

A Novel Adaptive Variable-Order Fractional Model for Speckle Noise Removal with Bayesian Optimization

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Abstract In this work, we propose a novel variable-order fractional total variation model tailored for speckle noise removal. We introduce a spatially adaptive fractional order derived from local image gradients, the model automatically transitions between near-first-order behaviour in boundary regions and near-second-order smoothing in homogeneous or textured areas. This adaptive mechanism, combined with a data fidelity term suited to multiplicative noise, enables superior recovery of diagnostically relevant features without artifacts. We establish the existence and uniqueness of the solution in a dedicated variable-order bounded variation space. We solve the non-convex optimization problem using the alternating direction method of multipliers with Grünwald–Letnikov discretization of variable-order derivatives. To ensure robust practical performance across varying noise levels, we tune automatically the hyperparameters of the proposed model via Bayesian optimization. Extensive experiments on synthetic and clinical ultrasound images demonstrate that the proposed VOFTV model outperforms state-of-the-art methods in terms of PSNR, SSIM, and visual quality, with enhanced preservation of fine anatomical details and reduced over-smoothing.

1 Introduction

Ultrasound imaging is a cornerstone of modern medical diagnostics owing to its radiation-free approach, ability to provide immediate visualization, and economic advantages. However, images acquired through coherent wave interference are inherently degraded by speckle noise—a multiplicative granular pattern that arises from random scattering within tissues [1, 2]. This noise obscures fine anatomical details, reduces contrast, and complicates tasks such as lesion detection, segmentation, and quantitative analysis, particularly in challenging applications like breast or cardiac imaging.

Over the past decades, numerous despeckling strategies have emerged. Local filtering techniques, such as Lee or Frost filters, adapt to local statistics but often blur edges or fail in highly textured regions [3, 4]. Nonlocal approaches, exemplified by Non-Local Means (NL-Means) and its variants (e.g., Block-Matching 3D or optimized Bayesian NL-Means), exploit self-similarity across the image to achieve superior texture preservation [5, 6]. Despite their effectiveness, these methods suffer from significant limitations: high computational complexity due to exhaustive patch searches, sensitivity to patch size and search window parameters, and a tendency to over-smooth subtle structures or introduce artifacts in regions with limited redundancy under strong speckle contamination [7, 8, 9].

Variational methods offer an attractive alternative by formulating despeckling as an energy minimization problem that balances data fidelity with prior constraints on image regularity. Classical Total Variation (TV) regularization, adapted to multiplicative noise via logarithmic transformations or dedicated fidelity terms, effectively preserves sharp boundaries while suppressing noise [10, 11]. Nevertheless, TV-based models frequently induce undesirable staircase artifacts in homogeneous areas and struggle to retain fine-scale oscillatory textures typical of

ultrasound data [12]. Higher-order extensions, such as Total Generalized Variation (TGV) [13], first- and second-order TV [14], infimal convolution model [15], nonstandard higher-order models [16, 17, 18] mitigate staircasing but may over-smooth important diagnostic features or require careful manual tuning of multiple parameters.

Fractional-order Total Variation (FOTV) has recently gained traction as a powerful regularization paradigm. By employing derivatives of non-integer order $\alpha \in (1, 2)$, FOTV bridges the gap between first-order edge preservation and second-order smoothness, offering enhanced capability to capture mid-scale textures without blocky effects [19, 20]. Applications to FOTV have demonstrated improved detail recovery over integer-order counterparts [21, 22, 23]. However, these models rely on a fixed global order α , which remains suboptimal for ultrasound images exhibiting heterogeneous regions—sharp organ boundaries coexist with granular parenchymal textures and smoothly varying intensities. A single α cannot simultaneously optimize edge fidelity (favoring $\alpha \approx 1$) and texture preservation in smoother zones (favoring $\alpha \approx 2$).

To address these shortcomings, we propose a novel variable-order fractional total variation (VOFTV) model specifically tailored for ultrasound despeckling. Building upon the adaptive fractional framework introduced in [24] for additive noise, our approach extends variable-order differentiation to the multiplicative setting for the first time. The fractional order $g(x)$ is spatially modulated automatically based on local image gradients, enabling near-first-order behavior at edges and near-second-order smoothing in homogeneous or textured areas. This spatially adaptive mechanism inherently captures the non-local memory properties of fractional operators while accommodating the structural diversity of ultrasound data, yielding superior preservation of diagnostically relevant features without staircase artifacts or over-smoothing. The proposed energy functional is:

$$TV^{g(x,y)}(u) + \lambda \int_{\Omega} \frac{(f - u)^2}{u} dx dy \quad (1.1)$$

where $TV^{g(x,y)}(u)$ denotes the variable-order fractional total variation, and the fidelity term corresponds to the multiplicative noise assumption common in speckle modeling [10, 25]. To ensure robust and reproducible performance, we employ Bayesian optimization for automatic hyperparameters selection (including the balancing parameter λ and indicators controlling $g(x, y)$), eliminating manual trial-and-error and adapting to varying noise levels [26, 27]. The main contributions of this work are:

