_0, \dots, a_{k-1} \rangle$.

Proof. Since A is not S -Noetherian, there exists an ideal I of A which is not S -finite. Let $a_0 \in I$. Since $s_1 I \not\subseteq \langle a_0 \rangle$, there exists $a_1 \in I$ such that, $s_1 a_1 \notin \langle a_0 \rangle$. I is not S -finite, then $s_2 s_1 I \not\subseteq \langle a_0, a_1 \rangle$. Hence there exists $a_2 \in I$ such that, $s_2 s_1 a_2 \notin \langle a_0, a_1 \rangle$. By induction, there exists a sequence $(a_n)_{n \geq 0}$ of elements of $I \subseteq A$ such that, $s_1 \cdots s_k a_k \notin \langle a_0, \dots, a_{k-1} \rangle$ for each $k \geq 1$. $\square$

Remark 2.6. With the same notations above, for a fixed integer $k \geq 1$, for every $n \geq k$, $s_k a_n \notin \langle a_0, \dots, a_{n-1} \rangle$ because if not, $s_1 \cdots s_n a_n = (s_1 \cdots s_{k-1} s_{k+1} \cdots s_n) s_k a_n \in \langle a_0, \dots, a_{n-1} \rangle$ absurd.

Theorem 2.7. Let A be a ring and $S = \{s_i, i \in \mathbb{N}^*\}$ a multiplicative subset of A . If A is not S -Noetherian, then $A[[X]]$ is not S - χ_0 -generated.

Proof. By Lemma 2.5, there exists a sequence $(a_n)_{n \geq 0}$ of elements of A such that, for every $k \geq 1$, $s_1 \cdots s_k a_k \notin \langle a_0, \dots, a_{k-1} \rangle$. As in [10], for $u = (u_n)_{n \geq 1} \in \mathcal{A}$, we define $F_u = a_0 + \sum_{n=1}^{+\infty} a_n X^{u_n}$. Let $I = \langle F_u, u \in \mathcal{A} \rangle A[[X]]$. Suppose that I is S - α -generated, then there exist $i \geq 1$ and $\mathcal{B} \subseteq \mathcal{A}$ such that, $|\mathcal{B}| \leq \chi_0$ and $s_i I \subseteq \langle F_u, u \in \mathcal{B} \rangle A[[X]]$. By Theorem 2.4, the set $(\mathcal{A}, \leq)$ is fathomless. Since $|\mathcal{B}| \leq \chi_0$, there exists $v \in \mathcal{A}$ such that, $v \ll \mathcal{B}$ (i.e, $v \ll u$ for each $u \in \mathcal{B}$). By definition of I , $F_v \in I$. Hence there exist $u^0, \dots, u^n \in \mathcal{B}$ such that, $s_i F_v = \sum_{j=0}^n h_j F_{u^j}$ with $h_0, \dots, h_n \in A[[X]]$. There exists $O \in U$ such that, $v_m < u_m^j$ for every $m \in O$, $0 \leq j \leq n$. For a fixed integer $m \in O$, $0 \leq j \leq n$ and $t \leq v_m$ the coefficient of X^t in F_{u^j} belongs to $\langle a_0, \dots, a_{m-1} \rangle A$. Then the coefficient of X^{v_m} in $h_j F_{u^j}$ belongs to $\langle a_0, \dots, a_{m-1} \rangle A$. Hence the coefficient of X^{v_m} in $\sum_{j=0}^n h_j F_{u^j} = s_i F_v$ belongs to $\langle a_0, \dots, a_{m-1} \rangle A$. Thus for each $m \in O$ the coefficient of X^{v_m} in $s_i F_v$ belongs to $\langle a_0, \dots, a_{m-1} \rangle A$. In particular, for every $m \in O \cap \{i, i+1, i+2, \dots\}$, $s_i a_m \in \langle a_0, \dots, a_{m-1} \rangle A$, which contradict Remark 2.7. Hence the ideal I of $A[[X]]$ is not S - α -generated. $\square$

The following Corollary is an easy consequence of Theorem 1.1 and Theorem 2.7.

Corollary 2.8. Let A be a ring and $S = \{s_i, i \in \mathbb{N}^*\}$ an anti-Archimedean multiplicative subset of A . The following statements are equivalent :

- (i) The ring A is S -Noetherian.
- (ii) The ring $A[[X]]$ is S -Noetherian.
- (iii) The ring $A[[X]]$ is S - χ_0 -generated.

