

A CONTROLLABILITY-BASED REGULARIZATION FOR THE ILL-POSED BACKWARD ADVECTION–DIFFUSION EQUATION

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Abstract We study a controllability-based regularization strategy for the backward (retrograde) advection–diffusion equation, a prototypical ill-posed parabolic problem with a non-self-adjoint structure. Rather than introducing an external regularization parameter as in classical Tikhonov or quasi-reversibility approaches, we reinterpret the retrograde dynamics as an inverse problem for the well-posed forward equation, and approximate it by a family of forward controllability problems. We establish the density of the reachable set at the final time via a backward uniqueness argument for the adjoint equation, and derive a necessary and sufficient boundedness criterion on the approximate controllability cost ensuring the existence of a regular solution to the backward equation.

As an application, we analyze a distributed-tracking optimal control problem constrained by the backward advection–diffusion equation. The controllability-based approach yields a decoupled singular optimality system, without invoking any Slater-type qualification assumption. The analysis accommodates anisotropic diffusion tensors and divergence-free drifts under standard boundedness conditions.

1 Introduction

The transport and dispersion of pollutants in water systems are commonly modeled by advection–diffusion equations. These equations model the combined effects of molecular diffusion and convective transport induced by the underlying flow field [1]. In environmental applications, such as river contamination, groundwater remediation, or coastal water management, the ability to influence the pollutant concentration through localized interventions is of considerable practical interest.

Mathematically, the advection–diffusion equation belongs to the class of parabolic partial differential equations. The forward problem is well-posed in suitable Sobolev spaces and generates an analytic semigroup on $L^2(\Omega)$ [2, 3, 4]. However, reconstructing a trajectory from prescribed terminal data is ill-posed in the sense of Hadamard [1]: small perturbations of the terminal data may produce arbitrarily large variations in the reconstructed solution. This instability is well known for parabolic equations and is closely related to the smoothing properties of analytic semigroups; see also [5] for a systematic treatment of inverse parabolic problems.

Backward parabolic equations arise naturally in optimal control and inverse problems. In particular, terminal observation problems and adjoint-based optimization procedures require solving equations backward in time. Without appropriate regularization, such problems may fail to admit stable solutions.

Classical approaches to regularization include Tikhonov-type penalization [6] and quasi-reversibility methods [7]. In both cases, the ill-posed problem is approximated by a family of well-posed problems parameterized by a regularization parameter $\alpha > 0$, and one recovers a solution in the limit $\alpha \rightarrow 0$ under suitable conditions. These methods are intrinsically differ-

ent from the approach proposed here: instead of modifying the backward equation by adding a penalty term or reversing the diffusion sign, we reinterpret the backward dynamics as an inverse problem for the *forward* operator. The solvability of the backward problem is then characterized by the behavior of the approximate controllability cost, a quantity that is intrinsic to the PDE structure and does not require the selection of an external parameter. This connection is made precise in Theorem 3.4 below.

In parallel, the theory of controllability for parabolic equations has undergone significant development. Exact and approximate controllability results for heat-type equations are well established [8, 9]. In many situations, controllability properties are derived from Carleman estimates and unique continuation principles; see [10, 11] for detailed treatments. The cost of approximate controllability for heat equations was analyzed quantitatively by Fernández-Cara and Zuazua [13]. Backward uniqueness for parabolic operators, which plays a central role in density arguments, has been extensively studied; see [14, 15].

The controllability perspective provides an alternative way to interpret certain inverse or ill-posed problems. Rather than directly solving a backward equation, one may seek initial data that steer the forward dynamics toward a prescribed terminal condition. For the heat equation, this idea leads to density results for reachable sets and to variational formulations that avoid classical qualification conditions in constrained optimization. The work of Guel et al. [16] addressed these challenges for the heat equation, introducing a controllability-based regularization that yields a decoupled optimality system without Slater-type conditions. A hierarchical extension is given in [17].

The objective of the present work is to extend this controllability-based framework to the advection–diffusion equation with anisotropic diffusion tensor and divergence-free drift. The presence of a first-order transport term introduces additional analytical difficulties, since the operator is generally non-self-adjoint. Nevertheless, under suitable regularity and incompressibility assumptions on the velocity field, the operator retains a parabolic structure and generates a strongly continuous semigroup on $L^2(\Omega)$ [3, 4]. For recent numerical investigations of optimal control for advection–diffusion equations using penalization, we also refer to [19].

We first analyze the forward advection–diffusion equation and establish its well-posedness in the classical energy space $\mathcal{V}$, which is continuously embedded in $C([0, T]; L^2(\Omega))$ [2, Chapter 1, Theorem 3.1]. We then reinterpret the backward problem as an exact controllability condition at the final time. By characterizing the orthogonal complement of the reachable set through an adjoint equation and invoking backward uniqueness results, we prove that the reachable set is dense in $L^2(\Omega)$. This density property yields a necessary and sufficient condition for the existence of regular solutions to the backward problem.

In a second step, we apply this analysis to an optimal control problem with distributed tracking cost, extending the approach of [16] to the advection–diffusion operator. We prove existence and uniqueness of approximate minimizers, establish uniform energy estimates via a key boundedness argument, and show convergence toward a singular optimality system as the penalization parameter tends to zero. The resulting system is decoupled and does not rely on Slater-type interiority assumptions, which may fail in advection-dominated regimes. Related optimal control results for parabolic equations with nonlocal or nonlinear features can be found in [20].