- Introduction of the first variable-order fractional variation model for multiplicative speckle noise removal in ultrasound imaging
- Theoretical analysis establishing existence and uniqueness of solutions in an appropriate variable-order bounded variation space.
- An efficient numerical scheme combining Alternating Direction Method of Multipliers (ADMM) with Grünwald-Letnikov discretization of variable-order derivatives.
- Automatic hyperparameter tuning via Bayesian optimization, enhancing practical applicability.
- Comprehensive experimental validation on clinical and synthetic ultrasound images, demonstrating superior quantitative metrics (PSNR, SSIM) and visual quality compared to state-of-the-art variational, nonlocal-based despecklers.

The rest of this manuscript is organized as follows. Section 2 provides the essential background before introducing the proposed VOF-TV model, and establishes its theoretical well-posedness. Section 3 describes the numerical discretization, the ADMM algorithm, and discusses convergence aspects of the implementation. Section 4 presents the adaptive model with parameter optimization and reports extensive experimental results, including comparisons with existing methods. Finally, Section 5 offers the main findings and outlines possible future directions.

2 Mathematical background and problem setting

2.1 Variable-order fractional derivatives and fractional total variation

To handle spatially varying image characteristics in ultrasound despeckling, we employ variable-order fractional calculus. Following established definitions in fractional variational models, we adopt the Grünwald–Letnikov formulation for variable-order derivatives, which generalizes integer-order operators [28].

Let $\Omega \subset \mathbb{R}^2$ be a bounded open domain with Lipschitz boundary, and let $g : \Omega \rightarrow \mathbb{R}$ be a measurable function satisfying $1 < g_{\min} \leq g(x, y) \leq g_{\max} < 2$ almost everywhere. For a function $f : [a, b] \rightarrow \mathbb{R}$, the left-sided variable-order Grünwald–Letnikov derivative is defined as

$${}_a D_x^{g(x,y)} f(x) = \lim_{h \rightarrow 0^+} h^{-g(x,y)} \sum_{j=0}^{\lfloor (x-a)/h \rfloor} (-1)^j \binom{g(x,y)}{j} f(x-jh),$$

the right-sided as

$$D_b^{g(x,y)} f(x) = \lim_{h \rightarrow 0^+} h^{-g(x,y)} \sum_{j=0}^{\lfloor (b-x)/h \rfloor} (-1)^j \binom{g(x,y)}{j} f(x+jh),$$

and the central variable-order derivative as their average:

$${}_{a,b} D_x^{g(x,y)} f(x) = \frac{1}{2} \left({}_a D_x^{g(x,y)} f(x) + D_b^{g(x,y)} f(x) \right).$$

These extend naturally to two dimensions along each direction. The variable-order gradient operator is then $\nabla^{g(x,y)} u = ({}^g \partial_x u, {}^g \partial_y u)$, where ${}^g \partial_x$ and ${}^g \partial_y$ denote suitable combinations of left/right/central approximations.

The regularization term in the energy functional (1.1) is the VOFTV defined by:

$$TV^{g(\cdot)}(u) := \sup \left\{ \int_{\Omega} u \operatorname{div}^{g(\cdot)} \phi \, dx \, dy \mid \phi \in C_0^1(\Omega; \mathbb{R}^2), \|\phi\|_{L^\infty(\Omega)} \leq 1 \right\}, \quad (2.1)$$

where the variable-order divergence is $\operatorname{div}^{g(\cdot)} \phi = {}^g \partial_x \phi_1 + {}^g \partial_y \phi_2$ [24].

We define the space of functions of variable-order bounded variation as

$$BV^{g(\cdot)}(\Omega) = \left\{ u \in L^1(\Omega) \mid TV^{g(\cdot)}(u) < +\infty \right\},$$

equipped with the norm

$$\|u\|_{BV^{g(\cdot)}(\Omega)} = \|u\|_{L^1(\Omega)} + TV^{g(\cdot)}(u).$$

We have the following proprieties [24]:

Lemma 2.1. *The functional (2.1) is lower semicontinuous with respect to $L^1(\Omega)$ -convergence in $BV^{g(\cdot)}(\Omega)$.*

Lemma 2.2. *The space $(BV^{g(\cdot)}(\Omega), \|\cdot\|_{BV^{g(\cdot)}(\Omega)})$ is a Banach space.*

These properties follow by adapting arguments for fixed-order cases to the bounded variation of g , ensuring uniform control over the variable-order operators [20, 24].