Example 2.9. An α -generated ring A is S - α -generated for every multiplicative subset S of A .

But the converse is false. Let p be a prime number and $A = \prod_{n=1}^{+\infty} \mathbb{Z}/p^n\mathbb{Z}$, the product ring. It is clear that A is not Noetherian. By Corollary 2.10, the ring $A[[X]]$ is not χ_0 -generated. Let $s = (\bar{1}, \bar{p}, \bar{p}, \bar{0}, \bar{0}, \dots) \in A$ and $S = \{1, s, s^2, s^3\}$. The set S is multiplicative subset of A and by [2, Example 2.5], A is S -Noetherian. For each $n \geq 3$, $s^n = s^3$, then $s^n A = s^3 A \subseteq s^2 A \subseteq sA$. Thus for every $0 \leq i \leq 3$, $s^3 \in \bigcap_{n=1}^{+\infty} (s^i)^n A \cap S$, hence S is anti-Archimedean. By Theorem 1.1, $A[[X]]$ is S -Noetherian and by Corollary 2.8, $A[[X]]$ is S - χ_0 -generated.

Now, we are going to extend the concept " S - α -generated" from rings to modules.

Definition 2.10. Let A be a ring, S a multiplicative subset of A , M an A -module, N an A -submodule of M and α an infinite cardinal number.

- (i) We say that N is S - α , if there exist $s \in S$ and $\{n_\lambda, \lambda \in \Lambda\} \subseteq N$ with $|\Lambda| \leq \alpha$ such that, $sN \subseteq \langle n_\lambda, \lambda \in \Lambda \rangle$.
- (ii) We say that M is S - α -generated, if every submodule of M is S - α .
- (iii) We say that M is α -generated, if it is S - α -generated with $S = \{1\}$.

Example 2.11. Every homomorphic image of a S - α -generated module is S - α -generated. In particular, let A be a ring, S a multiplicative subset of A , M a A -module, N a A -submodule of M and α an infinite cardinal number. If M is S - α -generated, so is the A -module M/N .

Lemma 2.12. Let A be a ring, S a multiplicative subset of A , M an A -module, N an A -submodule of M and α an infinite cardinal number. The following statements are equivalent :

- (i) The A -module M is S - α -generated.
- (ii) The A -modules N and M/N are S - α -generated.

Proof. "1) $\implies$ 2)" Clear.

"2) $\implies$ 1)" Let T be a submodule of M . Denote $\bar{T} = \{\bar{x}, x \in T\}$. It is clear that $\bar{T}$ is a submodule of M/N , hence it is S - α . Then there exist $s \in S$ and $\{t_\lambda, \lambda \in \Lambda\} \subseteq T$ with $|\Lambda| \leq \alpha$ such that, $s\bar{T} \subseteq \langle \bar{t}_\lambda, \lambda \in \Lambda \rangle$. On the other hand, $T \cap N$ is a submodule of N , then $T \cap N$ is S - α . Hence there exist $s' \in S$ and $\{l_\beta, \beta \in \Delta\} \subseteq T$ with $|\Delta| \leq \alpha$ such that,

$s'(T \cap N) \subseteq \langle l_\beta, \beta \in \Delta \rangle$. Let $x \in T$, $\bar{x} \in \bar{T}$. Then $s\bar{x} = \sum_{j=1}^{n_1} a_j \bar{t}_{\lambda_j}$ with $a_1, \dots, a_{n_1} \in A$. Thus

$sx - \sum_{j=1}^{n_1} a_j t_{\lambda_j} \in T \cap N$. Consequently, $s'sx - s' \sum_{j=1}^{n_1} a_j t_{\lambda_j} = \sum_{i=1}^{n_2} b_i l_{\beta_i}$ with $b_1, \dots, b_{n_2} \in A$.

Hence $s'sx = \sum_{j=1}^{n_1} s'a_j t_{\lambda_j} + \sum_{i=1}^{n_2} b_i l_{\beta_i}$. Thus $s'sT \subseteq \langle t_\lambda, l_\beta; \lambda \in \Lambda, \beta \in \Delta \rangle$ with $s's \in S$ and $|\Lambda \cup \Delta| \leq \alpha$. □

Proposition 2.13. Let A be a ring, S multiplicative subset of A , M a A -module and α an infinite cardinal number. If M is S - α -generated, so is M^n for each $n \geq 1$.