The paper is organized as follows. Section 2 presents the functional framework and introduces the forward advection–diffusion equation and the optimal control problem. Section 3 establishes the density of the reachable set and derives a necessary and sufficient criterion for solvability of the backward problem. Section 4 develops the controllability-based optimal control formulation and proves convergence toward the associated singular optimality system. Section 5 discusses future directions.

2 Functional Setting and Problem Formulation

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $\mathcal{C}^2$ boundary Γ , and let $T > 0$ be fixed. We define

$$Q = \Omega \times (0, T), \quad \Sigma = \Gamma \times (0, T),$$

and let $\mathcal{U}_{ad}$ be a nonempty, closed, convex subset of $L^2(Q)$.

Throughout the paper, we work under the following assumption.

Assumption 2.1. The diffusion tensor $D \in L^\infty(\Omega; \mathbb{R}^{n \times n})$ is symmetric and uniformly positive definite: there exists $\delta > 0$ such that

$$D(x)\xi \cdot \xi \geq \delta|\xi|^2 \quad \text{for a.e. } x \in \Omega, \forall \xi \in \mathbb{R}^n.$$

The velocity field $\mathbf{b} \in W^{1,\infty}(\Omega; \mathbb{R}^n)$ satisfies the incompressibility condition

$$\nabla \cdot \mathbf{b} = 0 \quad \text{in } \Omega.$$

The divergence-free condition ensures that $\mathbf{b} \cdot \nabla$ is skew-adjoint in $L^2(\Omega)$. Specifically, for all $y, p \in L^2(0, T; H_0^1(\Omega))$, an integration by parts gives

$$\int_Q (\mathbf{b} \cdot \nabla y) p \, dx \, dt = - \int_Q y (\mathbf{b} \cdot \nabla p) \, dx \, dt, \quad (2.1)$$

where we used $\nabla \cdot \mathbf{b} = 0$ and the homogeneous Dirichlet boundary conditions on Σ . Consequently, the $L^2(Q)$ -adjoint of the primal operator $\mathcal{L} := -\operatorname{div}(D\nabla \cdot) + \mathbf{b} \cdot \nabla$ is $\mathcal{L}^* := -\operatorname{div}(D\nabla \cdot) - \mathbf{b} \cdot \nabla$, i.e., the transport term changes sign in the adjoint equation.

We introduce the energy space

$$\mathcal{V} = L^2(0, T; H_0^1(\Omega)) \cap H^1(0, T; H^{-1}(\Omega)).$$

By a classical Lions–Magenes embedding theorem [2, Chapter 1, Theorem 3.1], the space $\mathcal{V}$ is continuously embedded in $C([0, T]; L^2(\Omega))$:

$$\mathcal{V} \hookrightarrow C([0, T]; L^2(\Omega)). \quad (2.2)$$

This embedding is essential: it ensures that pointwise evaluations $y(\cdot, t) \in L^2(\Omega)$ are well-defined for every $t \in [0, T]$ and depend continuously on t , so that the terminal condition $y(\cdot, T) = 0$ is meaningful.

2.1 Forward and Retrograde Dynamics

For a given source term $v \in L^2(Q)$ and initial datum $y_0 \in L^2(\Omega)$, we consider the forward advection–diffusion problem

$$\begin{cases} \frac{\partial y}{\partial t} - \operatorname{div}(D\nabla y) + \mathbf{b} \cdot \nabla y = v & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(\cdot, 0) = y_0 & \text{in } \Omega. \end{cases} \quad (2.3)$$

Under Assumption 2.1, classical parabolic theory [12, 2, 4] ensures the following result.

Proposition 2.2 (Well-posedness of the forward problem). *For every $(v, y_0) \in L^2(Q) \times L^2(\Omega)$, problem (2.3) admits a unique solution $y \in \mathcal{V}$. Moreover, there exists $C > 0$ independent of (v, y_0) such that*

$$\|y\|_{\mathcal{V}} \leq C(\|v\|_{L^2(Q)} + \|y_0\|_{L^2(\Omega)}). \quad (2.4)$$

The proof is classical and follows from coercivity of the bilinear form associated with $\mathcal{L}$ and the Galerkin method; see [4, Chapter 3] or [2, Vol. I, Chapter 3].

We study the backward (retrograde) problem: given $v \in L^2(Q)$, find $z \in \mathcal{V}$ satisfying

$$\begin{cases} \frac{\partial z}{\partial t} - \operatorname{div}(D\nabla z) + \mathbf{b} \cdot \nabla z = v & \text{in } Q, \\ z = 0 & \text{on } \Sigma, \\ z(\cdot, T) = 0 & \text{in } \Omega. \end{cases} \quad (2.5)$$

Remark 2.3. The backward problem (2.5) differs fundamentally from the standard initial-value problem (2.3): instead of an initial datum $z(\cdot, 0) = z_0$, we impose a terminal condition $z(\cdot, T) = 0$. Although (2.5) can formally be reduced to a forward parabolic problem by the time reversal $s = T - t$ – at the cost of replacing $\mathbf{b}$ by $-\mathbf{b}$ – the resulting problem is still generally ill-posed in the sense of Hadamard [1]: small perturbations in v may lead to arbitrarily large changes in the initial value $z(\cdot, 0)$. The present framework interprets (2.5) as a terminal compatibility condition for the forward dynamics, and characterizes when this condition can be satisfied within the energy space $\mathcal{V}$.