2.2 Proposed variational model

The key novelty lies in making the fractional order adaptive through $g(x, y)$, computed from the observed noisy image f . This function is designed to approach 1 near edges (promoting strong edge preservation akin to classical TV) and approach 2 in smooth or textured regions (favoring higher-order smoothing to suppress staircasing). We consider the following data-driven candidates, pre-smoothed with a Gaussian kernel $G_\sigma * f$ to mitigate noise sensitivity

$$g(x, y) = 1 + \exp(-\|\nabla(G_\sigma * f)(x, y)\|), \quad (2.2)$$

These ensure automatic adaptation without manual order selection, outperforming fixed-order fractional models in heterogeneous ultrasound data.

The despeckling model aims to consider the minimization problem:

$$\min_{\{u>0; u \in BV^{g(\cdot)}(\Omega)\}} TV^{g(\cdot)}(u) + \lambda \int_{\Omega} \frac{(f-u)^2}{u} dx dy, \quad (2.3)$$

where the second term is the data fidelity, and $\lambda > 0$ balances regularization. This formulation leverages VOFTV's adaptive nature to simultaneously preserve organ boundaries and parenchymal textures in ultrasound images.

Since the fidelity term $\Phi(u) = \int_{\Omega} \frac{(f-u)^2}{u} dx dy$ is only well-defined and finite for $u > 0$ almost everywhere, we consider its natural extension

$$\tilde{\Phi}(u) = \begin{cases} \Phi(u) & \text{if } u > 0 \text{ a.e. in } \Omega, \\ +\infty & \text{otherwise,} \end{cases}$$

and define the extended energy functional

$$\tilde{\mathcal{J}}(u) = TV^{g(\cdot)}(u) + \lambda \tilde{\Phi}(u), \quad u \in BV^{g(\cdot)}(\Omega).$$

Minimization is then performed over the whole space $BV^{g(\cdot)}(\Omega)$. The following existence and uniqueness result holds.

Theorem 2.3. *Let $f \in L^\infty(\Omega)$ with $f > 0$ a.e., $g : \Omega \rightarrow \mathbb{R}$ measurable with $1 < g_{\min} \leq g(x) \leq g_{\max} < 2$ a.e., and $\lambda > 0$. Then the functional $\tilde{\mathcal{J}}$ admits a unique global minimizer $u^* \in BV^{g(\cdot)}(\Omega)$. Moreover, any minimizer satisfies $u^* > 0$ a.e. in Ω .*

We notice that, the effective domain of $\tilde{\mathcal{J}}$ is $\text{dom}(\tilde{\mathcal{J}}) = \{u \in BV^{g(\cdot)}(\Omega) : u > 0 \text{ a.e. in } \Omega\}$. On this set we have $\tilde{\mathcal{J}}(u) = \mathcal{J}(u) := TV^{g(\cdot)}(u) + \lambda \Phi(u)$, while $\tilde{\mathcal{J}}(u) = +\infty$ outside. For $u \in \text{dom}(\tilde{\mathcal{J}})$, we recall that $\|u\|_{BV^{g(\cdot)}} = \|u\|_{L^1} + TV^{g(\cdot)}(u)$, boundedness of both terms implies boundedness in $BV^{g(\cdot)}(\Omega)$. For the fidelity term, rewrite

$$\Phi(u) = \int_{\Omega} u dx - 2 \int_{\Omega} f dx dy + \int_{\Omega} \frac{f^2}{u} dx dy.$$

Since the last term is nonnegative, we obtain $\Phi(u) \geq \int_{\Omega} u dx - C$, where $C = 2 \int_{\Omega} f dx dy$ is constant. Consequently,

$$\tilde{\mathcal{J}}(u) \geq TV^{g(\cdot)}(u) + \lambda \|u\|_{L^1(\Omega)} - \lambda C.$$

Therefore, if $\tilde{\mathcal{J}}(u_n)$ is bounded along a minimizing sequence, both $TV^{g(\cdot)}(u_n)$ and $\|u_n\|_{L^1(\Omega)}$ remain bounded, implying boundedness in $BV^{g(\cdot)}(\Omega)$. This proves coercivity of $\tilde{\mathcal{J}}$.

The term $TV^{g(\cdot)}(\cdot)$ is convex on $BV^{g(\cdot)}(\Omega)$. The pointwise integrand $t \mapsto (f(x) - t)^2/t$ for $t > 0$ has second derivative

$$\frac{d^2}{dt^2} \left(\frac{(f(x) - t)^2}{t} \right) = \frac{2f(x)^2}{t^3} > 0 \quad \forall t > 0.$$

Hence Φ (and thus $\tilde{\Phi}$ on the positive cone) is strictly convex. Therefore $\tilde{\mathcal{J}}$ is strictly convex on $\text{dom}(\tilde{\mathcal{J}})$.