Proof. By induction, if $n = 1$, M is S - α -generated. Let $n \geq 1$, assume that M^n is S - α -generated. Let $f : M^{n+1} \rightarrow M$ the first projection. It is clear that f is a surjective homomorphism of A -modules and that $\ker(f) = \{0\} \times M^n$. Since $\ker(f) \simeq M^n$ is S - α -generated and $M^{n+1}/\ker(f) \simeq M$ is S - α -generated. By Lemma 2.12, M^{n+1} is S - α -generated. □

Corollary 2.14. *Let A be a ring, S a multiplicative subset of A , M a finitely generated A -module and α an infinite cardinal number. If A is S - α -generated, so is M .*

Proof. Let $\{x_1, \dots, x_n\}$ be generating family of M and $f : A^n \rightarrow M$ the map defined by $f(a) = a_1x_1 + \dots + a_nx_n$ for every $a = (a_1, \dots, a_n) \in A^n$. By Proposition 2.13, A^n is a S - α -generated A -module. Thus $M \simeq A^n/\ker(f)$ is an S - α -generated A -module. $\square$

Corollary 2.15. *Let A be a ring, M a finitely generated A -module and α an infinite cardinal number. If A is α -generated, then M is a α -generated A -module.*

Corollary 2.16. *Let A be a ring, S a multiplicative subset of A , M an A -module and α an infinite cardinal number. If A is S - α -generated ring and M is a S -finite A -module, then M is a S - α -generated A -module.*

Proof. There exists $s \in S$ and a finitely generated submodule N of M such that, $sM \subseteq N$. By Corollary 2.15, the A -module N is S - α -generated. Then the submodule sM of N is S - α -generated. Hence the A -module M is S - α -generated. $\square$

Proposition 2.17. *Let $A \subseteq B$ be an extension of rings, $S \subseteq A$ a multiplicative set satisfying for each sequence $(s_n)_{n \geq 0}$ of elements of S , $\bigcap_{n=0}^{+\infty} s_0 \cdots s_n A \cap S \neq \emptyset$. If the ring A is S - α -generated and B is an S -finite A -module, then the ring $A + XB[X]$ is S - α -generated.*

Proof. By Proposition 2.2, the ring $A[X]$ is S - α -generated. Since B is an S -finite A -module, the $A[X]$ -module $B[X]$ is S -finite. By Corollary ??, the $A[X]$ -module $B[X]$ is S - α -generated. Hence the $(A + XB[X])$ -module $B[X]$ is S - α -generated. By Lemma 2.12, the $(A + XB[X])$ -submodule $A + XB[X]$ of $B[X]$ is S - α -generated. Thus $A + XB[X]$ is S - α -generated ring. $\square$

Corollary 2.18. *Let $A \subseteq B$ be an extension of rings and α an infinite cardinal number. If A is an α -generated ring and B is a finitely generated A -module, then the ring $A + XB[X]$ is α -generated.*

Notation. Let $\mathcal{A} = (A_n)_{n \geq 0}$ be an increasing sequence of rings and $A = \bigcup_{n=0}^{+\infty} A_n$. Recall that

$$\mathcal{A}[X] = \{f = \sum_{i=0}^n a_i X^i \in A[X], a_i \in A_i, \text{ for each } 0 \leq i \leq n\}.$$

It is well known that $\mathcal{A}[X]$ is a subring of $A[X]$.

Proposition 2.19. *Let $\mathcal{A} = (A_n)_{n \geq 0}$ be an increasing sequence of rings, $A = \bigcup_{n=0}^{+\infty} A_n$, $S \subseteq A_0$ a multiplicative set satisfying for each sequence $(s_n)_{n \geq 0}$ of elements of S , $\bigcap_{n=0}^{+\infty} s_0 \cdots s_n A \cap S \neq \emptyset$ and α an infinite cardinal number. If the sequence $\mathcal{A}$ is S -stationary, for each $n \geq 1$, the A_0 -module A_n is S -finite and the ring A_0 is S - α -generated, then the ring $\mathcal{A}[X]$ is S - α -generated.*