Problem (Backward solvability). *Given $v \in L^2(Q)$, characterize the conditions under which (2.5) admits a solution $z \in \mathcal{V}$.*

2.2 Optimal Control Problem

Given $z_d \in L^2(Q)$ and $N > 0$, the source term v in (2.5) plays the role of a distributed control variable. We call a pair (v, z) *admissible* if $v \in \mathcal{U}_{ad}$ and $z \in \mathcal{V}$ satisfies (2.5). The set of admissible pairs is

$$\mathcal{A} = \{(v, z) \in \mathcal{U}_{ad} \times \mathcal{V} : (2.5) \text{ holds}\}.$$

We assume throughout Section 4 that $\mathcal{A} \neq \emptyset$, i.e., there exists at least one admissible pair. We introduce the cost functional

$$J(v, z) = \frac{1}{2} \|z - z_d\|_{L^2(Q)}^2 + \frac{N}{2} \|v\|_{L^2(Q)}^2, \quad (2.6)$$

where the first term measures the distance of the state z to the target z_d , and the second term penalizes the control effort with weight $N > 0$. We consider the optimal control problem

$$\inf_{(v, z) \in \mathcal{A}} J(v, z). \quad (2.7)$$

Theorem 2.4 (Existence and uniqueness of the optimal control). *Under Assumption 2.1, if $\mathcal{A} \neq \emptyset$, the optimal control problem (2.7) admits a unique solution $(u, y) \in \mathcal{A}$, called the optimal control-state pair.*

Proof. The functional J is strictly convex (since $N > 0$), coercive on $\mathcal{U}_{ad}$ (from the term $\frac{N}{2} \|v\|_{L^2(Q)}^2$), and weakly lower semicontinuous on $L^2(Q)$. The set $\mathcal{A}$ is convex and closed in $L^2(Q) \times \mathcal{V}$ by linearity and continuity of the map $v \mapsto z(v)$. Hence existence and uniqueness follow from the standard theorem on minimization of strictly convex coercive functionals on closed convex sets [12, Chapter II, Theorem 1.1]. $\square$

The Euler–Lagrange first-order optimality condition gives

$$(y - z_d, z - y)_{L^2(Q)} + N(u, v - u)_{L^2(Q)} \geq 0, \quad \forall (v, z) \in \mathcal{A}. \quad (2.8)$$

Finding a decoupled form of this condition is the central challenge. The classical penalization approach of Lions [12] yields a decoupled optimality system only under a Slater-type condition, which requires $\mathcal{U}_{ad}$ to have nonempty interior in $L^2(Q)$. This condition may fail in advection-dominated regimes or when $\mathcal{U}_{ad}$ has a bang-bang structure. We therefore adopt the controllability-based approach of Guel et al. [16], adapted here to the advection–diffusion operator.

The formal first-order optimality system involves the adjoint state p solving

$$\begin{cases} -\frac{\partial p}{\partial t} - \operatorname{div}(D \nabla p) - \mathbf{b} \cdot \nabla p = y - z_d & \text{in } Q, \\ p = 0 & \text{on } \Sigma, \\ p(\cdot, T) = 0 & \text{in } \Omega, \end{cases} \quad (2.9)$$

together with the variational inequality $\langle p + Nu, v - u \rangle_{L^2(Q)} \geq 0$ for all $v \in \mathcal{U}_{ad}$.

Remark 2.5. The sign change $-\mathbf{b} \cdot \nabla p$ in (2.9) is a consequence of (2.1): the $L^2(Q)$ -adjoint of $\mathbf{b} \cdot \nabla$ is $-\mathbf{b} \cdot \nabla$. This sign change is specific to the advection–diffusion operator and does not arise for the self-adjoint heat equation studied in [16].

3 Controllability and Regularity Criterion

This section reinterprets the backward problem (2.5) as an approximate controllability problem for the forward dynamics, following the strategy of [16]. We first prove density of the reachable set, then derive a sharp criterion for the existence of regular backward solutions.

3.1 Reachable Set and Decomposition of the Forward Solution

Fix $v \in L^2(Q)$. For any $y_0 \in L^2(\Omega)$, let $y(\cdot; v, y_0)$ denote the unique solution of the forward problem (2.3). By linearity of (2.3) with respect to (v, y_0) , we decompose

$$y(\cdot; v, y_0) = y^v + y^0(\cdot; y_0),$$

where:

- $y^v \in \mathcal{V}$ solves (2.3) with source v and zero initial datum, $y^v(\cdot, 0) = 0$;
- $y^0(\cdot; y_0) \in \mathcal{V}$ solves the homogeneous equation

$$\begin{cases} \frac{\partial y^0}{\partial t} - \operatorname{div}(D\nabla y^0) + \mathbf{b} \cdot \nabla y^0 = 0 & \text{in } Q, \\ y^0 = 0 & \text{on } \Sigma, \\ y^0(\cdot, 0) = y_0 & \text{in } \Omega. \end{cases} \quad (3.1)$$

The *reachable set at time T* is

$$\mathcal{R}(T) = \{y^0(\cdot, T; y_0) \mid y_0 \in L^2(\Omega)\} \subset L^2(\Omega).$$

By the decomposition above, $y(\cdot, T; v, y_0) = y^v(\cdot, T) + y^0(\cdot, T; y_0)$. Hence the backward problem (2.5) admits a solution $z \in \mathcal{V}$ if and only if there exists $y_0 \in L^2(\Omega)$ such that $y^0(\cdot, T; y_0) = -y^v(\cdot, T)$, i.e., $-y^v(\cdot, T) \in \mathcal{R}(T)$. The question of solvability is therefore equivalent to whether the target $-y^v(\cdot, T)$ belongs to the reachable set.