By Lemma 2.1, $TV^{g(\cdot)}(\cdot)$ is lower semicontinuous with respect to strong $L^1(\Omega)$ -convergence on $BV^{g(\cdot)}(\Omega)$. Let $u_n \rightarrow u$ strongly in $L^1(\Omega)$ with $u_n > 0$ a.e. If $u > 0$ a.e., then by Fatou's lemma and continuity of $t \mapsto (f - t)^2/t$ on $(0, +\infty)$,

$$\liminf_{n \rightarrow \infty} \Phi(u_n) \geq \Phi(u).$$

If $u \leq 0$ on a set of positive measure, then $\tilde{\Phi}(u) = +\infty$ and the liminf is also $+\infty$. Thus $\tilde{\mathcal{J}}$ is lower semicontinuous on $BV^{g(\cdot)}(\Omega)$ with respect to strong L^1 -convergence.

Let $\{u_n\} \subset \text{dom}(\tilde{\mathcal{J}})$ be a minimizing sequence, i.e. $\tilde{\mathcal{J}}(u_n) \rightarrow \inf \tilde{\mathcal{J}} < +\infty$. By coercivity, $\{u_n\}$ is bounded in $BV^{g(\cdot)}(\Omega)$. Using the compactness property of bounded sequences in $BV^{g(\cdot)}(\Omega)$ with respect to $L^1(\Omega)$ convergence (see [20, 24]), there exists a subsequence (denoted u_n) and $u^* \in BV^{g(\cdot)}(\Omega)$ such that $u_n \rightarrow u^*$ strongly in $L^1(\Omega)$. Lower semicontinuity then gives

$$\tilde{\mathcal{J}}(u^*) \leq \liminf_{n \rightarrow \infty} \tilde{\mathcal{J}}(u_n) = \inf \tilde{\mathcal{J}}.$$

Hence u^* is a minimizer and, in particular, $\tilde{\mathcal{J}}(u^*) < +\infty$, so $u^* > 0$ a.e.

The uniqueness follows by the strict convexity of $\tilde{\mathcal{J}}$ on its effective domain. This completes the proof.

3 Alternating Direction Method of Multipliers

We discretize the image domain Ω on a uniform rectangular grid with unit spacing. The observed noisy image and the restored image are represented by matrices $\mathbf{F}, \mathbf{U} \in \mathbb{R}^{M \times N}$, whose entries are denoted by $\mathbf{F}_{i,j}$ and $\mathbf{U}_{i,j}$, respectively, for $i = 1, \dots, M$ and $j = 1, \dots, N$.

The spatially varying fractional order $g_{i,j} = g(i, j)$ is precomputed at each pixel from the image $\mathbf{F}$ using one of the indicator functions defined in the preliminaries. Following the Grünwald–Letnikov definition for variable-order derivatives, the coefficients at each pixel (i, j) are generated recursively:

$$w_0^{(i,j)} = 1, \quad w_k^{(i,j)} = \left(1 - \frac{g_{i,j} + 1}{k}\right) w_{k-1}^{(i,j)}, \quad k \geq 1.$$

The backward variable-order differences are approximated as:

$$D_x^{g-} \mathbf{U}_{i,j} = \sum_{k=0}^{i-1} w_k^{(i,j)} \mathbf{U}_{i-k,j}, \quad D_y^{g-} \mathbf{U}_{i,j} = \sum_{k=0}^{j-1} w_k^{(i,j)} \mathbf{U}_{i,j-k},$$

and the discrete forward differences are

$$D_x^{g+} \mathbf{U}_{i,j} = \sum_{p=0}^{M-i} w_p^{(i,j)} \mathbf{U}_{i+p,j}, \quad D_y^{g+} \mathbf{U}_{i,j} = \sum_{p=0}^{N-j} w_p^{(i,j)} \mathbf{U}_{i,j+p}.$$

We also discretized the combined operators $D_x^{g+} D_x^{g-}$ and $D_y^{g+} D_y^{g-}$ as:

$$D_x^{g+} D_x^{g-} \mathbf{U}_{i,j} = \sum_{p=i}^M w_{p-i}^{(i,j)} \sum_{l=0}^p w_l^{(p,j)} \mathbf{U}_{p-l,j},$$

$$D_y^{g+} D_y^{g-} \mathbf{U}_{i,j} = \sum_{p=j}^N w_{p-j}^{(i,j)} \sum_{l=0}^p w_l^{(i,p)} \mathbf{U}_{i,p-l}.$$

In the discrete setting, the discrete variable-order fractional gradient at pixel (i, j) is defined as $(\nabla^{g(\cdot)} \mathbf{U})_{i,j} = (D_x^{g-} \mathbf{U}_{i,j}, D_y^{g-} \mathbf{U}_{i,j})$. For the anisotropic discretization, the discrete VOFTV is then

$$V(\mathbf{U}) = \sum_{i,j} |D_x^{g-} \mathbf{U}_{i,j}| + |D_y^{g-} \mathbf{U}_{i,j}|. \quad (3.1)$$

