

Fractional Fuzzy Neural Networks Evolving in Hypercomplex Space: A Framework for Pseudo Almost Automorphic Solutions

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Abstract This article introduces a novel class of fractional-order fuzzy cellular neural networks evolving in hypercomplex spaces generated by the Cayley–Dickson construction, including quaternions, octonions, and higher-dimensional algebras. Using the framework of μ -pseudo almost automorphic functions in the sense of measure theory, together with a Banach fixed-point approach, we establish new sufficient conditions for the existence and uniqueness of μ -pseudo almost automorphic mild solutions. Furthermore, a generalized Gronwall inequality is employed to derive sufficient criteria guaranteeing finite-time stability of these solutions. A distinctive feature of this work is the explicit handling of analytical difficulties arising from non-commutativity (quaternions) and non-associativity (octonions), achieved through contraction coefficients based on structure constants. Numerical simulations in both quaternion and octonion settings are provided to illustrate and validate the theoretical results.

1 Introduction

Higher-dimensional number systems are generated recursively by the Cayley–Dickson process, a systematic doubling procedure that extends the real field to algebras of increasingly higher dimension. Beginning with the real numbers $\mathbb{R}$, this construction yields in succession the complex numbers $\mathbb{C}$ [1], the quaternions $\mathbb{Q}$, the octonions $\mathbb{O}$, and an infinite family of hypercomplex algebras collectively denoted $\mathbb{H}$. The first steps produce algebras of dimensions 1, 2, 4, and 8, while further iterations give rise to structures such as the sedenions $\mathbb{S}$ (dimension 16) and the pathions $\mathbb{P}$ (dimension 32) [2]:

$\mathbb{R} :$	a_0	real number, dimension 1,
$\mathbb{C} :$	$a_0 + a_1 e_1 = (a_0, a_1)$	complex number, dimension 2,
$\mathbb{Q} :$	$a_0 + a_1 e_1 + a_2 e_2 + a_3 e_3 = (a_0, a_1, a_2, a_3)$	quaternion, dimension 4,
$\mathbb{O} :$	$a_0 + a_1 e_1 + a_2 e_2 + a_3 e_3 + a_4 e_4 + a_5 e_5 + a_6 e_6 + a_7 e_7$ $= (a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7)$	octonion, dimension 8.

A fundamental trade-off governs this recursive construction: each doubling of dimension comes at the cost of algebraic properties.

Hypercomplex structures are increasingly used in the modeling of multidimensional dynamical systems, especially when state variables evolve in high-dimensional or noncommutative algebraic spaces [3, 4]. In this setting, fractional calculus provides an effective framework for describing memory effects and anomalous dynamics. In particular, fractional differential equations involving Caputo derivatives have attracted growing interest due to their wide applications [5],[6]. In general, fractional differential equations capture highly complex dynamical behaviors, motivating the development of numerous analytical and numerical methods. As a result, fractional calculus has attracted growing research interest across many scientific fields [7, 8, 9, 10]. When combined with hypercomplex-valued state spaces derived from Cayley–Dickson algebras, this approach offers new perspectives for modeling and analyzing high-dimensional and nonclassical dynamical systems.

Beyond abstract dynamical systems, fractional calculus has proven highly effective in modeling artificial neural networks (ANNs) [11, 12, 13]. When described by fractional-order differential equations, these models are known as fractional artificial neural networks. Moreover, integrating fractional dynamics with fuzzy logic leads to fuzzy fractional neural networks [14], offering a more accurate and efficient framework for complex physical and engineering applications. Such models provide several advantages, including enhanced memory capacity, improved uncertainty handling, faster information processing, and better convergence and stability properties [15]. As a result, fractional and fuzzy fractional neural networks have attracted growing research interest [16, 17].

There are very few works addressing the dynamic analysis of μ -pseudo almost automorphic solutions for fractional-order fuzzy cellular neural networks. This research direction remains largely unexplored and constitutes an open and challenging problem. Motivated by this gap, the present article investigates the following model:

$$\begin{aligned}
 {}^C D_t^\alpha x_i(t) = & -a_i x_i(t) + \sum_{j=1}^n b_{ij}(t) f_j(x_j(t)) + \sum_{j=1}^n c_{ij}(t) u_j(t) + I_i(t) \\
 & + \bigwedge_{j=1}^n \alpha_{ij}^1(t) g_j(x_j(t - \tau_{ij}(t))) + \bigvee_{j=1}^n \alpha_{ij}^2(t) g_j(x_j(t - \tau_{ij}(t))) \\
 & + \bigwedge_{j=1}^n \alpha_{ij}^3(t) \int_0^{+\infty} K_{ij}(s) h_j(x_j(t - s)) ds + \bigvee_{j=1}^n \alpha_{ij}^4(t) \int_0^{+\infty} K_{ij}(s) h_j(x_j(t - s)) ds \\
 & + \bigwedge_{j=1}^n \beta_{ij}^1(t) u_j(t) + \bigvee_{j=1}^n \beta_{ij}^2(t) u_j(t), \quad i = 1, \dots, n,
 \end{aligned} \tag{1.1}$$

with initial conditions:

$$x_i(s) = \phi_i(s), \quad s \in (-\infty, t_0], \tag{1.2}$$

where $\phi_i(\cdot) \in C\left((-\infty, t_0], \mathbb{H}\right)$, such that $\mathbb{H}$ is the hypercomplex space.

The system's parameters are:

- The constant $a_i > 0$ characterizes the decay rate of the i -th neuronal unit.
- The hypercomplex-valued functions $\alpha_{ij}^1(\cdot)$, $\alpha_{ij}^2(\cdot)$, $\alpha_{ij}^3(\cdot)$, $\alpha_{ij}^4(\cdot)$, $\beta_{ij}^1(\cdot)$, and $\beta_{ij}^2(\cdot)$ represent the fuzzy MIN and MAX templates for feedback and feed-forward connections, respectively.
- The delay kernel $K_{ij}(\cdot) \in \mathbb{R}^+$ models the distribution of transmission delays in the network.
- Elements $b_{ij}(\cdot)$ and $c_{ij}(\cdot)$, both belonging to $\mathbb{H}$, constitute the feedback and feed-forward templates.
- The operators $\bigwedge$ and $\bigvee$ are the fuzzy AND and OR operations, respectively.

- The activation functions $f_j(\cdot)$, $g_j(\cdot)$, and $h_j(\cdot)$ take values in $\mathbb{H}$.
- Transmission delays are described by the positive-valued functions $\tau_{ij}(\cdot) \in \mathbb{R}^+$.
- The i^{th} neuron's state, external input, and bias are denoted by $x_i(\cdot)$, $u_i(\cdot)$, and $I_i(\cdot) \in \mathbb{H}$, respectively.

1.1 Main contributions of this work

The present work establishes several contributions:

- (i) Our paper proposes a novel class of hypercomplex fractional-order fuzzy cellular neural networks defined within Cayley-Dickson algebras, including quaternions, octonions, and higher-dimensional extensions.
- (ii) Establishment of sufficient conditions ensuring the existence and uniqueness of μ -pseudo almost automorphic solutions for the proposed class of networks. The existence result is derived via a fixed-point approach based on the Banach contraction principle.
- (iii) Derivation of sufficient conditions guaranteeing the finite-time stability of the obtained μ -pseudo almost automorphic solutions. This analysis relies on a generalized Gronwall inequality adapted to fractional-order systems.
- (iv) Unlike classical commutative frameworks, the proposed model explicitly addresses and overcomes the analytical difficulties arising from non-commutativity and non-associativity. This is achieved through the use of contraction coefficients based on structure constants.
- (v) Provision of two numerical simulations to illustrate and validate the theoretical findings.

Remark 1.1. The novelty of this work does not consist solely in extending the framework from real numbers to hypercomplex systems, but rather in addressing and overcoming the fundamental analytical challenges induced by non-commutativity (quaternions) and non-associativity (octonions). Table (1.1) illustrates how, as the dimension of numerical systems increases, there is a progressive weakening of the classical algebraic properties of multiplication:

Table 1. Properties of the multiplication operation in different numerical systems.

Number System	Commutativity	Associativity	Alternativeness
$\mathbb{R}$	+	+	+
$\mathbb{C}$	+	+	+
$\mathbb{Q}$	–	+	+
$\mathbb{O}$	–	–	+

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The remainder of this paper is organized as follows. Section 2 presents the necessary preliminaries on hypercomplex algebras, fractional calculus, and function spaces including μ -pseudo almost automorphic functions. Section 3 establishes the main results: the existence and uniqueness of μ -pseudo almost automorphic solutions via Banach contraction principle, followed by the uniform stability analysis. Section 4 provides a numerical simulation in quaternion space

to validate the theoretical findings. Finally, Section 5 concludes the paper and discusses future research directions.

2 Definitions and terminology

2.1 Hypercomplex algebras

A hypercomplex number a over the real field can be expressed as

$$a = a_0 + a_1 e_1 + a_2 e_2 + \cdots + a_m e_m,$$

where m is a positive integer, $a_0, a_1, \dots, a_m \in \mathbb{R}$, and $e_1, e_2, \dots, e_m$ are hyperimaginary units satisfying the multiplication rules:

$$e_i e_j = k_{ij}^0 + k_{ij}^1 e_1 + k_{ij}^2 e_2 + \cdots + k_{ij}^m e_m, \quad i, j = 1, 2, \dots, m,$$

where the real coefficients k_{ij}^ℓ , $\ell = 0, \dots, m$ are uniquely determined by subscripts i, j of hyperimaginary units e_i, e_j .

For two hypercomplex numbers

$$a = a_0 + a_1 e_1 + \cdots + a_m e_m, \quad b = b_0 + b_1 e_1 + \cdots + b_m e_m,$$

and any scalar $\lambda \in \mathbb{R}$, the algebraic operations are defined as follows:

Addition:

$$a + b = (a_0 + b_0) + (a_1 + b_1)e_1 + \cdots + (a_m + b_m)e_m.$$

Scalar multiplication:

$$\lambda a = (\lambda a_0) + (\lambda a_1)e_1 + \cdots + (\lambda a_m)e_m.$$

Multiplication: The product ab can be written using the structure constants k_{ij}^ℓ :

$$\begin{aligned} ab = & \left(a_0 b_0 + \sum_{i,j=1}^m a_i b_j k_{ij}^0 \right) + \left(a_0 b_1 + a_1 b_0 + \sum_{i,j=1}^m a_i b_j k_{ij}^1 \right) e_1 \\ & + \left(a_0 b_2 + a_2 b_0 + \sum_{i,j=1}^m a_i b_j k_{ij}^2 \right) e_2 + \cdots + \left(a_0 b_m + a_m b_0 + \sum_{i,j=1}^m a_i b_j k_{ij}^m \right) e_m. \end{aligned}$$

Order of multiplication: We define the product of three hypercomplex elements $a, b, c \in \mathbb{H}$, by left-associativity: $abc = (ab)c$, and assume that

$$\|ab\|_{\mathbb{H}} \leq \|a\|_{\mathbb{H}} \|b\|_{\mathbb{H}}.$$

Imaginary parts: For $a = a_0 + a_1 e_1 + a_2 e_2 + \cdots + a_m e_m \in \mathbb{H}$, we denote its pure imaginary part by

$$\tilde{a} = a_1 e_1 + a_2 e_2 + \cdots + a_m e_m.$$

Norms: For $a \in \mathbb{H}$, we define the norm

$$\|a\|_{\mathbb{H}} := |a_0| + |a_1| + \cdots + |a_m|.$$

and for $d = (d_1, \dots, d_n)^\top \in \mathbb{H}^n$, we set

$$\|d\|_{\mathbb{H}^n} := \max_{1 \leq i \leq n} \left\{ \|d_i\|_{\mathbb{H}} \right\}.$$

The pair $(\mathbb{H}, \|\cdot\|_{\mathbb{H}})$ constitutes a Banach space, and likewise $(\mathbb{H}^n, \|\cdot\|_{\mathbb{H}^n})$ is itself a Banach space when equipped with the induced norm.