3.2 Density of the Reachable Set

Proposition 3.1 (Density of the reachable set). *Under Assumption 2.1, the reachable set $\mathcal{R}(T)$ is dense in $L^2(\Omega)$.*

Proof. Since $\mathcal{R}(T)$ is a linear subspace of $L^2(\Omega)$, it suffices to show $\mathcal{R}(T)^\perp = \{0\}$. Let $k \in \mathcal{R}(T)^\perp$. By definition,

$$\langle y^0(\cdot, T; y_0), k \rangle_{L^2(\Omega)} = 0 \quad \forall y_0 \in L^2(\Omega).$$

We introduce the backward homogeneous adjoint problem

$$\begin{cases} -\frac{\partial \varphi}{\partial t} - \operatorname{div}(D\nabla \varphi) - \mathbf{b} \cdot \nabla \varphi = 0 & \text{in } Q, \\ \varphi = 0 & \text{on } \Sigma, \\ \varphi(\cdot, T) = k & \text{in } \Omega. \end{cases} \quad (3.2)$$

This problem is well-posed in $\mathcal{V}$: indeed, the change of variables $s = T - t$ transforms it into a forward problem for the operator $\mathcal{L}^* = \partial_s - \operatorname{div}(D\nabla \cdot) - \mathbf{b} \cdot \nabla$ with initial datum $k \in L^2(\Omega)$, so Proposition 2.2 applies.

Multiplying the weak formulation of (3.1) by φ and integrating over Q , an integration by parts in time gives

$$\begin{aligned} \langle y^0(\cdot, T; y_0), \varphi(\cdot, T) \rangle_{L^2(\Omega)} - \langle y_0, \varphi(\cdot, 0) \rangle_{L^2(\Omega)} &= \int_Q \left[-\frac{\partial \varphi}{\partial t} y^0 - \operatorname{div}(D\nabla \varphi) y^0 \right] dx dt \\ &\quad + \int_Q (\mathbf{b} \cdot \nabla y^0) \varphi dx dt. \end{aligned}$$

Using (2.1) and the fact that φ satisfies (3.2), the right-hand side vanishes, yielding

$$\langle y^0(\cdot, T; y_0), k \rangle_{L^2(\Omega)} = \langle y_0, \varphi(\cdot, 0) \rangle_{L^2(\Omega)}.$$

Since the left-hand side is zero for all $y_0 \in L^2(\Omega)$, we deduce $\varphi(\cdot, 0) = 0$ in $L^2(\Omega)$. By the backward uniqueness property for non-self-adjoint parabolic operators with bounded measurable coefficients and divergence-free drift [14, 8, 15], we conclude $\varphi \equiv 0$ in Q , hence $k = \varphi(\cdot, T) = 0$. This shows $\mathcal{R}(T)^\perp = \{0\}$. $\square$

Remark 3.2. The backward uniqueness property used in the proof holds under the present assumptions: D bounded, measurable, uniformly elliptic; $\mathbf{b} \in W^{1,\infty}$, divergence-free; Dirichlet boundary conditions. This follows from the unique continuation principle for parabolic equations; see [14] for the general case and [11, Chapter 2, Theorem 2.1] for a Carleman-estimate-based proof covering non-self-adjoint operators with advection terms of the form $\mathbf{b} \cdot \nabla$. For lower regularity drift ($\mathbf{b} \in L^\infty$), additional assumptions on the divergence structure may be needed; see [10].

3.3 A Necessary and Sufficient Criterion for Backward Solvability

As an immediate consequence of Proposition 3.1, we obtain the following approximate null controllability result.

Corollary 3.3 (Approximate null controllability). *For every $v \in L^2(Q)$ and every $\varepsilon > 0$, there exists $y_{0,\varepsilon} \in L^2(\Omega)$ such that the solution $y_\varepsilon := y(\cdot; v, y_{0,\varepsilon}) \in \mathcal{V}$ of (2.3) satisfies*

$$\|y_\varepsilon(\cdot, T)\|_{L^2(\Omega)} < \varepsilon.$$

Proof. By Proposition 3.1, $\mathcal{R}(T)$ is dense in $L^2(\Omega)$. Since $-y^v(\cdot, T) \in L^2(\Omega)$, there exists $y_{0,\varepsilon} \in L^2(\Omega)$ such that

$$\|y^0(\cdot, T; y_{0,\varepsilon}) - (-y^v(\cdot, T))\|_{L^2(\Omega)} < \varepsilon.$$

Then, using the decomposition $y_\varepsilon = y^v + y^0(\cdot; y_{0,\varepsilon})$,

$$\|y_\varepsilon(\cdot, T)\|_{L^2(\Omega)} = \|y^v(\cdot, T) + y^0(\cdot, T; y_{0,\varepsilon})\|_{L^2(\Omega)} < \varepsilon. \quad \square$$

The following theorem is the main result of this section. It characterizes backward solvability through the boundedness of the family $(y_{0,\varepsilon})_{\varepsilon>0}$ provided by Corollary 3.3. This is the advection–diffusion analogue of the controllability cost criterion of Guel et al. [16] for the heat equation, and is related in spirit to the cost analysis of Fernández-Cara and Zuazua [13].