The discrete model is

$$\min_{u_{i,j} > 0} \left\{ V(\mathbf{U}) + \lambda \sum_{i,j} \frac{(\mathbf{F}_{i,j} - \mathbf{U}_{i,j})^2}{u_{i,j}} \right\}. \quad (3.2)$$

The considered model (3.2) is composed of two terms. The first one is a nondifferentiable regularization term involving an L^1 norm with a fractional-order derivative. The second one is a

data fidelity term, which is differentiable but still nonlinear and non-convex. Because of these properties, the minimization problem is difficult to solve directly. To overcome this difficulty, we use the Alternating Direction Method of Multipliers (ADMM). This method splits the original problem into simpler subproblems that are solved alternately. Multipliers are used to link these subproblems and guide the iterations toward the optimal solution. ADMM is efficient for optimization problems containing nonlinear terms.

We then introduce auxiliary variables $\mathbf{d} = (d^x, d^y)$ such that $d_{i,j}^x \approx D_x^{g-} \mathbf{U}_{i,j}$ and $d_{i,j}^y \approx D_y^{g-} \mathbf{U}_{i,j}$. The constrained problem becomes

$$\begin{aligned} \min_{\mathbf{U}_{i,j} > 0, \mathbf{d}} \quad & \sum_{i,j} (|d_{i,j}^x| + |d_{i,j}^y|) + \lambda \sum_{i,j} \frac{(\mathbf{F}_{i,j} - \mathbf{U}_{i,j})^2}{\mathbf{U}_{i,j}} \\ \text{s.t.} \quad & \mathbf{d}_{i,j} = (\nabla^{g(\cdot)} \mathbf{U})_{i,j} \quad \forall i, j. \end{aligned} \quad (3.3)$$

Furthermore, we use the augmented Lagrangian method to transform the constrained problem into an unconstrained one. We then form the augmented Lagrangian $\mathcal{L}(\cdot, \cdot, \cdot)$, as follow:

$$\begin{aligned} \mathcal{L}(\mathbf{U}, \mathbf{d}, \boldsymbol{\mu}) = & \sum_{i,j} (|d_{i,j}^x| + |d_{i,j}^y|) + \lambda \sum_{i,j} \frac{(\mathbf{F}_{i,j} - \mathbf{U}_{i,j})^2}{\mathbf{U}_{i,j}} \\ & + \sum_{i,j} \langle (\nabla^{g(\cdot)} \mathbf{U})_{i,j} - \mathbf{d}_{i,j}, \boldsymbol{\mu}_{i,j} \rangle + \frac{\rho}{2} \sum_{i,j} \|(\nabla^{g(\cdot)} \mathbf{U})_{i,j} - \mathbf{d}_{i,j}\|_2^2, \end{aligned} \quad (3.4)$$

where $\boldsymbol{\mu} = (\mu^x, \mu^y)$ is the vector-valued Lagrange multiplier field (one vector per pixel) associated with the constraint $\mathbf{d} = \nabla^{g(\cdot)} \mathbf{U}$, and $\rho > 0$ is the penalty parameter.

The objective of ADMM is to compute a saddle point of the augmented Lagrangian $\mathcal{L}(\mathbf{U}, \mathbf{d}, \boldsymbol{\mu})$ by alternately minimizing with respect to the primal variables $\mathbf{U}$ and $\mathbf{d}$ and maximizing with respect to the dual variable $\boldsymbol{\mu}$. We say that $(\mathbf{U}, \mathbf{d}, \boldsymbol{\mu})$ is a saddle point of $\mathcal{L}$ if

$$\mathcal{L}(\mathbf{U}, \mathbf{d}, \boldsymbol{\xi}) \leq \mathcal{L}(\mathbf{U}, \mathbf{d}, \boldsymbol{\mu}) \leq \mathcal{L}(\mathbf{W}, \mathbf{q}, \boldsymbol{\mu}), \quad \forall (\mathbf{W}, \mathbf{q}, \boldsymbol{\xi}). \quad (3.5)$$

Lemma 3.1. *Let $(\mathbf{U}, \mathbf{d}, \boldsymbol{\mu})$ be a saddle point of $\mathcal{L}$ with $\rho > 0$. Then $\mathbf{d} = \nabla^{g(\cdot)} \mathbf{U}$, and $\mathbf{U}$ is a solution of the discrete minimization problem (3.2).*

From the saddle-point property (3.5), we have

$$\mathcal{L}(\mathbf{U}, \mathbf{d}, \boldsymbol{\xi}) \leq \mathcal{L}(\mathbf{U}, \mathbf{d}, \boldsymbol{\mu}) \quad \forall \boldsymbol{\xi}.$$