Example 2.1. Octonions form a fundamental class of hypercomplex algebras. As an eight-dimensional, non-associative division algebra over the real numbers, they exhibit rich algebraic structures and possess remarkable mathematical properties. For practical computations, the multiplication rules of octonions are summarized in the following Table [19]:

Table 2. Multiplication table of octonions

$\times$	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_0	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	$-e_0$	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
e_2	e_2	$-e_3$	$-e_0$	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_3	e_2	$-e_1$	$-e_0$	e_7	$-e_6$	e_5	$-e_4$
e_4	e_4	$-e_5$	$-e_6$	$-e_7$	$-e_0$	e_1	e_2	e_3
e_5	e_5	e_4	$-e_7$	e_6	$-e_1$	$-e_0$	$-e_3$	e_2
e_6	e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	$-e_0$	$-e_1$
e_7	e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	$-e_0$

Due to the non-associativity of octonions, this table is essential, the product $e_i e_j$, is obtained from the entry at the intersection of row i and column j .

2.2 Fractional Calculus

Definition 2.2. [20] For a function $f \in C([t_0, \infty))$, $t \in \mathbb{R}^+$ and $\alpha \in \mathbb{R}^+$, the Riemann-Liouville fractional integral is given by:

$${}_{{t_0}}I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_{{t_0}}^t (t - \tau)^{\alpha-1} f(\tau) d\tau,$$

where

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt. \tag{2.1}$$

Definition 2.3. [20] Let $\alpha > 0$ and $m = \lceil \alpha \rceil$. For $f \in C^m([t_0, \infty), \mathbb{R})$, the Caputo fractional derivative of order α is defined as:

$${}_{{t_0}}^C D_t^\alpha f(t) = \frac{1}{\Gamma(m - \alpha)} \int_{{t_0}}^t (t - \tau)^{m-\alpha-1} f^{(m)}(\tau) d\tau.$$

Γ is defined in (2.1).

If $0 < \alpha \leq 1$ then

$${}_{{t_0}}^C D_t^\alpha f(t) = \frac{1}{\Gamma(1 - \alpha)} \int_{{t_0}}^t (t - \tau)^{-\alpha} f'(\tau) d\tau.$$

When $\alpha = n \in \mathbb{N}$:

$${}_{{t_0}}^C D_t^n f(t) = f^{(n)}(t).$$

Remark 2.4. The Caputo fractional derivative exhibits linearity, vanishes when applied to constants, and generally does not possess the semigroup property, distinguishing it from classical integer-order differentiation.

2.3 Functions space

For a Banach space $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$, let $C(\mathbb{R}, \mathcal{Y})$ be the set of all continuous functions from $\mathbb{R}$ to $\mathcal{Y}$.

Definition 2.5. [21] A function $f \in C(\mathbb{R}, \mathcal{Y})$ is called almost automorphic if for every real sequence $(\mathcal{T}_n)_{n \in \mathbb{N}}$, there exists a subsequence $(s_n)_{n \in \mathbb{N}}$ such that $\lim_{n \rightarrow \infty} f(t + s_n)$ exists for each $t \in \mathbb{R}$. Denote this limit by $g(t)$. Conversely, for every $t \in \mathbb{R}$, $\lim_{n \rightarrow \infty} g(t - s_n) = f(t)$. The collection of all such functions is denoted by $AA(\mathbb{R}, \mathcal{Y})$.

Theorem 2.6. [21] The function space $AA(\mathbb{R}, \mathcal{Y})$, equipped with the norm $\|\cdot\|_{\infty}$, constitutes a Banach space.

Let $\mathcal{B}$ be the Lebesgue σ -algebra on $\mathbb{R}$. Define

$$\mathcal{M} = \left\{ \mu > 0 \text{ on } \mathcal{B} : \mu(\mathbb{R}) = \infty, \mu([a, b]) < \infty, \forall a, b \in \mathbb{R} \right\}.$$

Within $\mathcal{M}$, we consider the subclass of measures μ that satisfy the following hypotheses:

(C.0) :

$$\left\{ \begin{array}{l} \text{(i)} \quad \forall a, b, c \in \mathbb{R}, 0 \leq a < b \leq c, \exists \varepsilon_0 > 0, \delta_0 > 0 \text{ such that} \\ \quad |\theta| \geq \varepsilon_0 \implies \mu((a + \theta, b + \theta)) \geq \delta_0 \mu([\theta, c + \theta]), \\ \text{(ii)} \quad \forall \theta \in \mathbb{R}, \exists \theta_1 > 0 \text{ and a bounded interval } J \text{ such that} \\ \quad \mu(\{c_1 + \theta : c_1 \in J_1\}) \leq \theta_1 \mu(J_1), \quad \text{for all } J_1 \in \mathcal{B} \text{ such that } J \cap J_1 = \emptyset. \end{array} \right.$$

Remark 2.7. [22] Hypothesis (ii) implies (i).

Definition 2.8. [22] For a fixed measure $\mu \in \mathcal{M}$:

(i) A bounded continuous function $f : \mathbb{R} \rightarrow \mathcal{Y}$ is μ -ergodic if

$$\lim_{r \rightarrow +\infty} \frac{1}{\mu([-r, r])} \int_{[-r, r]} \|f(t)\| d\mu(t) = 0.$$

Denote by $\xi(\mathbb{R}, \mathcal{Y}, \mu)$ the corresponding space.

(ii) A continuous function $f : \mathbb{R} \rightarrow \mathcal{Y}$ is μ -pseudo almost automorphic if

$$f = f_1 + f_2$$

with $f_1 \in AA(\mathbb{R}, \mathcal{Y})$ (almost automorphic) and $f_2 \in \xi(\mathbb{R}, \mathcal{Y}, \mu)$ (μ -ergodic). This space is denoted $PAA(\mathbb{R}, \mathcal{Y}, \mu)$.

Theorem 2.9. [22] Let $\mathcal{M}$ satisfy (C.0). The function space $PAA(\mathbb{R}, \mathcal{Y}, \mu)$, equipped with the norm $\|\cdot\|_{\infty}$, constitutes a Banach space.

2.4 Fundamental assumptions and basic lemmas

For the remainder of this work, we introduce the following notation. Let $h : \mathbb{R} \rightarrow \mathbb{H}$ be a bounded and continuous function, where $\mathbb{H}$ denotes a hypercomplex Banach space equipped with the norm $\|\cdot\|_{\mathbb{H}}$. We define

$$h^* := \sup_{t \in \mathbb{R}} \|h(t)\|_{\mathbb{H}} < +\infty.$$

In addition, throughout the sequel, we assume that hypotheses (C.1), (C.2) and (C.3) are satisfied.

(C.1) Let the parameter functions $b_{ij}(\cdot)$, $c_{ij}(\cdot)$, $\beta_{ij}^1(\cdot)$, $\beta_{ij}^2(\cdot)$, $I_i(\cdot)$, $u_i(\cdot)$ belong to the space $PAA(\mathbb{R}, \mathbb{H}, \mu)$, while the functions $\tau_{ij}(\cdot)$, $\alpha_{ij}^1(\cdot)$, $\alpha_{ij}^2(\cdot)$, $\alpha_{ij}^3(\cdot)$, $\alpha_{ij}^4(\cdot)$ belong to the space $AA(\mathbb{R}, \mathbb{H})$.

(C.2) Let $K_{ij}(\cdot) : [0, +\infty) \rightarrow \mathbb{R}$ be a continuous and Lebesgue integrable kernel. For analytical convenience we assume that

$$\int_0^{+\infty} K_{ij}(m) dm = \int_{-\infty}^s K_{ij}(s-m) ds = 1, \quad 1 \leq i < j \leq n.$$

(C.3) For all $1 \leq j \leq n$, let the nonlinear functions $f_j(\cdot)$, $g_j(\cdot)$, $h_j(\cdot) \in C(\mathbb{H}, \mathbb{H})$. Assume that there exist positive Lipschitz constant $L_j^f, L_j^g, L_j^h > 0$, such that for all $\varrho_1, \varrho_2 \in \mathbb{H}$:

$$\begin{aligned} \|f_j(\varrho_1) - f_j(\varrho_2)\|_{\mathbb{H}} &\leq L_j^f \|\varrho_1 - \varrho_2\|_{\mathbb{H}}, \quad \|g_j(\varrho_1) - g_j(\varrho_2)\|_{\mathbb{H}} \leq L_j^g \|\varrho_1 - \varrho_2\|_{\mathbb{H}}, \\ \|h_j(\varrho_1) - h_j(\varrho_2)\|_{\mathbb{H}} &\leq L_j^h \|\varrho_1 - \varrho_2\|_{\mathbb{H}}, \end{aligned}$$

Moreover, the activation functions satisfy the zero-condition at the origin.

The proofs of our results rely on the following lemmas

Lemma 2.10. [22] Let $\mu \in \mathcal{M}$ satisfy condition (C.0). Then the space $PAA(\mathbb{R}, \mathbb{H}, \mu)$ is translation invariant; i.e., for every $f \in PAA(\mathbb{R}, \mathbb{H}, \mu)$ and every $\lambda \in \mathbb{R}$, the function

$$f_\lambda(t) := f(t + \lambda)$$

also belongs to $PAA(\mathbb{R}, \mathbb{H}, \mu)$.

Lemma 2.11. [22] Assume that $\mu \in \mathcal{M}$ fulfills (C.0). If $\phi(\cdot)$ and $\psi(\cdot)$ are both in $PAA(\mathbb{R}, \mathbb{H}, \mu)$, then their product $(\phi\psi)(\cdot)$ also belongs to this space.

Lemma 2.12. Assume that Hypothesis (C.3) holds. Suppose that:

- (i) $x : \mathbb{R} \rightarrow \mathbb{H}$ belongs to $PAA(\mathbb{R}, \mathbb{H}, \mu)$, the space of μ -pseudo almost automorphic functions.
- (ii) $\tau : \mathbb{R} \rightarrow \mathbb{R}^+$ is continuous, bounded, and the function $\theta(t) := t - \tau(t)$ is a C^1 -diffeomorphism from $\mathbb{R}$ onto $\mathbb{R}$ satisfying $\inf_{t \in \mathbb{R}} \theta'(t) > 0$. Equivalently, $1 - \tau'(t) \geq \delta > 0$ for all $t \in \mathbb{R}$.