Theorem 3.4 (Backward solvability criterion via controllability cost). *Let $v \in L^2(Q)$. The backward problem (2.5) admits a solution $z \in \mathcal{V}$ if and only if there exists a family $(y_{0,\varepsilon})_{\varepsilon>0} \subset L^2(\Omega)$ satisfying $\|y(\cdot; v, y_{0,\varepsilon})(\cdot, T)\|_{L^2(\Omega)} < \varepsilon$ for all $\varepsilon > 0$, and such that*

$$\sup_{\varepsilon>0} \|y_{0,\varepsilon}\|_{L^2(\Omega)} < \infty.$$

Proof. ($\Rightarrow$) *Necessity.* Suppose $z \in \mathcal{V}$ solves (2.5). Set $y_{0,\varepsilon} := z(\cdot, 0) \in L^2(\Omega)$ for all $\varepsilon > 0$ (recall that $z \in \mathcal{V} \hookrightarrow C([0, T]; L^2(\Omega))$ by (2.2)). The unique solution of (2.3) with initial datum $z(\cdot, 0)$ and source v coincides with z (by uniqueness in Proposition 2.2), so $y(\cdot; v, y_{0,\varepsilon})(\cdot, T) = z(\cdot, T) = 0$. The family is constant, hence uniformly bounded.

($\Leftarrow$) *Sufficiency.* Suppose $(y_{0,\varepsilon})_\varepsilon$ is bounded in $L^2(\Omega)$. By Proposition 2.2, the energy estimate (2.4) gives

$$\|y_\varepsilon\|_{\mathcal{V}} \leq C(\|v\|_{L^2(Q)} + \|y_{0,\varepsilon}\|_{L^2(\Omega)}) \leq C',$$

so $(y_\varepsilon)_\varepsilon$ is uniformly bounded in $\mathcal{V}$. Since $\mathcal{V}$ is reflexive, there exist a subsequence $\varepsilon_k \rightarrow 0$ and $z \in \mathcal{V}$ such that $y_{\varepsilon_k} \rightharpoonup z$ weakly in $\mathcal{V}$. By the embedding (2.2), the map $w \mapsto w(\cdot, T)$ is linear and continuous from $\mathcal{V}$ to $L^2(\Omega)$, hence weakly continuous. Combining with the strong convergence $\|y_{\varepsilon_k}(\cdot, T)\|_{L^2(\Omega)} < \varepsilon_k \rightarrow 0$, we deduce $z(\cdot, T) = 0$ in $L^2(\Omega)$. Passing to the limit

$\varepsilon_k \rightarrow 0$ in the weak formulation of (2.3) (which is stable under weak convergence in $\mathcal{V}$), we conclude that z satisfies

$$\begin{cases} \frac{\partial z}{\partial t} - \operatorname{div}(D\nabla z) + \mathbf{b} \cdot \nabla z = v & \text{in } Q, \\ z = 0 & \text{on } \Sigma, \\ z(\cdot, T) = 0 & \text{in } \Omega. \end{cases}$$

Hence $z \in \mathcal{V}$ solves (2.5). $\square$

Remark 3.5. The family $(y_{0,\varepsilon})_\varepsilon$ is chosen optimally (minimal $L^2(\Omega)$ -norm for the ε -approximate terminal condition), so its boundedness is precisely the finiteness of the approximate controllability cost. A bounded family corresponds to a finite controllability cost, which is equivalent to the solvability of the backward problem. This is the exact advection–diffusion analogue of the criterion established in [16] for the heat equation. The parameter ε controls only the approximation accuracy; the uniform boundedness is a qualitative threshold condition, not a rate condition.

4 Controllability-Based Optimal Control and Singular Optimality System

We now apply the controllability perspective developed in Section 3 to the optimal control problem (2.7), following [16] adapted to the advection–diffusion operator. We recall that $\mathcal{A} \neq \emptyset$ and $(u, y) \in \mathcal{A}$ denotes the unique optimal pair from Theorem 2.4.

4.1 Penalized Formulation

For each $v \in \mathcal{U}_{ad}$ and $\varepsilon > 0$, Corollary 3.3 yields $y_{0,\varepsilon}(v) \in L^2(\Omega)$ such that the forward solution $y_\varepsilon^v := y(\cdot; v, y_{0,\varepsilon}(v)) \in \mathcal{V}$ satisfies $\|y_\varepsilon^v(\cdot, T)\|_{L^2(\Omega)} < \varepsilon$. We choose $y_{0,\varepsilon}(v)$ to have minimal $L^2(\Omega)$ -norm among all initial data achieving this property.