Using definition (3.4), this gives

$$\sum_{i,j} \langle \nabla^{g(\cdot)} \mathbf{U}_{i,j} - \mathbf{d}_{i,j}, \boldsymbol{\xi}_{i,j} - \boldsymbol{\mu}_{i,j} \rangle \leq 0 \quad \forall \boldsymbol{\xi}.$$

This inequality holds for arbitrary multipliers only if $\nabla^{g(\cdot)} \mathbf{U}_{i,j} - \mathbf{d}_{i,j} = 0$, for all i, j , hence $\mathbf{d} = \nabla^{g(\cdot)} \mathbf{U}$. Substituting this relation into the constrained formulation (3.3) shows that $\mathbf{U}$ minimizes the unconstrained discrete functional (3.2).

The ADMM scheme consists of the following alternating minimizations and multiplier update:

$$\mathbf{U}^{k+1} = \arg \min_{\mathbf{U} > 0} \mathcal{L}(\mathbf{U}, \mathbf{d}^k, \boldsymbol{\mu}^k), \quad (3.6a)$$

$$\mathbf{d}^{k+1} = \arg \min_{\mathbf{d}} \mathcal{L}(\mathbf{U}^{k+1}, \mathbf{d}, \boldsymbol{\mu}^k), \quad (3.6b)$$

$$\boldsymbol{\mu}_{i,j}^{k+1} = \boldsymbol{\mu}_{i,j}^k + \rho (\nabla^{g(\cdot)} \mathbf{U}_{i,j}^{k+1} - \mathbf{d}_{i,j}^{k+1}). \quad (3.6c)$$

3.1 The U-subproblem

For fixed $\mathbf{d}^k$ and $\boldsymbol{\mu}^k$, the U-subproblem (3.6a) reads

$$\mathbf{U}^{k+1} = \arg \min_{\mathbf{U} > 0} \sum_{i,j} \langle \nabla^{g(\cdot)} \mathbf{U}_{i,j}, \boldsymbol{\mu}_{i,j}^k \rangle + \lambda \sum_{i,j} \frac{(\mathbf{F}_{i,j} - \mathbf{U}_{i,j})^2}{\mathbf{U}_{i,j}} + \frac{\rho}{2} \sum_{i,j} \|(\nabla^{g(\cdot)} \mathbf{U})_{i,j} - \mathbf{d}_{i,j}^k\|_2^2. \quad (3.7)$$

This subproblem is nonlinear due to the fidelity term $\frac{(f-u)^2}{u}$. Moreover, the positivity constraint $U_{i,j} > 0$ must be enforced. A common strategy is to introduce an auxiliary variable $\mathbf{V} = \mathbf{U}$ and treat the positivity and fidelity term separately via an inner ADMM loop.

We thus reformulate the $\mathbf{U}$ -subproblem as the following constrained problem, for fixed outer iterates $\mathbf{d}^k, \boldsymbol{\mu}^k$:

$$\begin{aligned} \min_{\mathbf{U}, \mathbf{V}} \quad & \sum_{i,j} \langle (\nabla^{g(\cdot)} \mathbf{U})_{i,j}, \boldsymbol{\mu}_{i,j}^k \rangle + \lambda \sum_{i,j} \frac{(\mathbf{F}_{i,j} - \mathbf{V}_{i,j})^2}{\mathbf{V}_{i,j}} + \frac{\rho}{2} \sum_{i,j} \|(\nabla^{g(\cdot)} \mathbf{U})_{i,j} - \mathbf{d}_{i,j}^k\|_2^2 \\ \text{s.t.} \quad & \mathbf{V} = \mathbf{U}, \quad \mathbf{V}_{i,j} > 0 \quad \forall i, j. \end{aligned} \quad (3.8)$$

We now apply an inner ADMM with penalty parameter $\rho_1 > 0$ and inner Lagrange multiplier field $\boldsymbol{\eta}$. The inner augmented Lagrangian is

$$\begin{aligned} \mathcal{L}_{\text{inner}}(\mathbf{U}, \mathbf{V}, \boldsymbol{\eta}) = & \sum_{i,j} \langle (\nabla^{g(\cdot)} \mathbf{U})_{i,j}, \boldsymbol{\mu}_{i,j}^k \rangle + \lambda \sum_{i,j} \frac{(\mathbf{F}_{i,j} - \mathbf{V}_{i,j})^2}{\mathbf{V}_{i,j}} + \frac{\rho}{2} \sum_{i,j} \|(\nabla^{g(\cdot)} \mathbf{U})_{i,j} - \mathbf{d}_{i,j}^k\|_2^2 \\ & + \sum_{i,j} \langle \mathbf{U}_{i,j} - \mathbf{V}_{i,j}, \boldsymbol{\eta}_{i,j} \rangle + \frac{\rho_1}{2} \sum_{i,j} (\mathbf{U}_{i,j} - \mathbf{V}_{i,j})^2. \end{aligned} \quad (3.9)$$