Then the composed function with time-varying delay

$$t \mapsto f(x(t - \tau(t)))$$

belongs to $PAA(\mathbb{R}, \mathbb{H}, \mu)$.

Remark 2.13. By the uniqueness of the decomposition in [22], we write $x = x^a + x^\mu$ with $x^a \in AA(\mathbb{R}, \mathbb{H})$ and $x^\mu \in \xi(\mathbb{R}, \mathbb{H}, \mu)$. Define

$$\Xi(t) = \underbrace{f(x^a(t - \tau(t)))}_{\Xi_1(t)} + \underbrace{[f(x^a(t - \tau(t)) + x^\mu(t - \tau(t))) - f(x^a(t - \tau(t)))]}_{\Xi_2(t)}.$$

Then $\Xi_1 \in AA(\mathbb{R}, \mathbb{H})$ and $\Xi_2 \in \xi(\mathbb{R}, \mathbb{H}, \mu)$ (see [23]). Hence $\Xi \in PAA(\mathbb{R}, \mathbb{H}, \mu)$.

Lemma 2.14. Under assumptions (C.1) – (C.3) and $x_{ij}(\cdot) \in PAA(\mathbb{R}, \mathbb{H}, \mu)$, the function

$$\Pi(t) = \int_0^{+\infty} K_{ij}(s) h_j(x_{ij}(t-s)) ds \quad 1 \leq i, j \leq n,$$

belongs to $PAA(\mathbb{R}, \mathbb{H}, \mu)$.

Remark 2.15. By Lemma 2.12, $h_j(x_{ij}(\cdot)) \in PAA(\mathbb{R}, \mathbb{H}, \mu)$. Thus there exist $p_{ij} \in AA(\mathbb{R}, \mathbb{H})$ and $q_{ij} \in \xi(\mathbb{R}, \mathbb{H}, \mu)$ such that $h_j(x_{ij}(s)) = p_{ij}(s) + q_{ij}(s)$. Consequently,

$$\Pi(t) = \int_{-\infty}^t K_{ij}(t-s) h_j(x_{ij}(s)) ds = \underbrace{\int_{-\infty}^t K_{ij}(t-s) p_{ij}(s) ds}_{P_{ij}(t)} + \underbrace{\int_{-\infty}^t K_{ij}(t-s) q_{ij}(s) ds}_{Q_{ij}(t)}.$$

One has $P_{ij} \in AA(\mathbb{R}, \mathbb{H})$ and $Q_{ij} \in \xi(\mathbb{R}, \mathbb{H}, \mu)$ (see Theorem 1 in [23]). Hence $\Pi \in PAA(\mathbb{R}, \mathbb{H}, \mu)$.

Lemma 2.16. Let $\mu \in \mathcal{M}$ satisfy condition (C.0). Assume that for all $i, j \in \{1, \dots, n\}$, the functions $\alpha_{ij}^1(\cdot), \alpha_{ij}^3(\cdot)$ belong to $AA(\mathbb{R}, \mathbb{H})$, the delay functions $\tau_{ij}(\cdot) \in AA(\mathbb{R}, \mathbb{R}^+)$, the activation functions $g_j, h_j : \mathbb{H} \rightarrow \mathbb{H}$ are Lipschitz continuous, and $x_j \in PAA(\mathbb{R}, \mathbb{H}, \mu)$. Then, for each $i = 1, \dots, n$, the following functions belong to $PAA(\mathbb{R}, \mathbb{H}, \mu)$:

$$\Phi_i(t) := \bigwedge_{j=1}^n \alpha_{ij}^1(t) g_j(x_j(t - \tau_{ij}(t))), \quad \Psi_i(t) := \bigwedge_{j=1}^n \alpha_{ij}^3(t) \int_0^\infty K_{ij}(s) h_j(x_j(t-s)) ds.$$

Remark 2.17. Lemma 2.16 can be proved by adapting the reasoning of Lemma 2.9 in [11], using the composition properties established in Lemma 2.12 and the closure of PAA under Lipschitz functions and fuzzy operations.

3 Main results

3.1 Existence of μ -pseudo almost automorphic solutions

Definition 3.1. [24] A function $x = (x_1, \dots, x_n)^T \in C(\mathbb{R}, \mathbb{H})$ is said to be a mild solution of (1.1) if it satisfies

$$\begin{aligned} x_i(t) &= S_i(t-t_0)x_i(t_0) \\ &+ \int_{t_0}^t (t-s)^{\alpha-1} \mathcal{K}_i(t-s) \left(\sum_{j=1}^n b_{ij}(s) f_j(\phi_j(s)) + \sum_{j=1}^n c_{ij}(s) u_j(s) + I_i(s) \right. \\ &+ \bigwedge_{j=1}^n \alpha_{ij}^1(s) g_j(\phi_j(s - \tau_{ij}(s))) + \bigvee_{j=1}^n \alpha_{ij}^2(s) g_j(\phi_j(s - \tau_{ij}(s))) \\ &+ \bigwedge_{j=1}^n \alpha_{ij}^3(s) \int_0^{+\infty} K_{ij}(m) h_j(\phi_j(s-m)) dm \\ &+ \bigvee_{j=1}^n \alpha_{ij}^4(s) \int_0^{+\infty} K_{ij}(m) h_j(\phi_j(s-m)) dm \\ &\left. + \bigwedge_{j=1}^n \beta_{ij}^1(s) u_j(s) + \bigvee_{j=1}^n \beta_{ij}^2(s) u_j(s) \right) ds, \quad t \geq t_0, \end{aligned} \quad (3.1)$$

where

$$\begin{aligned} \zeta_\alpha(\theta) &= \frac{1}{\alpha} \theta^{-1-\frac{1}{\alpha}} p_\alpha(\theta^{-\frac{1}{\alpha}}), \\ S_i(t) &= \int_0^\infty \zeta_\alpha(\theta) e^{-a_i t^\alpha \theta} d\theta, \\ \mathcal{K}_i(t) &= \alpha \int_0^\infty \theta \zeta_\alpha(\theta) e^{-a_i t^\alpha \theta} d\theta, \\ p_\alpha(\theta) &= \frac{1}{\pi} \sum_{k=1}^\infty (-1)^{k-1} \theta^{-\alpha k-1} \frac{\Gamma(\alpha k+1)}{k!} \sin(k\pi\alpha), \quad \theta \in (0, \infty), \end{aligned}$$

and

$$\zeta_\alpha(\theta) \geq 0, \theta \in (0, \infty), \quad \int_0^{+\infty} \zeta_\alpha(\theta) d\theta = 1, \quad \int_0^{+\infty} \theta \zeta_\alpha(\theta) d\theta = \frac{1}{\Gamma(\alpha + 1)}.$$

Letting $t_0 \rightarrow \infty$, via (3.1) we have

$$\begin{aligned} x_i(t) &= \int_{-\infty}^t (t-s)^{\alpha-1} \mathcal{K}_i(t-s) \left(\sum_{j=1}^n b_{ij}(s) f_j(\phi_j(s)) + \sum_{j=1}^n c_{ij}(s) u_j(s) + I_i(s) \right. \\ &\quad + \bigwedge_{j=1}^n \alpha_{ij}^1(s) g_j(\phi_j(s - \tau_{ij}(s))) + \bigvee_{j=1}^n \alpha_{ij}^2(s) g_j(\phi_j(s - \tau_{ij}(s))) \\ &\quad + \bigwedge_{j=1}^n \alpha_{ij}^3(s) \int_0^{+\infty} K_{ij}(m) h_j(\phi_j(s-m)) dm \\ &\quad + \bigvee_{j=1}^n \alpha_{ij}^4(s) \int_0^{+\infty} K_{ij}(m) h_j(\phi_j(s-m)) dm \\ &\quad \left. + \bigwedge_{j=1}^n \beta_{ij}^1(s) u_j(s) + \bigvee_{j=1}^n \beta_{ij}^2(s) u_j(s) \right) ds, \quad 1 \leq i \leq n. \end{aligned} \quad (3.2)$$

Lemma 3.2. Assume that (C.0)-(C.3) hold. If $\phi = (\phi_1, \dots, \phi_n)^\top \in PAA(\mathbb{R}, \mathbb{H}^n, \mu)$, then function $\Omega_{\phi_i}(t) : \mathbb{R} \rightarrow \mathbb{H}$ defined by

$$\Omega_{\phi_i}(t) = \int_{-\infty}^t (t-s)^{\alpha-1} \mathcal{K}_i(t-s) \Lambda_i(s, \phi) ds, \quad 1 \leq i \leq n, \quad (3.3)$$

where

$$\begin{aligned} \Lambda_i(s, \phi) &= \sum_{j=1}^n b_{ij}(s) f_j(\phi_j(s)) + \sum_{j=1}^n c_{ij}(s) u_j(s) + I_i(s) \\ &\quad + \bigwedge_{j=1}^n \alpha_{ij}^1(s) g_j(\phi_j(s - \tau_{ij}(s))) + \bigvee_{j=1}^n \alpha_{ij}^2(s) g_j(\phi_j(s - \tau_{ij}(s))) \\ &\quad + \bigwedge_{j=1}^n \alpha_{ij}^3(s) \int_0^{+\infty} K_{ij}(m) h_j(\phi_j(s-m)) dm \\ &\quad + \bigvee_{j=1}^n \alpha_{ij}^4(s) \int_0^{+\infty} K_{ij}(m) h_j(\phi_j(s-m)) dm \\ &\quad + \bigwedge_{j=1}^n \beta_{ij}^1(s) u_j(s) + \bigvee_{j=1}^n \beta_{ij}^2(s) u_j(s), \end{aligned}$$

belong to $PAA(\mathbb{R}, \mathbb{H}^n, \mu)$.

For all $1 \leq i \leq n$, it follows from Lemmas 2.10, 2.11, 2.12, 2.14, and 2.16 that $\Lambda_i(\cdot) \in PAA(\mathbb{R}, \mathbb{H}^n, \mu)$. Consequently, we can decompose Λ_i as

$$\Lambda_i(\cdot) = \Lambda_i^1(\cdot) + \Lambda_i^2(\cdot), \quad (3.4)$$

where $\Lambda_i^1(\cdot) \in AA(\mathbb{R}, \mathbb{H})$ and $\Lambda_i^2(\cdot) \in \xi(\mathbb{R}, \mathbb{H}, \mu)$. To prove that the fractional integral preserves μ -pseudo almost automorphy, we treat the almost automorphic and μ -ergodic components separately. Using (3.4), write

$$\Omega_{\phi_i}(t) = \left[\underbrace{\int_{-\infty}^t (t-s)^{\alpha-1} \mathcal{K}_i(t-s) \Lambda_i^1(s) ds}_{\Omega_{\phi_i}^1(t)} + \underbrace{\int_{-\infty}^t (t-s)^{\alpha-1} \mathcal{K}_i(t-s) \Lambda_i^2(s) ds}_{\Omega_{\phi_i}^2(t)} \right].$$

Step 1: We prove that $\Omega_{\phi_i}^1(\cdot) \in AA(\mathbb{R}, \mathbb{H})$.