Proof. There exist $s \in S$ and $N \in \mathbb{N}$ such that, for each $n \geq N$, $sA_n \subseteq A_N$. For $1 \leq i \leq N$, there exist $s_i \in S$ and $a_{i,1}, \dots, a_{i,k_i} \in A_i$ such that, $s_i A_i \subseteq \langle a_{i,j}, 1 \leq j \leq k_i \rangle A_0$. Let $t = s_1 \cdots s_N$. Hence $tA_i \subseteq \langle a_{i,j}, 1 \leq j \leq k_i \rangle A_0$ for each $1 \leq i \leq N$. Then $tsA = \bigcup_{n=0}^{+\infty} tsA_n \subseteq \bigcup_{n=0}^N tA_n \subseteq \langle a_{i,j}, 1 \leq i \leq N, 1 \leq j \leq k_i \rangle A_0$. Thus the A_0 -module A is S -finite. It follows that the $A_0[X]$ -module $A[X]$ is S -finite. By Proposition 2.2, the ring $A_0[X]$ is S - α -generated and by Corollary 2.16, the $A_0[X]$ -module $A[X]$ is S - α -generated. Then the $\mathcal{A}[X]$ -module $A[X]$ is S - α -generated. Hence the ring $\mathcal{A}[X]$ is S - α -generated. $\square$

Corollary 2.20. *Let $\mathcal{A} = (A_n)_{n \geq 0}$ an increasing sequence of rings and α an infinite cardinal number. If $\mathcal{A}$ is stationary, the ring A_0 is α -generated and for every $n \geq 1$, the A_0 -module A_n is finitely generated, then the ring $\mathcal{A}[X]$ is α -generated.*

Now our purpose is to give a partial answer of the Cohen's theorem type of the $S - \alpha$ -generated rings. Let A be a ring and S a multiplicative subset of A . We shall call that A satisfies the property $(*)$, if each increasing sequence of non- $S - \chi_0$ -generated ideals of A is S -stationary, i.e, if $(I_n)_{n \geq 1}$ is an increasing sequence of ideals of A such that for each $n \geq 1$, I_n is not $S - \chi_0$ -generated, then there exist $s \in S$ and $k \geq 1$ such that $sI_n \subseteq I_k$ for each $n \geq k$.

Lemma 2.21. *Let A be a ring, S a multiplicative subset of A and α an infinite cardinal number. A maximal element in the set of non- $S - \alpha$ -generated ideals of A is a prime ideal.*

Proof. Let P be a such element. It clear that $P \cap S = \emptyset$ (because in the other case $sP \subseteq sA \subseteq P$ for some $s \in P \cap S$). Assume that P is not a prime ideal. Then there exist $a, b \in A \setminus P$ such that $ab \in P$. By maximality of P , since $P \subset P + aA$, there exist $s \in S$ and $\{x_\lambda, \lambda \in \Lambda\} \subseteq P$ such that $s(P + aA) \subseteq \langle x_\lambda + ay_\lambda, \lambda \in \Lambda \rangle$ and $|\Lambda| \leq \alpha$ where $y_\lambda \in A$ for every $\lambda \in \Lambda$. Let $J = (P : a) = \{x \in A, xa \in P\}$. Since $b \in J$ and $Pa \subseteq P$, we have $P \subset J$. By maximality of P , there exist $t \in S$ and $\{u_\beta, \beta \in \Gamma\} \subseteq J$ such that $tJ \subseteq \langle u_\beta, \beta \in \Gamma \rangle$ and $|\Gamma| \leq \alpha$.

Let $x \in P$. We have $sx \in sP \subseteq s(P + aA)$. Then there exist $d_1, \dots, d_n \in A$ and $\lambda_1, \dots, \lambda_n \in \Lambda$ such that $sx = \sum_{i=1}^n d_i(x_{\lambda_i} + ay_{\lambda_i})$. Since $a \sum_{i=1}^n d_i y_{\lambda_i} = sx - \sum_{i=1}^n d_i x_{\lambda_i} \in P$, then $\sum_{i=1}^n d_i y_{\lambda_i} \in J$.

Hence, there exist $c_1, \dots, c_k \in A$ and $\beta_1, \dots, \beta_k \in \Gamma$ such that $t \sum_{i=1}^n d_i y_{\lambda_i} = \sum_{j=1}^k c_j u_{\beta_j}$.