The inner ADMM iterations (starting from some initial $\mathbf{v}^0, \boldsymbol{\eta}^0$) are:

$$\begin{aligned} \mathbf{U}^{k+1} = \arg \min_{\mathbf{U}} \quad & \sum_{i,j} \langle \nabla^{g(\cdot)} \mathbf{U}_{i,j}, \boldsymbol{\mu}_{i,j}^k \rangle + \frac{\rho}{2} \sum_{i,j} \|\nabla^{g(\cdot)} \mathbf{U}_{i,j} - \mathbf{d}_{i,j}^k\|_2^2 \\ & + \sum_{i,j} \langle \mathbf{U}_{i,j}, \boldsymbol{\eta}_{i,j}^k \rangle + \frac{\rho_1}{2} \sum_{i,j} (\mathbf{U}_{i,j} - \mathbf{V}_{i,j}^k)^2, \end{aligned} \quad (3.10a)$$

$$\mathbf{V}^{k+1} = \arg \min_{\mathbf{V} > 0} \quad \lambda \sum_{i,j} \frac{(\mathbf{F}_{i,j} - \mathbf{V}_{i,j})^2}{\mathbf{V}_{i,j}} - \sum_{i,j} \langle \mathbf{V}_{i,j}, \boldsymbol{\eta}_{i,j}^k \rangle + \frac{\rho_1}{2} \sum_{i,j} (\mathbf{U}_{i,j}^{k+1} - \mathbf{V}_{i,j})^2, \quad (3.10b)$$

$$\boldsymbol{\eta}_{i,j}^{k+1} = \boldsymbol{\eta}_{i,j}^k + \rho_1 (\mathbf{U}_{i,j}^{k+1} - \mathbf{V}_{i,j}^{k+1}). \quad (3.10c)$$

We now solve the inner $\mathbf{U}$ -subproblem (3.10a). This subproblem is quadratic in $\mathbf{U}$. It can be rewritten as

$$\mathbf{U}^{k+1} = \arg \min_{\mathbf{U}} \quad \frac{\rho}{2} \sum_{i,j} \left\| (\nabla^{g(\cdot)} \mathbf{U})_{i,j} - \left(\mathbf{d}_{i,j}^k + \frac{\boldsymbol{\mu}_{i,j}^k}{\rho} \right) \right\|_2^2 + \frac{\rho_1}{2} \sum_{i,j} \left(\mathbf{U}_{i,j} - \left(\mathbf{V}_{i,j}^k - \frac{\boldsymbol{\eta}_{i,j}^k}{\rho_1} \right) \right)^2. \quad (3.11)$$

The corresponding Euler–Lagrange equation is

$$\begin{aligned} \rho (D_x^{g+} D_x^{g-} \mathbf{U}_{i,j} + D_y^{g+} D_y^{g-} \mathbf{U}_{i,j}) + \rho_1 \mathbf{U}_{i,j} = & \rho (D_x^{g+} (d_{i,j}^{x,k} + \mu_{i,j}^{x,k} / \rho) \\ & + D_y^{g+} (d_{i,j}^{y,k} + \mu_{i,j}^{y,k} / \rho)) + \rho_1 (v_{i,j}^k - \eta_{i,j}^k / \rho_1). \end{aligned} \quad (3.12)$$

Furthermore, we solve the inner $\mathbf{v}$ -subproblem (3.10b). This subproblem decouples completely across pixels. For each (i, j) , we solve

$$v^{k+1} = \arg \min_{v > 0} \quad \lambda \frac{(f_{i,j} - v)^2}{v} + \frac{\rho_1}{2} (v - c_{i,j})^2, \quad (3.13)$$

where $c_{i,j} := u_{i,j}^{k+1} - \eta_{i,j}^k / \rho_1$ is the proximal center (shifted by the inner dual variable). Taking the derivative w.r.t. v on the objective of the subproblem (3.13) and setting it to zero yields the cubic equation

$$\rho_1 v^3 + (\lambda - \rho_1 c_{i,j}) v^2 - \lambda f_{i,j}^2 = 0. \quad (3.14)$$

Lemma 3.2. *The cubic polynomial (3.14) has exactly one real positive root $v^* > 0$.*

Let $P(v) = \rho_1 v^3 + (\lambda - \rho_1 c_{i,j})v^2 - \lambda f_{i,j}^2$. Since $\rho_1 > 0$ and $f_{i,j} > 0$, we have $P(0) = -\lambda f_{i,j}^2 \leq 0$ and $\lim_{v \rightarrow +\infty} P(v) = +\infty$. By the Intermediate Value Theorem, there exists at least one root $v^* > 0$.