By definition, for every real sequence $(S_n)_{n \in \mathbb{N}}$, there exists a subsequence $(s_n)_{n \in \mathbb{N}}$ such that

$$\lim_{n \rightarrow \infty} \|\Lambda_i^1(t + s_n) - \bar{\Lambda}_i^1(t)\|_{\mathbb{H}} = 0, \quad t \in \mathbb{R},$$

and

$$\lim_{n \rightarrow \infty} \|\bar{\Lambda}_i^1(t - s_n) - \Lambda_i^1(t)\|_{\mathbb{H}} = 0, \quad t \in \mathbb{R}.$$

Fix $\varepsilon > 0$. Then there exists $N_0 \in \mathbb{N}$ such that for $n > N_0$, and $\forall t \in \mathbb{R}$

$$\|\Lambda_i^1(t + s_n) - \bar{\Lambda}_i^1(t)\|_{\mathbb{H}} < \varepsilon.$$

Define the candidate limit function:

$$\bar{\Omega}_{\phi_i}^1(t) = \int_{-\infty}^t (t-s)^{\alpha-1} \mathcal{K}_i(t-s) \bar{\Lambda}_i^1(s) ds.$$

For $n > N_0$, we can see that

$$\begin{aligned} \|\Omega_{\phi_i}^1(t + s_n) - \bar{\Omega}_{\phi_i}^1(t)\|_{\mathbb{H}} &= \left\| \int_{-\infty}^{t+s_n} (t+s_n-s)^{\alpha-1} \mathcal{K}_i(t+s_n-s) \Lambda_i^1(s) ds \right. \\ &\quad \left. - \int_{-\infty}^t (t-s)^{\alpha-1} \mathcal{K}_i(t-s) \bar{\Lambda}_i^1(s) ds \right\|_{\mathbb{H}}. \end{aligned}$$

Let $u = s - s_n$, then

$$\left\| \Omega_{\phi_i}^1(t + s_n) - \bar{\Omega}_{\phi_i}^1(t) \right\|_{\mathbb{H}} = \int_{-\infty}^t (t-u)^{\alpha-1} \mathcal{K}_i(t-u) \left\| \Lambda_i^1(u + s_n) - \bar{\Lambda}_i^1(u) \right\|_{\mathbb{H}} du$$

By the choice of N_0 , for $n > N_0$ and all $u \in \mathbb{R}$, we have

$$\left\| \Omega_{\phi_i}^1(t + s_n) - \bar{\Omega}_{\phi_i}^1(t) \right\|_{\mathbb{H}} \leq \varepsilon \int_{-\infty}^t (t-u)^{\alpha-1} \mathcal{K}_i(t-u) du.$$

A standard estimate yields:

$$\int_{-\infty}^t (t-u)^{\alpha-1} \mathcal{K}_i(t-u) du < \frac{1}{a_i}. \quad (3.5)$$

Therefore,

$$\lim_{n \rightarrow \infty} \left\| \Omega_{\phi_i}^1(t + s_n) - \bar{\Omega}_{\phi_i}^1(t) \right\|_{\mathbb{H}} \leq \varepsilon \frac{1}{a_i}.$$

Since ε is arbitrary, we conclude

$$\lim_{n \rightarrow \infty} \|\Omega_{\phi_i}^1(t + s_n) - \bar{\Omega}_{\phi_i}^1(t)\|_{\mathbb{H}} = 0, \quad \forall t \in \mathbb{R}.$$

A completely analogous argument shows:

$$\lim_{n \rightarrow \infty} \|\bar{\Omega}_{\phi_i}^1(t - s_n) - \Omega_{\phi_i}^1(t)\|_{\mathbb{H}} = 0 \quad \forall t \in \mathbb{R}.$$

Thus, by Definition 2.5 $\Omega_{\phi_i}^1(\cdot) \in AA(\mathbb{R}, \mathbb{H})$.

Step 2: We prove that $\Omega_{\phi_i}^2(\cdot) \in \xi(\mathbb{R}, \mathbb{H}, \mu)$.

Let $\Lambda_i^2 \in \xi(\mathbb{R}, \mathbb{H}, \mu)$. By definition,

$$\lim_{r \rightarrow \infty} \frac{1}{\mu([-r, r])} \int_{[-r, r]} \|\Lambda_i^2(s)\|_{\mathbb{H}} d\mu(s) = 0.$$

We start from the expression of $\Omega_{\phi_i}^2$ after the change of variable $u = t - s$:

$$\Omega_{\phi_i}^2(t) = \int_0^\infty u^{\alpha-1} \mathcal{K}_i(u) \Lambda_i^2(t-u) du.$$

Hence,

$$\|\Omega_{\phi_i}^2(t)\|_{\mathbb{H}} \leq \int_0^\infty u^{\alpha-1} \mathcal{K}_i(u) \|\Lambda_i^2(t-u)\|_{\mathbb{H}} du. \quad (3.6)$$

Integrating (3.6) over $t \in [-r, r]$ with respect to μ and applying Fubini's theorem we obtain:

$$\int_{[-r, r]} \|\Omega_{\phi_i}^2(t)\|_{\mathbb{H}} d\mu(t) \leq \int_0^\infty u^{\alpha-1} \mathcal{K}_i(u) \left(\int_{[-r, r]} \|\Lambda_i^2(t-u)\|_{\mathbb{H}} d\mu(t) \right) du. \quad (3.7)$$

Divide both sides by $\mu([-r, r])$:

$$\frac{1}{\mu([-r, r])} \int_{[-r, r]} \|\Omega_{\phi_i}^2(t)\|_{\mathbb{H}} d\mu(t) \leq \int_0^\infty u^{\alpha-1} \mathcal{K}_i(u) \Phi_r(u) du, \quad (3.8)$$

where

$$\Phi_r(u) := \frac{1}{\mu([-r, r])} \int_{[-r, r]} \|\Lambda_i^2(t-u)\|_{\mathbb{H}} d\mu(t). \quad (3.9)$$

Since $\Lambda_i^2 \in \xi(\mathbb{R}, \mathbb{H}, \mu)$ and ξ is translation-invariant under the assumptions on μ (see [22]), for each fixed $u \geq 0$, the function $t \mapsto \Lambda_i^2(t-u)$ also belongs to ξ . Hence,

$$\lim_{r \rightarrow \infty} \Phi_r(u) = 0 \quad \forall u \geq 0.$$

Moreover, $\Phi_r(u) \leq \sup_{t \in \mathbb{R}} \|\Lambda_i^2(t)\|_{\mathbb{H}} < \infty$. Therefore, applying the Lebesgue Dominated Convergence Theorem to (3.8), we conclude:

$$\lim_{r \rightarrow \infty} \frac{1}{\mu([-r, r])} \int_{[-r, r]} \|\Omega_{\phi_i}^2(t)\|_{\mathbb{H}} d\mu(t) = 0. \quad (3.10)$$

Hence $\Omega_{\phi_i}^2 \in \xi(\mathbb{R}, \mathbb{H}, \mu)$. Via steps 1 and 2, $\Omega_{\phi_i}(\cdot) \in PAA(\mathbb{R}, \mathbb{H}^n, \mu)$.

Lemma 3.3. Let $x_j, y_j \in \mathbb{H}$ and let $\alpha_{ij}^1(\cdot), \alpha_{ij}^2(\cdot) : \mathbb{R} \rightarrow \mathbb{H}$ be continuous functions. For each $t \in \mathbb{R}$ and for any activation functions $g_j : \mathbb{H} \rightarrow \mathbb{H}$, we have

$$\begin{aligned} \left\| \bigwedge_{j=1}^n \alpha_{ij}^1(t) g_j(x_j) - \bigwedge_{j=1}^n \alpha_{ij}^1(t) g_j(y_j) \right\|_{\mathbb{H}} &\leq \sum_{j=1}^n \|\alpha_{ij}^1(t)\|_{\mathbb{H}} \|g_j(x_j) - g_j(y_j)\|_{\mathbb{H}}, \\ \left\| \bigvee_{j=1}^n \alpha_{ij}^2(t) g_j(x_j) - \bigvee_{j=1}^n \alpha_{ij}^2(t) g_j(y_j) \right\|_{\mathbb{H}} &\leq \sum_{j=1}^n \|\alpha_{ij}^2(t)\|_{\mathbb{H}} \|g_j(x_j) - g_j(y_j)\|_{\mathbb{H}}. \end{aligned}$$

Remark 3.4. Lemma 3.3 is inspired by the fundamental result established in Corollary 1 of paper [14], which provides a Lipschitz-type estimate for fuzzy MIN and MAX operations in real-valued fuzzy cellular neural networks. The adaptation to the hypercomplex setting presented here replaces absolute values with the hypercomplex norm $\|\cdot\|_{\mathbb{H}}$ and extends the inequality to vector-valued activation functions while preserving the linear dependence on the norms of the fuzzy template coefficients. This generalization is essential for the contraction argument developed in the proof of Theorem 3.5.

For any function $\phi : \mathbb{R} \rightarrow \mathbb{H}^n$, we define the uniform norm

$$\|\phi\|_u := \sup_{t \in \mathbb{R}} \|\phi(t)\|_{\mathbb{H}^n}, \quad \text{where } \|\phi(t)\|_{\mathbb{H}^n} := \max_{1 \leq i \leq n} \|\phi_i(t)\|_{\mathbb{H}}.$$

Theorem 3.5. *Under hypotheses (C.0)–(C.3), let*

$$\beta_1 := \max_{1 \leq i \leq n} \frac{1}{a_i} \left(\sum_{j=1}^n b_{ij}^* L_j^f + \sum_{j=1}^n (\alpha_{ij}^{1*} + \alpha_{ij}^{2*}) L_j^g + \sum_{j=1}^n (\alpha_{ij}^{3*} + \alpha_{ij}^{4*}) L_j^h \right).$$

If $\beta_1 < 1$, then (1.1) possesses a unique μ -pseudo almost automorphic solution belonging to

$$\nabla = \left\{ \phi \in PAA(\mathbb{R}, \mathbb{H}^n, \mu) : \|\phi - \phi_0\|_u \leq \frac{\beta_0 \beta_1}{1 - \beta_1} \right\},$$

with $\phi_0 \in PAA(\mathbb{R}, \mathbb{H}^n, \mu)$, and $\beta_0 > 0$ defined in the proof.

Recall that ϕ_0 is defined by

$$\phi_0(t) = \int_{-\infty}^t (t-s)^{\alpha-1} \mathcal{K}_i(t-s) G_i(s) ds,$$

where

$$G_i(s) = \sum_{j=1}^n c_{ij}(s) u_j(s) + I_i(s) + \bigwedge_{j=1}^n \beta_{ij}^1(s) u_j(s) + \bigvee_{j=1}^n \beta_{ij}^2(s) u_j(s).$$

We obtain

$$\|\phi_0\|_u \leq \max_{1 \leq i \leq n} \frac{1}{a_i} \left\{ \sum_{j=1}^n c_{ij}^* u_j^* + I_i^* + \sum_{j=1}^n (\beta_{ij}^{1*} + \beta_{ij}^{2*}) u_j^* \right\} = \beta_0.$$

Define the set

$$\nabla = \left\{ \phi \in PAA(\mathbb{R}, \mathbb{H}^n, \mu), \|\phi - \phi_0\|_u \leq \frac{\beta_0 \beta_1}{1 - \beta_1} \right\}$$

a closed convex subset of $PAA(\mathbb{R}, \mathbb{H}^n, \mu)$.