We introduce the penalized cost functional

$$J_\varepsilon(v) = \frac{1}{2} \|y_\varepsilon^v - z_d\|_{L^2(Q)}^2 + \frac{N}{2} \|v\|_{L^2(Q)}^2 + \frac{1}{2\varepsilon} \|y_{0,\varepsilon}(v) - y(\cdot, 0)\|_{L^2(\Omega)}^2. \quad (4.1)$$

Here $y(\cdot, 0) \in L^2(\Omega)$ is the initial value of the optimal state (well-defined by (2.2)). The third term in (4.1) penalizes the deviation of the approximate initial datum $y_{0,\varepsilon}(v)$ from the optimal initial value, with weight $\frac{1}{\varepsilon}$. As $\varepsilon \rightarrow 0$, this penalization forces $y_{0,\varepsilon}(u_\varepsilon)$ to converge to $y(\cdot, 0)$, which is the mechanism driving the convergence to the singular optimality system.

The penalized problem is

$$\inf_{v \in \mathcal{U}_{ad}} J_\varepsilon(v). \quad (4.2)$$

Proposition 4.1 (Existence and key uniform bound). *For every $\varepsilon > 0$, problem (4.2) admits a unique minimizer $u_\varepsilon \in \mathcal{U}_{ad}$. Denoting $y_\varepsilon := y_\varepsilon^{u_\varepsilon} \in \mathcal{V}$ and $y_{0,\varepsilon} := y_{0,\varepsilon}(u_\varepsilon) \in L^2(\Omega)$, one has*

$$J_\varepsilon(u_\varepsilon) \leq J(u, y), \quad (4.3)$$

and there exist constants $C_1, C_2, C_3 > 0$, independent of ε , such that

$$\|y_\varepsilon\|_{L^2(Q)} \leq C_1, \quad \|u_\varepsilon\|_{L^2(Q)} \leq C_2, \quad \|y_{0,\varepsilon}\|_{L^2(\Omega)} \leq C_3. \quad (4.4)$$

Proof. Existence. The functional J_ε is strictly convex (from the term $\frac{N}{2} \|v\|_{L^2(Q)}^2$ with $N > 0$), coercive on $\mathcal{U}_{ad}$, and weakly lower semicontinuous. Standard convex minimization theory [12, Chapter II, Theorem 1.1] yields a unique minimizer $u_\varepsilon \in \mathcal{U}_{ad}$.

Key bound. Since $(u, y) \in \mathcal{A}$, we have $y(\cdot, T) = 0$, so setting $y_{0,\varepsilon}^* := y(\cdot, 0)$ and $y_\varepsilon^{u,*} := y$, we have $\|y(\cdot, T)\|_{L^2(\Omega)} = 0 < \varepsilon$, hence u is admissible for (4.2) with this selection. Computing:

$$J_\varepsilon(u) = \frac{1}{2} \|y - z_d\|_{L^2(Q)}^2 + \frac{N}{2} \|u\|_{L^2(Q)}^2 + \frac{1}{2\varepsilon} \underbrace{\|y(\cdot, 0) - y(\cdot, 0)\|_{L^2(\Omega)}^2}_{=0} = J(u, y).$$

Hence $J_\varepsilon(u_\varepsilon) \leq J_\varepsilon(u) = J(u, y)$, which is independent of ε . This proves (4.3).

Uniform bounds. From (4.3), we extract

$$\frac{N}{2} \|u_\varepsilon\|_{L^2(Q)}^2 \leq J(u, y),$$

giving $\|u_\varepsilon\|_{L^2(Q)} \leq C_2$. By the energy estimate (2.4), $\|y_\varepsilon\|_{\mathcal{V}} \leq C(\|u_\varepsilon\|_{L^2(Q)} + \|y_{0,\varepsilon}\|_{L^2(\Omega)})$. From (4.3),

$$\frac{1}{2\varepsilon} \|y_{0,\varepsilon} - y(\cdot, 0)\|_{L^2(\Omega)}^2 \leq J(u, y),$$

so $\|y_{0,\varepsilon}\|_{L^2(\Omega)} \leq \|y(\cdot, 0)\|_{L^2(\Omega)} + (2\varepsilon J(u, y))^{1/2} \leq C_3$ for $\varepsilon \leq 1$. The bound on y_ε then follows. $\square$

4.2 Convergence of the Penalized Method

Proposition 4.2 (Weak convergence and identification of the limit). *There exist $\hat{u} \in L^2(Q)$, $\hat{y} \in \mathcal{V}$, and $\hat{y}_0 \in L^2(\Omega)$ such that, up to a subsequence,*

$$u_\varepsilon \rightharpoonup \hat{u} \text{ in } L^2(Q), \quad y_\varepsilon \rightharpoonup \hat{y} \text{ in } \mathcal{V}, \quad y_{0,\varepsilon} \rightharpoonup \hat{y}_0 \text{ in } L^2(\Omega).$$

Moreover, $\hat{y}_0 = y(\cdot, 0)$ in Ω , $\hat{y}(\cdot, T) = 0$ in Ω , and $(\hat{u}, \hat{y}) \in \mathcal{A}$.

Proof. The uniform bounds (4.4) and reflexivity of $L^2(Q)$, $\mathcal{V}$, and $L^2(\Omega)$ yield the weak convergences along a subsequence.