Therefore,

$$(ts)x = \sum_{i=1}^n (td_i)x_{\lambda_i} + a(t \sum_{i=1}^n d_i y_{\lambda_i}) = \sum_{i=1}^n (td_i)x_{\lambda_i} + a \sum_{j=1}^k c_j u_{\beta_j} = \sum_{i=1}^n (td_i)x_{\lambda_i} + \sum_{j=1}^k c_j (au_{\beta_j}).$$

Thus $(ts)x \in \langle \{x_\lambda, \lambda \in \Lambda\} \cup \{au_\beta, \beta \in \Gamma\} \rangle \subseteq P$ with $|\{x_\lambda, \lambda \in \Lambda\} \cup \{au_\beta, \beta \in \Gamma\}| \leq |\Lambda| + |\Gamma| \leq \alpha$. Moreover, $(ts)P \subseteq \langle \{x_\lambda, \lambda \in \Lambda\} \cup \{au_\beta, \beta \in \Gamma\} \rangle \subseteq P$. Consequently, P is $S - \alpha$ -generated. Contradiction. This end the proof. $\square$

Theorem 2.22. *Let S be a multiplicative subset of a ring A such that A satisfies $(*)$. Then the ring A is $S - \chi_0$ -generated if and only if each prime ideal of A (disjoint with S) is $S - \chi_0$ -generated.*

Proof. Let $\mathcal{F}$ be the set of all ideals of A which are not $S - \chi_0$ -generated. Assume that A is not an $S - \chi_0$ -generated ring, then $\mathcal{F} \neq \emptyset$. Let $(I_\alpha)_{\alpha \in \Lambda}$ be a totally ordered family of $(\mathcal{F}, \subseteq)$ and $I = \bigcup_{\alpha \in \Lambda} I_\alpha$. Suppose that I is $S - \chi_0$ -generated. Therefore, there exist $s \in S$ and $(a_n)_{n \geq 1} \subseteq I$ such that

$$sI \subseteq \langle a_n, n \geq 1 \rangle.$$

Since $a_1 \in I = \bigcup_{\alpha \in \Lambda} I_\alpha$, there exists $\alpha_1 \in \Lambda$ such that $a_1 \in I_{\alpha_1}$. By the same reason, there

exists $\alpha_2 \in \Lambda$ such that $a_2 \in I_{\alpha_2}$. But the set $(I_\alpha)_{\alpha \in \Lambda}$ is totally ordered. Hence we can assume that $I_{\alpha_1} \subseteq I_{\alpha_2}$. By induction, let $n \geq 2$ and $I_{\alpha_1} \subseteq I_{\alpha_2} \subseteq \dots \subseteq I_{\alpha_n}$ such that $a_i \in I_{\alpha_i}$ where $\alpha_i \in \Lambda$ and $1 \leq i \leq n$. Since $a_{n+1} \in I$, there exists $\alpha_{n+1} \in \Lambda$ such that $a_{n+1} \in I_{\alpha_{n+1}}$. Since $(I_\alpha)_{\alpha \in \Lambda}$ is totally ordered, we have two cases, either $I_{\alpha_n} \subseteq I_{\alpha_{n+1}}$ or $I_{\alpha_{n+1}} \subseteq I_{\alpha_n}$. In the second one, we change $I_{\alpha_{n+1}}$ by I_{α_n} . Therefore, we have an increasing sequence $(I_{\alpha_n})_{n \geq 1}$ such that $a_1, \dots, a_n \in I_{\alpha_n}$ for every $n \geq 1$. By hypothesis, there exist $t \in S$ and $n \geq 1$ an integer such that $tI_{\alpha_n} \subseteq I_{\alpha_n}$ for all $k \geq n$. Moreover, $(ts)I_{\alpha_n} \subseteq (ts)I = t(sI) \subseteq t\langle a_i, i \geq 1 \rangle \subseteq tI \subseteq I_{\alpha_n}$. Thus I_{α_n} is $S - \chi_0$ -generated : Absurd. Then $I \in \mathcal{F}$. Consequently, $(\mathcal{F}, \subseteq)$ is inductive. By Zorn's Lemma, $\mathcal{F}$ has a maximal element, which is a prime ideal of A by Lemma 2.21, contradiction with the hypothesis. Then A is an $S - \chi_0$ -generated ring. $\square$

Example 2.23. Let A be a ring. If each increasing sequence of non- χ_0 -generated ideals is stationary, then the ring A is χ_0 generated if and only if each prime ideal of A is χ_0 -generated.

Let A be an integral domain. We recall the following Lemma [5, Lemma 5.5]. A right chain domain is right isonoetherian if and only if it satisfies the ACC for infinitely generated right ideals. It follows that if A is a valuation domain, then A is an isonoetherian domain if and only if it satisfies the ACC on infinitely generated ideals.

Example 2.24. Let V be an isonoetherian valuation domain, then V is a χ_0 -generated ring if and only if each prime ideal of V is χ_0 -generated.

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