Therefore, for any $\phi(\cdot) \in \nabla$, we have

$$\begin{aligned} \left\| \Omega_\phi - \phi_0 \right\|_{\mathbb{H}} &= \left\| \int_{-\infty}^t (t-s)^{\alpha-1} \mathcal{K}_i(t-s) \left(\sum_{j=1}^n b_{ij}(s) f_j(\phi_j(s)) \right. \right. \\ &\quad + \bigwedge_{j=1}^n \alpha_{ij}^1(s) g_j(\phi_j(s - \tau_{ij}(s))) + \bigvee_{j=1}^n \alpha_{ij}^2(s) g_j(\phi_j(s - \tau_{ij}(s))) \\ &\quad + \bigwedge_{j=1}^n \alpha_{ij}^3(s) \int_0^\infty K_{ij}(m) h_j(\phi_j(s - m)) dm \\ &\quad \left. \left. + \bigvee_{j=1}^n \alpha_{ij}^4(s) \int_0^\infty K_{ij}(m) h_j(\phi_j(s - m)) dm \right) ds \right\|_{\mathbb{H}} \\ &\leq \int_{-\infty}^t (t-s)^{\alpha-1} \mathcal{K}_i(t-s) \left\{ \sum_{j=1}^n \|b_{ij}(s)\|_{\mathbb{H}} \|f_j(\phi_j(s))\|_{\mathbb{H}} \right. \\ &\quad \left. + \bigwedge_{j=1}^n \|\alpha_{ij}^1(s)\|_{\mathbb{H}} \|g_j(\phi_j(s - \tau_{ij}(s)))\|_{\mathbb{H}} \right. \end{aligned}$$

$$\begin{aligned}
& + \bigvee_{j=1}^n \|\alpha_{ij}^2(s)\|_{\mathbb{H}} \|g_j(\phi_j(s - \tau_{ij}(s)))\|_{\mathbb{H}} \\
& + \bigwedge_{j=1}^n \|\alpha_{ij}^3(s)\|_{\mathbb{H}} \int_0^\infty |K_{ij}(m)| \|h_j(\phi_j(s - m))\|_{\mathbb{H}} dm \\
& + \bigvee_{j=1}^n \|\alpha_{ij}^4(s)\|_{\mathbb{H}} \int_0^\infty |K_{ij}(m)| \|h_j(\phi_j(s - m))\|_{\mathbb{H}} dm \Big\} ds.
\end{aligned}$$

An application of Lemma 3.3, combined with assumption (C.3), leads to

$$\begin{aligned}
\|\Omega_\phi - \phi_0\|_{\mathbb{H}} & \leq \int_{-\infty}^t (t-s)^{\alpha-1} \mathcal{K}_i(t-s) \left\{ \sum_{j=1}^n \|b_{ij}(s)\|_{\mathbb{H}} L_j^f \|\phi_j(s)\|_{\mathbb{H}} \right. \\
& + \sum_{j=1}^n \left(\|\alpha_{ij}^1(s)\|_{\mathbb{H}} + \|\alpha_{ij}^2(s)\|_{\mathbb{H}} \right) L_j^g \|\phi_j(s - \tau_{ij}(s))\|_{\mathbb{H}} \\
& + \left. \sum_{j=1}^n \left(\|\alpha_{ij}^3(s)\|_{\mathbb{H}} + \|\alpha_{ij}^4(s)\|_{\mathbb{H}} \right) L_j^h \int_0^{+\infty} |K_{ij}(m)| \|\phi_j(s - m)\|_{\mathbb{H}} dm \right\} ds
\end{aligned}$$

Then

$$\begin{aligned}
\|\Omega_\phi - \phi_0\|_{\mathbb{H}} & \leq \int_{-\infty}^t (t-s)^{\alpha-1} \mathcal{K}_i(t-s) \left\{ \sum_{j=1}^n b_{ij}^* L_j^f \|\phi\|_u \right. \\
& + \sum_{j=1}^n (\alpha_{ij}^{1*} + \alpha_{ij}^{2*}) L_j^g \|\phi\|_u \\
& + \left. \sum_{j=1}^n (\alpha_{ij}^{3*} + \alpha_{ij}^{4*}) L_j^h \|\phi\|_u \underbrace{\int_0^{+\infty} |K_{ij}(m)| dm}_{=1} \right\} ds \\
& \leq \|\phi\|_u \int_{-\infty}^t (t-s)^{\alpha-1} \mathcal{K}_i(t-s) ds \\
& \times \left\{ \sum_{j=1}^n b_{ij}^* L_j^f + \sum_{j=1}^n (\alpha_{ij}^{1*} + \alpha_{ij}^{2*}) L_j^g + \sum_{j=1}^n (\alpha_{ij}^{3*} + \alpha_{ij}^{4*}) L_j^h \right\} \\
& \leq \frac{1}{a_i} \cdot \|\phi\|_u \cdot \left\{ \sum_{j=1}^n b_{ij}^* L_j^f + \sum_{j=1}^n (\alpha_{ij}^{1*} + \alpha_{ij}^{2*}) L_j^g + \sum_{j=1}^n (\alpha_{ij}^{3*} + \alpha_{ij}^{4*}) L_j^h \right\}.
\end{aligned}$$

Taking the supremum over t and then the maximum over i , we get:

$$\|\Omega_\phi - \phi_0\|_u \leq \beta_1 \|\phi\|_u,$$

where

$$\beta_1 := \max_{1 \leq i \leq n} \frac{1}{a_i} \left(\sum_{j=1}^n b_{ij}^* L_j^f + \sum_{j=1}^n (\alpha_{ij}^{1*} + \alpha_{ij}^{2*}) L_j^g + \sum_{j=1}^n (\alpha_{ij}^{3*} + \alpha_{ij}^{4*}) L_j^h \right).$$

But

$$\|\phi\|_u \leq \|\phi - \phi_0\|_u + \|\phi_0\|_u \leq \frac{\beta_0 \beta_1}{1 - \beta_1} + \beta_0 = \frac{\beta_0}{1 - \beta_1}.$$

Therefore,

$$\|\Omega\phi - \phi_0\|_u \leq \beta_1 \cdot \frac{\beta_0}{1 - \beta_1} = \frac{\beta_0\beta_1}{1 - \beta_1}.$$

Hence, $\Omega(\nabla) \subseteq \nabla$.

Take any two solutions $\phi(\cdot), \psi(\cdot) \in \nabla$, we obtain:

$$\begin{aligned} & \|\Omega\phi - \Omega\psi\|_{\mathbb{H}} \\ & \leq \int_{-\infty}^t (t-s)^{\alpha-1} \mathcal{K}_i(t-s) \left\{ \sum_{j=1}^n \|b_{ij}(s)\|_{\mathbb{H}} L_j^f \|\phi_j(s) - \psi_j(s)\|_{\mathbb{H}} \right. \\ & + \sum_{j=1}^n \left(\|\alpha_{ij}^1(s)\|_{\mathbb{H}} + \|\alpha_{ij}^2(s)\|_{\mathbb{H}} \right) L_j^g \|\phi_j(s - \tau_{ij}(s)) - \psi_j(s - \tau_{ij}(s))\|_{\mathbb{H}} \\ & + \left. \sum_{j=1}^n \left(\|\alpha_{ij}^3(s)\|_{\mathbb{H}} + \|\alpha_{ij}^4(s)\|_{\mathbb{H}} \right) L_j^h \|\phi_j(s - m) - \psi_j(s - m)\|_{\mathbb{H}} \int_0^\infty |K_{ij}(m)| dm \right\} ds \end{aligned}$$

which implies that

$$\begin{aligned} \|\Omega\phi - \Omega\psi\|_{\mathbb{H}} & \leq \int_{-\infty}^t (t-s)^{\alpha-1} \mathcal{K}_i(t-s) ds \left\{ \sum_{j=1}^n b_{ij}^* L_j^f + \sum_{j=1}^n \left(\alpha_{ij}^{1*} + \alpha_{ij}^{2*} \right) L_j^g \right. \\ & + \left. \sum_{j=1}^n \left(\alpha_{ij}^{3*} + \alpha_{ij}^{4*} \right) L_j^h \right\} \|\phi - \psi\|_u \\ & \leq \frac{1}{a_i} \left\{ \sum_{j=1}^n b_{ij}^* L_j^f + \sum_{j=1}^n \left(\alpha_{ij}^{1*} + \alpha_{ij}^{2*} \right) L_j^g + \sum_{j=1}^n \left(\alpha_{ij}^{3*} + \alpha_{ij}^{4*} \right) L_j^h \right\} \|\phi - \psi\|_u. \end{aligned}$$

Taking the supremum over t and the maximum over i :

$$\|\Omega\phi - \Omega\psi\|_u \leq \beta_1 \|\phi - \psi\|_u.$$

Since $\beta_1 < 1$, Ω is a contraction on ∇ . By the Banach fixed-point theorem, Ω has a unique fixed point in ∇ . This fixed point is a μ -pseudo almost automorphic solution of system (1.1).

3.2 Finite-time stability of μ -pseudo almost automorphic solutions

Definition 3.6. Let $x^*(t)$ be a mild solution of system (1.1) with initial condition $\phi^*(t)$. The solution $x^*(t)$ is said to be finite-time stable with respect to $\{\beta_2, \beta_3, t_0, T\}$, where $0 < \beta_2 < \beta_3$ and $T > t_0$, if for any mild solution $x(t)$ of (1.1) with initial condition $\phi(t)$ satisfies

$$\|\phi - \phi^*\|_* \leq \beta_2 \implies \|x(t) - x^*(t)\|_{\mathbb{H}^n} \leq \beta_3, \quad \forall t \in [t_0, T],$$

Lemma 3.7. [25] Let $\alpha > 0$ and let a be a nonnegative function locally integrable on $[t_0, T)$ for some $T \leq +\infty$. Let g be a nonnegative, nondecreasing function defined on $[t_0, T)$, and let u be a nonnegative locally integrable function on $[t_0, T)$ satisfying

$$u(t) \leq a(t) + g(t) \int_{t_0}^t (t-s)^{\alpha-1} u(s) ds, \quad t_0 \leq t < T.$$

Then

$$u(t) \leq a(t) + \int_{t_0}^t \left(\sum_{k=1}^{\infty} \frac{(g(t)\Gamma(\alpha))^k}{\Gamma(k\alpha)} (t-s)^{k\alpha-1} a(s) \right) ds, \quad t_0 \leq t < T.$$

Moreover, if a is nondecreasing on $[t_0, T)$, then

$$u(t) \leq a(t) E_\alpha(g(t)\Gamma(\alpha) t^\alpha), \quad t \geq t_0,$$

where $E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}$ is the Mittag-Leffler function.