Identification $\hat{y}_0 = y(\cdot, 0)$. From (4.3),

$$\|y_{0,\varepsilon} - y(\cdot, 0)\|_{L^2(\Omega)}^2 \leq 2\varepsilon J(u, y) \rightarrow 0,$$

so $y_{0,\varepsilon} \rightarrow y(\cdot, 0)$ strongly in $L^2(\Omega)$, hence $\hat{y}_0 = y(\cdot, 0)$.

Terminal condition. Since $\|y_\varepsilon(\cdot, T)\|_{L^2(\Omega)} < \varepsilon \rightarrow 0$ and the evaluation map $w \mapsto w(\cdot, T)$ is weakly continuous from $\mathcal{V}$ to $L^2(\Omega)$ (using (2.2)), we get $\hat{y}(\cdot, T) = 0$.

Admissibility. Passing to the limit in the weak formulation of (2.3) shows that $\hat{y}$ satisfies $\partial_t \hat{y} - \operatorname{div}(D \nabla \hat{y}) + \mathbf{b} \cdot \nabla \hat{y} = \hat{u}$ in Q , $\hat{y} = 0$ on Σ , $\hat{y}(\cdot, T) = 0$ in Ω . Hence $(\hat{u}, \hat{y}) \in \mathcal{A}$. $\square$

Theorem 4.3 (Strong convergence to the optimal pair). *The pair $(\hat{u}, \hat{y})$ coincides with the unique optimal pair (u, y) , and*

$$u_\varepsilon \rightarrow u \text{ strongly in } L^2(Q), \quad y_\varepsilon \rightarrow y \text{ strongly in } L^2(Q).$$

Proof. Optimality. Since $(\hat{u}, \hat{y}) \in \mathcal{A}$, we have $J(u, y) \leq J(\hat{u}, \hat{y})$. On the other hand, by weak lower semicontinuity of J and (4.3):

$$J(\hat{u}, \hat{y}) \leq \liminf_{\varepsilon \rightarrow 0} J_\varepsilon(u_\varepsilon) \leq J(u, y).$$

Hence $J(\hat{u}, \hat{y}) = J(u, y)$, and by uniqueness (Theorem 2.4), $(\hat{u}, \hat{y}) = (u, y)$.

Strong convergence. From $J(u, y) = \lim_{\varepsilon \rightarrow 0} J_\varepsilon(u_\varepsilon)$ and Proposition 4.2:

$$\|y - z_d\|_{L^2(Q)}^2 + N \|u\|_{L^2(Q)}^2 = \lim_{\varepsilon \rightarrow 0} (\|y_\varepsilon - z_d\|_{L^2(Q)}^2 + N \|u_\varepsilon\|_{L^2(Q)}^2).$$

Combining with weak convergence $y_\varepsilon \rightharpoonup y$ and $u_\varepsilon \rightharpoonup u$ in $L^2(Q)$, and weak lower semicontinuity of the norm, we obtain

$$\|y_\varepsilon - z_d\|_{L^2(Q)} \rightarrow \|y - z_d\|_{L^2(Q)}, \quad \|u_\varepsilon\|_{L^2(Q)} \rightarrow \|u\|_{L^2(Q)}.$$

Since $L^2(Q)$ is a Hilbert space, norm convergence combined with weak convergence implies strong convergence. $\square$

4.3 Approximate Optimality System

For each $\varepsilon > 0$, let $\varphi_\varepsilon := y(v - u_\varepsilon, 0)$ denote the solution of (2.3) with source $v - u_\varepsilon$ and zero initial datum ($\varphi_\varepsilon(\cdot, 0) = 0$). By the Euler–Lagrange condition at the minimizer u_ε :

$$(y_\varepsilon - z_d, \varphi_\varepsilon)_{L^2(Q)} + N(u_\varepsilon, v - u_\varepsilon)_{L^2(Q)} \geq 0, \quad \forall v \in \mathcal{U}_{ad}. \quad (4.5)$$

We introduce the adjoint state $p_\varepsilon \in \mathcal{V}$ satisfying

$$\begin{cases} -\frac{\partial p_\varepsilon}{\partial t} - \operatorname{div}(D\nabla p_\varepsilon) - \mathbf{b} \cdot \nabla p_\varepsilon = y_\varepsilon - z_d & \text{in } Q, \\ p_\varepsilon = 0 & \text{on } \Sigma, \\ p_\varepsilon(\cdot, T) = 0 & \text{in } \Omega. \end{cases} \quad (4.6)$$

Testing (4.6) against φ_ε , using $\varphi_\varepsilon(\cdot, 0) = 0$, $p_\varepsilon(\cdot, T) = 0$, and identity (2.1):

$$(y_\varepsilon - z_d, \varphi_\varepsilon)_{L^2(Q)} = (p_\varepsilon, v - u_\varepsilon)_{L^2(Q)}.$$

Substituting into (4.5) gives the variational inequality

$$(p_\varepsilon + Nu_\varepsilon, v - u_\varepsilon)_{L^2(Q)} \geq 0, \quad \forall v \in \mathcal{U}_{ad}. \quad (4.7)$$

Theorem 4.4 (Approximate optimality system). *For each $\varepsilon > 0$, u_ε solves (4.2) if and only if the quadruplet $\{y_{0,\varepsilon}, u_\varepsilon, y_\varepsilon, p_\varepsilon\} \in L^2(\Omega) \times (L^2(Q))^3$ is the unique solution of the coupled system consisting of*

$$\begin{cases} \frac{\partial y_\varepsilon}{\partial t} - \operatorname{div}(D\nabla y_\varepsilon) + \mathbf{b} \cdot \nabla y_\varepsilon = u_\varepsilon & \text{in } Q, \\ y_\varepsilon = 0 & \text{on } \Sigma, \\ y_\varepsilon(\cdot, 0) = y_{0,\varepsilon} & \text{in } \Omega, \\ \|y_\varepsilon(\cdot, T)\|_{L^2(\Omega)} < \varepsilon, \end{cases} \quad (4.8)$$

together with (4.6) and the variational inequality (4.7).