Theorem 3.8. Assume that hypotheses (C.0)–(C.3) hold. Furthermore, suppose that the following condition is satisfied: (C.4) There exist positive constants β_2, β_3 with $\beta_2 < \beta_3$ such that

$$\beta_2 E_\alpha(\kappa T^\alpha) \leq \beta_3,$$

where

$$\kappa = \max_{1 \leq i \leq n} \left\{ \sum_{j=1}^n b_{ij}^* L_j^f + \sum_{j=1}^n (\alpha_{ij}^{1*} + \alpha_{ij}^{2*}) L_j^g + \sum_{j=1}^n (\alpha_{ij}^{3*} + \alpha_{ij}^{4*}) L_j^h \right\}.$$

Then the μ -pseudo almost automorphic solution of system (1.1) is finite-time stable with respect to $\{\beta_2, \beta_3, t_0, T\}$.

Assume that hypotheses (C.0)–(C.4) hold. By Theorem 3.5, system (1.1) admits a unique μ -pseudo almost automorphic solution $x^*(t) = (x_1^*(t), \dots, x_n^*(t))^\top$ with initial condition $\phi^*(t) = (\phi_1^*(t), \dots, \phi_n^*(t))^\top$ satisfying the integral equation (1.2). Let $x(t) = (x_1(t), \dots, x_n(t))^\top$ be an arbitrary mild solution of (1.1) with initial condition $\phi(t) = (\phi_1(t), \dots, \phi_n(t))^\top$ satisfying the same equation (1.2). Define the error between the two solutions as $y(t) = x^*(t) - x(t)$.

For each $i = 1, \dots, n$, we have

$$\begin{aligned} y_i(t) &= S_i(t - t_0)(\phi_i^*(t_0) - \phi_i(t_0)) \\ &+ \int_{t_0}^t (t - s)^{\alpha-1} \mathcal{K}_i(t - s) \left[\sum_{j=1}^n b_{ij}(s) [f_j(x_j^*(s)) - f_j(x_j(s))] \right] \\ &+ \bigwedge_{j=1}^n \alpha_{ij}^1(s) g_j(x_j^*(s - \tau_{ij}(s))) - \bigwedge_{j=1}^n \alpha_{ij}^1(s) g_j(x_j(s - \tau_{ij}(s))) \\ &+ \bigvee_{j=1}^n \alpha_{ij}^2(s) g_j(x_j^*(s - \tau_{ij}(s))) - \bigvee_{j=1}^n \alpha_{ij}^2(s) g_j(x_j(s - \tau_{ij}(s))) \\ &+ \bigwedge_{j=1}^n \alpha_{ij}^3(s) \int_0^{+\infty} K_{ij}(m) [h_j(x_j^*(s - m)) - h_j(x_j(s - m))] dm \\ &+ \bigvee_{j=1}^n \alpha_{ij}^4(s) \int_0^{+\infty} K_{ij}(m) [h_j(x_j^*(s - m)) - h_j(x_j(s - m))] dm \Big] ds. \end{aligned} \quad (3.11)$$

For any $t \in [t_0, T]$ and $i = 1, \dots, n$, we find

$$\begin{aligned} \|S_i(t - t_0)(\phi_i^*(t_0) - \phi_i(t_0))\|_{\mathbb{H}} &\leq \int_0^\infty p_\alpha(r) e^{-a_i(t-t_0)r} \|\phi_i^*(t_0) - \phi_i(t_0)\|_{\mathbb{H}} dr \\ &\leq \int_0^\infty p_\alpha(r) \|\phi_i^*(t_0) - \phi_i(t_0)\|_{\mathbb{H}} dr \\ &\leq \|\phi_i^* - \phi_i\|_{t_0}, \end{aligned} \quad (3.12)$$

where $\|\phi_i^* - \phi_i\|_{t_0} = \sup_{t \in (-\infty, t_0]} \|\phi_i^*(t) - \phi_i(t)\|_{\mathbb{H}}$.

Let $\pi(t) = \max_{1 \leq i \leq n} \{\pi_i(t)\}$, where $\pi_i(t) = \sup_{\theta \in (-\infty, t]} \left\{ \|y_i(\theta)\|_{\mathbb{H}} \right\}$, for any $t \in [t_0, T]$. Then by (3.11) and (3.12):

$$\|y_i(t)\|_{\mathbb{H}} \leq \left\| S_i(t - t_0)(\phi_i^*(t_0) - \phi_i(t_0)) \right\|_{\mathbb{H}} \quad (3.13)$$

$$\begin{aligned}
& + \int_{t_0}^t (t-s)^{\alpha-1} \mathcal{K}_i(t-s) \left\| \sum_{j=1}^n b_{ij}(s) [f_j(x_j^*(s)) - f_j(x_j(s))] \right. \\
& + \bigwedge_{j=1}^n \alpha_{ij}^1(s) g_j(x_j^*(s - \tau_{ij}(s))) - \bigwedge_{j=1}^n \alpha_{ij}^1(s) g_j(x_j(s - \tau_{ij}(s))) \\
& + \bigvee_{j=1}^n \alpha_{ij}^2(s) g_j(x_j^*(s - \tau_{ij}(s))) - \bigvee_{j=1}^n \alpha_{ij}^2(s) g_j(x_j(s - \tau_{ij}(s))) \\
& + \bigwedge_{j=1}^n \alpha_{ij}^3(s) \int_0^{+\infty} K_{ij}(m) [h_j(x_j^*(s-m)) - h_j(x_j(s-m))] dm \\
& + \left. \bigvee_{j=1}^n \alpha_{ij}^4(s) \int_0^{+\infty} K_{ij}(m) [h_j(x_j^*(s-m)) - h_j(x_j(s-m))] dm \right\|_{\mathbb{H}} ds \\
& \leq \|\phi_i^* - \phi_i\|_{t_0} + \int_{t_0}^t (t-s)^{\alpha-1} \mathcal{K}_i(t-s) \left\{ \sum_{j=1}^n b_{ij}^* L_j^f + \sum_{j=1}^n (\alpha_{ij}^{1*} + \alpha_{ij}^{2*}) L_j^g \right. \\
& + \left. \sum_{j=1}^n (\alpha_{ij}^{3*} + \alpha_{ij}^{4*}) L_j^h \right\} \pi_i(s) ds.
\end{aligned} \tag{3.14}$$

On the other hand, we have

$$\mathcal{K}_i(t-s) = \alpha \int_0^\infty \theta \zeta_\alpha(\theta) e^{-a_i(t-s)\alpha\theta} d\theta,$$

Since $a_i > 0$, $t > s$ and $\theta \in (0, +\infty)$, it follows that $-a_i(t-s)\alpha\theta < 0$, and consequently $e^{-a_i(t-s)\alpha\theta} \leq 1$.

Moreover, from Definition 3.1 the following identity holds

$$\int_0^\infty \theta \zeta_\alpha(\theta) d\theta = \frac{1}{\Gamma(1+\alpha)}.$$

Therefore, we obtain

$$\begin{aligned}
\mathcal{K}_i(t-s) & \leq \alpha \frac{1}{\Gamma(1+\alpha)} = \frac{\alpha}{\alpha\Gamma(\alpha)} \quad (\text{using the Gamma function property}). \\
& = \frac{1}{\Gamma(\alpha)}.
\end{aligned}$$

Define

$$\kappa = \max_{1 \leq i \leq n} \left\{ \sum_{j=1}^n b_{ij}^* L_j^f + \sum_{j=1}^n (\alpha_{ij}^{1*} + \alpha_{ij}^{2*}) L_j^g + \sum_{j=1}^n (\alpha_{ij}^{3*} + \alpha_{ij}^{4*}) L_j^h \right\}.$$

Therefore,

$$\|y_i(t)\|_{\mathbb{H}} \leq \|\phi_i^* - \phi_i\|_{t_0} + \frac{\kappa}{\Gamma(\alpha)} \int_{t_0}^t (t-s)^{\alpha-1} \pi_i(s) ds, \quad 1 \leq i \leq n. \tag{3.15}$$

It follows that

$$\|y(t)\|_{\mathbb{H}^n} \leq \|\phi^* - \phi\|_* + \frac{\kappa}{\Gamma(\alpha)} \int_{t_0}^t (t-s)^{\alpha-1} \pi_i(s) ds, \quad 1 \leq i \leq n, \tag{3.16}$$

where $\|\phi^* - \phi\|_* = \max_{1 \leq i \leq n} \|\phi_i^* - \phi_i\|_{t_0}$.

For every $t_1 \in [t_0, t]$, we have

$$\|y(t_1)\|_{\mathbb{H}^n} \leq \|\phi^* - \phi\|_* + \frac{\kappa}{\Gamma(\alpha)} \int_{t_0}^t (t-s)^{\alpha-1} \pi_i(s) ds. \quad (3.17)$$

Consequently, for any $t \in [t_0, T]$, we have

$$\begin{aligned} \pi(t) &\leq \max \left\{ \sup_{\theta \in (-\infty, t_0]} \|y(t_1)\|_{\mathbb{H}^n}, \sup_{\theta \in (-\infty, t]} \|y(t_1)\|_{\mathbb{H}^n} \right\} \\ &\leq \max \left\{ \|\phi^* - \phi\|_*, \|\phi^* - \phi\|_* + \frac{\kappa}{\Gamma(\alpha)} \int_{t_0}^t (t-s)^{\alpha-1} \pi(s) ds \right\} \\ &\leq \|\phi^* - \phi\|_* + \frac{\kappa}{\Gamma(\alpha)} \int_{t_0}^t (t-s)^{\alpha-1} \pi(s) ds. \end{aligned}$$

Therefore, according to Lemma 3.7, we obtain

$$\pi(t) \leq \|\psi - \phi\|_* E_\alpha \left(\frac{\kappa}{\Gamma(\alpha)} \cdot \Gamma(\alpha) t^\alpha \right) = \|\psi - \phi\|_* E_\alpha(\kappa t^\alpha).$$

Consequently,

$$\|y(t)\|_{\mathbb{H}^n} \leq \|\phi^* - \phi\|_* E_\alpha(\kappa t^\alpha), \quad t \geq t_0.$$

where $E_\alpha(\cdot)$ denotes the one-parameter Mittag-Leffler. Then, for any $t \in [t_0, T]$

$$\|y(t)\|_{\mathbb{H}^n} \leq \beta_2 E_\alpha(\kappa T^\alpha).$$

By (C.4), we obtain that

$$\|y(t)\|_{\mathbb{H}^n} \leq \beta_3, \text{ for all } t \in [t_0, T].$$

Therefore, the μ -pseudo almost automorphic solution of system (1.1) is finite-time stable.

4 Numerical Simulation

We present two numerical examples to illustrate and validate the theoretical results established in Section 3. The first example is set in the quaternion algebra $\mathbb{Q}$ (dimension 4) [11, 23], and the second in the octonion algebra $\mathbb{O}$ (dimension 8). Let μ be the Lebesgue measure on $\mathbb{R}$, i.e., $d\mu(t) = dt$. Then μ satisfies condition (C.0).

Example 4.1. Consider system (1.1) with $n = 2$ neurons, fractional order $\alpha = 0.8$ evolving in the quaternion space $\mathbb{Q}$. The Caputo fractional order derivative is

$${}^C D_t^\alpha x(t) = {}^C D_t^\alpha x_0(t) + e_1 ({}^C D_t^\alpha x_1(t)) + e_2 ({}^C D_t^\alpha x_2(t)) + e_3 ({}^C D_t^\alpha x_3(t)), \quad 0 < \alpha < 1.$$

The network parameters are carefully selected to satisfy all theoretical requirements:

$$\begin{aligned} a_1 &= 4.0, \quad a_2 = 5.0. \\ b_{11}(t) &= 0.3 + 0.1e_1 + 0.2e_2 + 0.1e_3, \quad b_{12}(t) = 0.2 + 0.2e_1 + 0.1e_2 + 0.3e_3, \\ b_{21}(t) &= 0.4 + 0.1e_1 + 0.3e_2 + 0.2e_3, \quad b_{22}(t) = 0.3 + 0.2e_1 + 0.2e_2 + 0.3e_3. \end{aligned}$$

$$\alpha_{ij}^1(t) = 0.1, \quad \alpha_{ij}^2(t) = 0.2, \quad \alpha_{ij}^3(t) = 0.15, \quad \alpha_{ij}^4(t) = 0.25,$$

$$K_{ij}(s) = e^{-s}, \quad s \geq 0, \quad \text{satisfying} \quad \int_0^\infty K_{ij}(s) ds = 1,$$

$$\begin{aligned}
 \tau_{ij}(t) &= 1 + 0.5 \sin(t), \quad \text{with maximum delay } \tau^* = 1.5, \\
 u_1(t) &= (0.2 + 0.1 \sin(t)) + 0.1 \cos(t)e_1 + 0.05 \frac{t}{1+t^2}e_2, \\
 u_2(t) &= (0.3 + 0.2 \cos(t)) + 0.15 \sin(t)e_1 + 0.1 \frac{t}{1+t^2}e_2 + 0.05e_3, \\
 I_1(t) &= 0.1 + 0.05 \sin(2t)e_1, \quad I_2(t) = 0.2 + 0.1 \cos(2t)e_1 + 0.05e_2, \\
 \beta_{ij}^1(t) &= 0.05, \quad \beta_{ij}^2(t) = 0.1.
 \end{aligned}$$

Initial condition:

$$\begin{aligned}
 \phi_1(s) &= 0.1e^{-s}(1 + e_1 + e_2 + e_3), \quad s \leq 0, \\
 \phi_2(s) &= 0.2e^{-s}(1 + 2e_1 + e_2 + 0.5e_3), \quad s \leq 0.
 \end{aligned}$$

The activation functions :

$$\begin{aligned}
 f_j(x) &= \frac{1}{4} \tanh(x_0) + \frac{1}{4} \tanh(x_1)e_1 + \frac{1}{4} \tanh(x_2)e_2 + \frac{1}{4} \tanh(x_3)e_3, \\
 g_j(x) &= \frac{1}{5} \arctan(x_0) + \frac{1}{5} \arctan(x_1)e_1 + \frac{1}{5} \arctan(x_2)e_2 + \frac{1}{5} \arctan(x_3)e_3, \\
 h_j(x) &= \frac{1}{6} \sin(x_0) + \frac{1}{6} \sin(x_1)e_1 + \frac{1}{6} \sin(x_2)e_2 + \frac{1}{6} \sin(x_3)e_3,
 \end{aligned}$$

such that

$$L_j^f = \frac{1}{4}, \quad L_j^g = \frac{1}{5} = 0.2, \quad L_j^h = \frac{1}{6} \approx 0.1667. \quad (4.1)$$

For the constants of Theorems 3.5, we have $\beta_1 = 0.1571 < 1$. Form (C.4), there exist positive constants β_2, β_3 with $\beta_2 < \beta_3$ such that $\beta_2 E_\alpha(\kappa T^\alpha) \leq \beta_3$, where

$$\kappa = \max_{1 \leq i \leq n} \left\{ \sum_{j=1}^n b_{ij}^* L_j^f + \sum_{j=1}^n (\alpha_{ij}^{1*} + \alpha_{ij}^{2*}) L_j^g + \sum_{j=1}^n (\alpha_{ij}^{3*} + \alpha_{ij}^{4*}) L_j^h \right\} = 0.75333.$$

Choose $T = 5$, $\beta_2 = 0.1$. Thus (C.4) is satisfied with $\beta_3 = 1.3$. The unique μ -pseudo almost automorphic solution exists and is finite-time stable. Figures (1) and (2) illustrate the temporal evolution of all quaternion components for both neurons:

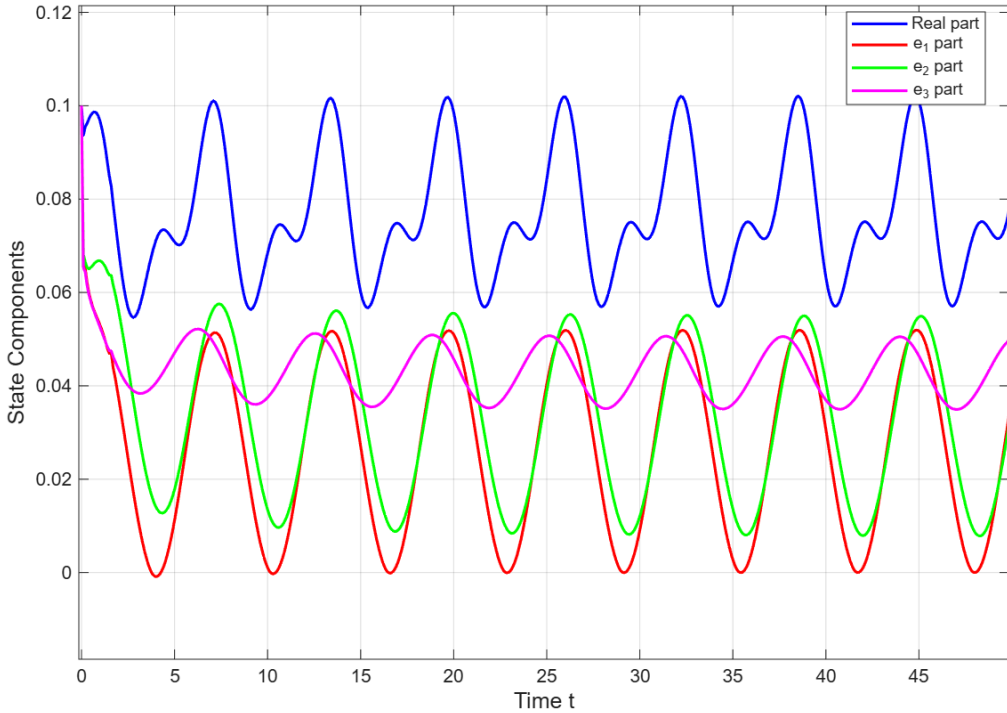


Figure 1. Time evolution of the four components of quaternion-valued neuron 1.

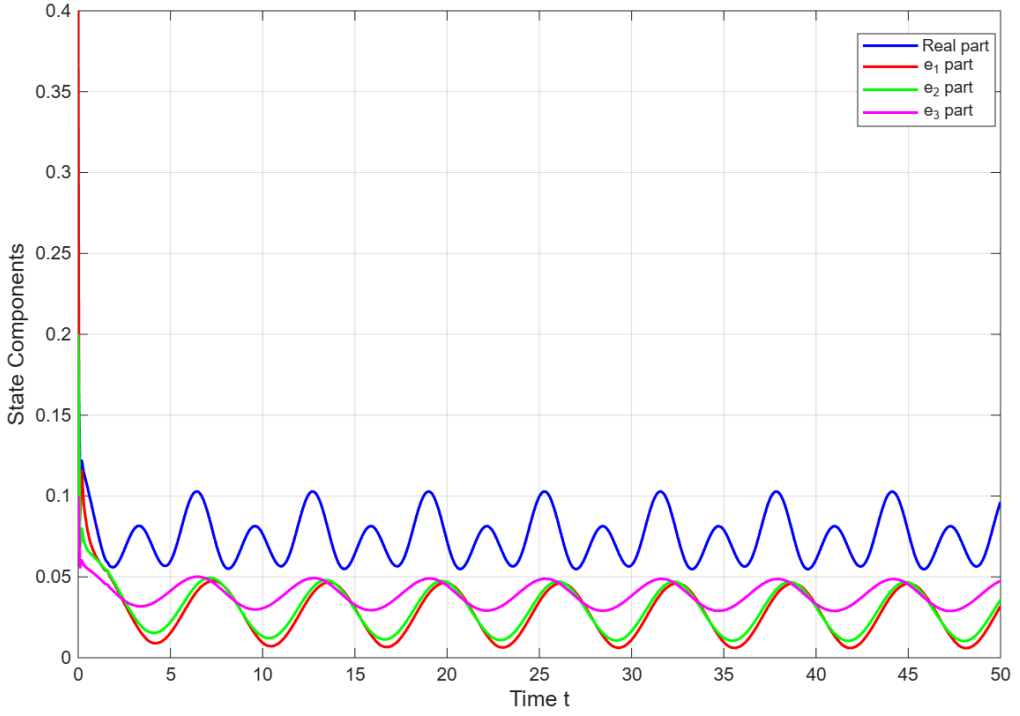


Figure 2. Time evolution of the four components of quaternion-valued neuron 2.

Example 4.2. We consider a fractional-order fuzzy neural network consisting of two neurons ($n = 2$) evolving in the octonion algebra $\mathbb{O}$ (dimension 8). The system is simulated over the time interval $t \in [0, 50]$, with fractional order $\alpha = 0.75$. The distributed delay kernel is chosen as

$$K_{ij}(s) = e^{-s}, \quad s \geq 0, \quad \int_0^{+\infty} K_{ij}(s) ds = 1.$$

The transmission delays are defined by

$$\begin{aligned} \tau_{11}(t) &= 0.8 + 0.4 \sin(t), & \tau_{12}(t) &= 0.8 + 0.4 \cos(t), \\ \tau_{21}(t) &= 0.8 + 0.4 \sin\left(t + \frac{\pi}{2}\right), & \tau_{22}(t) &= 0.8 + 0.4 \cos\left(t + \frac{\pi}{2}\right), \end{aligned}$$

yielding an effective maximum delay $\tau^* \approx 1.2$. The decay rates are given by $a_1 = 2.5$, $a_2 = 3$. Let

$$\begin{aligned} b_{11} &= 0.5 + 0.3e_1 + 0.2e_2 + 0.25e_3 + 0.15e_4 + 0.2e_5 + 0.1e_6 + 0.15e_7, \\ b_{12} &= 0.4 + 0.25e_1 + 0.3e_2 + 0.2e_3 + 0.2e_4 + 0.15e_5 + 0.2e_6 + 0.1e_7, \\ b_{21} &= 0.45 + 0.20.3e_1 + 0.25e_2 + 0.3e_3 + 0.15e_4 + 0.2e_5 + 0.15e_6 + 0.2e_7, \\ b_{22} &= 0.4 + 0.250.3e_1 + 0.2e_2 + 0.2e_3 + 0.25e_4 + 0.15e_5 + 0.2e_6 + 0.15e_7, \\ c_{11} &= 0.3 + 0.1e_1 + 0.15e_2 + 0.1e_3 + 0.05e_4 + 0.05e_5 + 0.05e_6 + 0.05e_7, \\ c_{12} &= 0.25 + 0.15e_1 + 0.1e_2 + 0.1e_3 + 0.05e_4 + 0.05e_5 + 0.05e_6 + 0.05e_7, \\ c_{21} &= 0.3 + 0.1e_1 + 0.1e_2 + 0.15e_3 + 0.05e_4 + 0.05e_5 + 0.05e_6 + 0.05e_7, \\ c_{22} &= 0.25 + 0.1e_1 + 0.15e_2 + 0.1e_3 + 0.05e_4 + 0.05e_5 + 0.05e_6 + 0.05e_7, \end{aligned}$$