4.4 Singular Optimality System

From Theorem 4.3, $y_\varepsilon \rightarrow y$ and $u_\varepsilon \rightarrow u$ strongly in $L^2(Q)$, so $y_\varepsilon - z_d \rightarrow y - z_d$ strongly. Since the adjoint equation (4.6) is well-posed (Proposition 2.2 applied after time reversal), the strong convergence of the right-hand side implies $p_\varepsilon \rightarrow p$ strongly in $L^2(Q)$ for some $p \in \mathcal{V}$. Passing to the limit $\varepsilon \rightarrow 0$ in Theorem 4.4 yields the main result.

Theorem 4.5 (Singular optimality system). *The pair (u, y) is the unique solution of the optimal control problem (2.7) if and only if there exists $p \in \mathcal{V}$ such that the triplet $\{u, y, p\} \in (L^2(Q))^2 \times \mathcal{V}$ satisfies the forward state equation*

$$\begin{cases} \frac{\partial y}{\partial t} - \operatorname{div}(D\nabla y) + \mathbf{b} \cdot \nabla y = u & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(\cdot, T) = 0 & \text{in } \Omega, \end{cases} \quad (4.9)$$

the backward adjoint equation

$$\begin{cases} -\frac{\partial p}{\partial t} - \operatorname{div}(D\nabla p) - \mathbf{b} \cdot \nabla p = y - z_d & \text{in } Q, \\ p = 0 & \text{on } \Sigma, \\ p(\cdot, T) = 0 & \text{in } \Omega, \end{cases} \quad (4.10)$$

and the variational inequality

$$(p + Nu, v - u)_{L^2(Q)} \geq 0, \quad \forall v \in \mathcal{U}_{ad}. \quad (4.11)$$

Remark 4.6. The system (4.9)–(4.11) is called *singular* because neither the state equation (4.9) nor the adjoint equation (4.10) carries an initial condition at $t = 0$. Instead, both equations carry terminal conditions at $t = T$. This singular structure arises from the backward nature of the constraint (2.5) and is an intrinsic feature of the controllability penalization: the penalization forces the terminal condition $y(\cdot, T) = 0$ directly, bypassing any need for an initial datum.

This structure should be contrasted with the classical optimality system for a *forward*-constrained control problem, where the state satisfies an initial condition and the adjoint satisfies a terminal condition. For the present backward constraint, both the state and the adjoint satisfy terminal conditions, and neither requires an initial datum.

In particular, the system is valid for:

- arbitrary closed convex admissible sets $\mathcal{U}_{ad} \subset L^2(Q)$, including $\mathcal{U}_{ad} = L^2(Q)^+$ (non-negative controls) or bang-bang constraints;
- advection-dominated regimes where $\|\mathbf{b}\|_\infty \gg \delta$;
- any case where $\mathcal{A} \neq \emptyset$.

No Slater-type interiority condition on $\mathcal{U}_{ad}$ is required.

5 Conclusion and Perspectives

We have studied the controllability-based regularization framework for the backward advection–diffusion equation with an anisotropic diffusion tensor and a divergence-free drift. The main mathematical difficulty, compared to the heat equation, lies in the non-self-adjoint structure of the transport operator, which requires a careful treatment of the adjoint equations throughout the analysis.

The main analytical contribution is a necessary and sufficient boundedness criterion for the existence of regular solutions to the backward advection–diffusion equation. This characterization is obtained via a density argument for the reachable set associated with the homogeneous forward dynamics, combined with a weak compactness argument. The divergence-free condition $\nabla \cdot \mathbf{b} = 0$ plays a fundamental role: it ensures the skew-adjointness identity, which underpins both the density argument and the derivation of the adjoint equations.

On the control side, the penalized controllability approach yields a decoupled singular optimality system for the optimal control–state pair, without requiring any Slater-type qualification condition. The state equation is supplemented with a terminal condition $y(\cdot, T) = 0$, while the adjoint equation satisfies $p(\cdot, T) = 0$; in particular, neither equation requires an initial condition at $t = 0$.

Several perspectives naturally arise from this analysis. First, the controllability-based criterion may be extended to more general transport operators, including variable-density flows or non-divergence-free drifts, along the lines of [21]. Second, extensions to semilinear or quasilinear advection–diffusion equations remain an open and challenging direction; see [20] for related nonlinear optimal control results. Third, the present approach suggests potential applications to inverse source problems and data assimilation, where backward parabolic stability issues play a central role [5]. Finally, the cost analysis of Fernández-Cara and Zuazua [13] for the heat equation suggests that quantitative bounds on the controllability cost—and hence on the threshold for backward solvability—could be derived for the advection–diffusion equation, thereby complementing the qualitative criterion of Theorem 3.4.

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