and

$$\begin{aligned} \alpha_{ij}^1(t) &= 0.25, & \alpha_{ij}^2 &= 0.35, & \alpha_{ij}^3 &= 0.20, & \alpha^4 &= 0.30, \\ \beta_{ij}^1(t) &= 0.15, & \beta_{ij}^2 &= 0.25, \end{aligned}$$

$$\begin{aligned}
u_1(t) &= 0.5 \sin(0.5t) + 0.4 \cos(0.6t)e_1 + 0.3 \sin(0.7t)e_2 + 0.2 \cos(0.8t)e_3 \\
&\quad + 0.4 \cos(0.6t)e_4 + 0.3 \sin(0.7t)e_5 + 0.2 \cos(0.8t)e_6 + 0.2 \cos(0.8t)e_7, \\
u_2(t) &= 0.5 \cos(0.5t) + 0.4 \sin(0.6t)e_1 + 0.3 \cos(0.7t)e_2 + 0.2 \sin(0.8t)e_3 \\
&\quad + 0.4 \sin(0.6t)e_4 + 0.3 \cos(0.7t)e_5 + 0.2 \sin(0.8t)e_6 + 0.2 \sin(0.8t)e_7.
\end{aligned}$$

The bias terms $I_1(t)$ and $I_2(t)$ are

$$\begin{aligned}
I_1(t) &= 0.1 + 0.05 \sin(2t)e_1 + 0.02e_2 + 0.2e_3 + (0.1 \sin(t) + 0.05)e_3 \\
&\quad + (0.2 \cos(t) + \frac{1}{1+t^2})e_4 + (0.2 \cos(t) + \frac{1}{1+t^2})e_5 \\
&\quad + (0.2 \sin(t) + \frac{1}{1+t^2})e_6 + 0.03e_7, \\
I_2(t) &= 0.1 + 0.05 \cos(2t)e_1 + 0.02e_2 + 0.2e_3 + (0.1 \cos(t) + 0.05)e_3 \\
&\quad + (0.2 \sin(t) + \frac{1}{1+t^2})e_4 + 0.2 \sin(t) + \frac{1}{1+t^2})e_5 \\
&\quad + (0.2 \cos(t) + \frac{1}{1+t^2})e_6 + 0.03e_7,
\end{aligned}$$

Let $x = x_0 + x_1e_1 + \dots + x_7e_7 \in \mathbb{O}$. Define

$$\begin{aligned}
f_j(x) &= \frac{1}{4} \tanh(x_0) + \frac{1}{4} \tanh(x_1)e_1 + \frac{1}{4} \tanh(x_2)e_2 + \frac{1}{4} \tanh(x_3)e_3 \\
&\quad + \frac{1}{4} \tanh(x_4)e_4 + \frac{1}{4} \tanh(x_5)e_5 + \frac{1}{4} \tanh(x_6)e_6 + \frac{1}{4} \tanh(x_7)e_7, \\
g_j(x) &= \frac{1}{5} \arctan(x_0) + \frac{1}{5} \arctan(x_1)e_1 + \frac{1}{5} \arctan(x_2)e_2 + \frac{1}{5} \arctan(x_3)e_3 \\
&\quad + \frac{1}{5} \arctan(x_4)e_4 + \frac{1}{5} \arctan(x_5)e_5 + \frac{1}{5} \arctan(x_6)e_6 + \frac{1}{5} \arctan(x_7)e_7, \\
h_j(x) &= \frac{1}{6} \sin(x_0) + \frac{1}{6} \sin(x_1)e_1 + \frac{1}{6} \sin(x_2)e_2 + \frac{1}{6} \sin(x_3)e_3 \\
&\quad + \frac{1}{6} \sin(x_4)e_4 + \frac{1}{6} \sin(x_5)e_5 + \frac{1}{6} \sin(x_6)e_6 + \frac{1}{6} \sin(x_7)e_7.
\end{aligned}$$

Lipschitz constants: $L_j^f = \frac{1}{4}$, $L_j^g = \frac{1}{5}$, $L_j^h = \frac{1}{6}$.

The chosen parameters satisfy hypotheses (C.0)-(C.3) by construction. The key constants arising from the theoretical analysis are then computed as $\beta_1 = 0.5077$ (ensuring the contraction condition $\beta_1 < 1$) and $\kappa = 1.2692$. Furthermore, selecting $\beta_2 = 0.1$, $\beta_3 = 2.0$, and $T = 5$ fulfills condition (C.4), since $\beta_2 E_\alpha(\kappa T^\alpha) \approx 1.5 \leq \beta_3$. Figures (3) and (4) illustrate the temporal evolution of all quaternion components for both neurons:

Remark 4.3. The parameter selection demonstrates that conditions (C.0) – (C.4) are not merely abstract mathematical constructs but can be realistically satisfied in practical scenarios.

5 Conclusions

In this work, we introduced a new class of fractional-order fuzzy cellular neural networks formulated in hypercomplex spaces derived from the Cayley-Dickson algebra construction, including quaternions, octonions, and higher-dimensional extensions. By combining fractional Caputo derivatives, fuzzy logic operations the proposed model provides a unified and powerful framework for analyzing high-dimensional, non-commutative, and non-associative dynamical systems. Using the concept of μ -pseudo almost automorphic functions within the framework of measure theory, we established, via a Banach contraction principle, sufficient conditions for

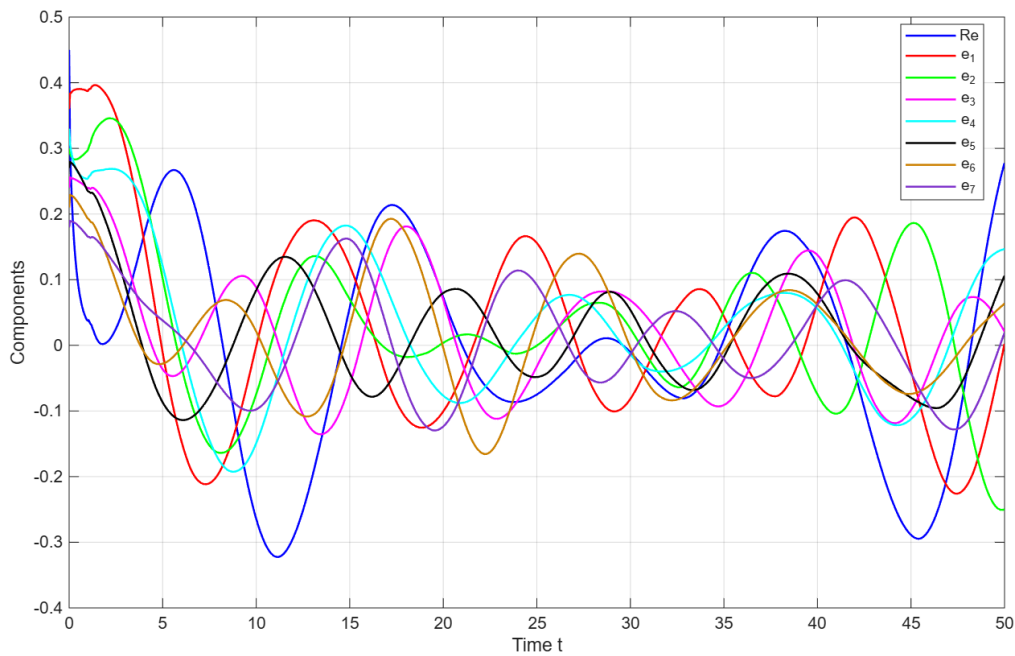


Figure 3. Temporal evolution of the eight octonionic components for neuron 1.

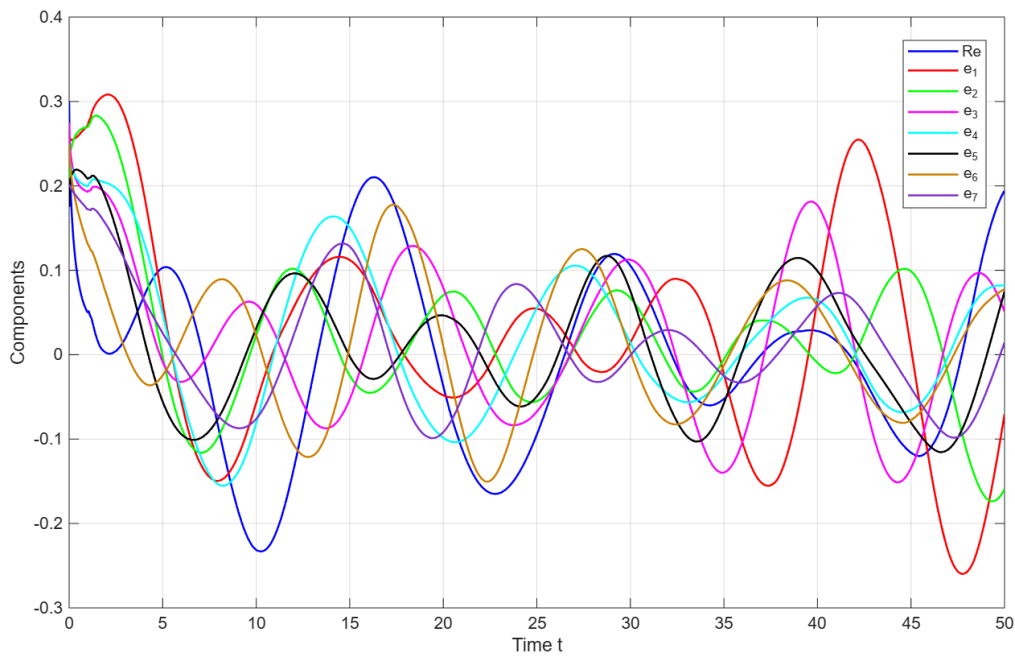


Figure 4. Temporal evolution of the eight octonionic components for neuron 2.

the existence and uniqueness of μ -pseudo almost automorphic mild solutions. Furthermore, a generalized Gronwall inequality enabled us to derive criteria ensuring the finite-time stability of these solutions. A distinctive methodological contribution lies in the explicit handling of the algebraic challenges induced by non-commutativity (quaternions) and non-associativity (octonions), achieved through contraction coefficients based on structure constants. Numerical simulations carried out in both quaternion and octonion settings corroborate the theoretical results and demonstrate the practical feasibility of the proposed approach. Future research directions include extending this framework to stochastic fractional fuzzy neural networks, exploring more general hypercomplex algebras such as sedenions $\mathbb{S}$, and investigating other stability concepts like Mittag-Leffler stability or synchronization.

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