To prove uniqueness, consider the derivative:

$$P'(v) = 3\rho_1 v^2 + 2(\lambda - \rho_1 c_{i,j})v = v[3\rho_1 v + 2(\lambda - \rho_1 c_{i,j})].$$

The critical points are $v = 0$ and $v = -\frac{2(\lambda - \rho_1 c_{i,j})}{3\rho_1}$. We now distinguish two cases:

Case 1: $\lambda - \rho_1 c_{i,j} \geq 0$. Then $P'(v) \geq 0$ for all $v \geq 0$, so $P(v)$ is non-decreasing on $[0, +\infty)$. Since $P(0) \leq 0$ and $P(v) \rightarrow +\infty$, there is exactly one positive root.

Case 2: $\lambda - \rho_1 c_{i,j} < 0$. The second critical point $v_c := -\frac{2(\lambda - \rho_1 c_{i,j})}{3\rho_1} > 0$ is a local minimum. Evaluating P at this point:

$$P(v_c) = \rho_1 \left(-\frac{2(\lambda - \rho_1 c_{i,j})}{3\rho_1} \right)^3 + (\lambda - \rho_1 c_{i,j}) \left(-\frac{2(\lambda - \rho_1 c_{i,j})}{3\rho_1} \right)^2 - \lambda f_{i,j}^2.$$

Simplifying gives

$$P(v_c) = \frac{4(\lambda - \rho_1 c_{i,j})^3}{27\rho_1^2} - \lambda f_{i,j}^2 < 0,$$

since both terms are negative. Thus $P(0) \leq 0$, $P(v_c) < 0$, and $P(v) \rightarrow +\infty$ as $v \rightarrow +\infty$. By continuity and the fact that P decreases on $(0, v_c)$ and increases on $(v_c, +\infty)$, there is exactly one root in $(v_c, +\infty) \subset (0, +\infty)$.

In both cases, the cubic has exactly one positive real root.

Remark 3.3. The positive real root $v^* > 0$ of the cubic polynomial (3.14) corresponds to the unique minimizer of the subproblem (3.10b). The uniqueness follows from the strict convexity of the functional in (3.10b) for $v > 0$, ensured by the $\frac{1}{v}$ term combined with the quadratic regularization, and is consistent with Lemma 3.2, which guarantees exactly one positive real root. All other roots (negative or complex) are mathematically valid but physically irrelevant for denoising: negative v would violate the image domain $v > 0$, while complex roots lack interpretation. The degenerate case $\lambda f^2 = 0$ occurs only when $\lambda = 0$ (eliminate the data fidelity term) or $f \equiv 0$ (trivial solution), both of which are excluded in practical denoising scenarios.

3.2 The d-subproblem

For fixed $\mathbf{U}^{k+1}$ and $\boldsymbol{\mu}^k$, the d-subproblem (3.6b) reads

$$\begin{aligned} \mathbf{d}^{k+1} = \arg \min_{\mathbf{d}} \sum_{i,j} (|d_{i,j}^x| + |d_{i,j}^y|) &+ \sum_{i,j} \langle (\nabla^{g(\cdot)} \mathbf{U}^{k+1})_{i,j} - \mathbf{d}_{i,j}, \boldsymbol{\mu}_{i,j}^k \rangle \\ &+ \frac{\rho}{2} \sum_{i,j} \|(\nabla^{g(\cdot)} \mathbf{U}^{k+1})_{i,j} - \mathbf{d}_{i,j}\|_2^2. \end{aligned} \quad (3.15)$$

Collecting terms depending on $\mathbf{d}$ and completing the square gives the equivalent proximal form

$$\mathbf{d}^{k+1} = \arg \min_{\mathbf{d}} \sum_{i,j} (|d_{i,j}^x| + |d_{i,j}^y|) + \frac{\rho}{2} \sum_{i,j} \left\| \mathbf{d}_{i,j} - \left(\nabla^{g(\cdot)} \mathbf{U}_{i,j}^{k+1} - \frac{\boldsymbol{\mu}_{i,j}^k}{\rho} \right) \right\|_2^2. \quad (3.16)$$

Since we use the anisotropic discretization of the VOFTV, the d-subproblem decouples across directions and pixels. Each component can therefore be solved independently via the one-dimensional soft-thresholding operator [29]:

$$d_{i,j}^{x,k+1} = \text{sign}(p_{i,j}^x) \cdot \max(|p_{i,j}^x| - \frac{1}{\rho}, 0), \quad (3.17)$$

$$d_{i,j}^{y,k+1} = \text{sign}(p_{i,j}^y) \cdot \max(|p_{i,j}^y| - \frac{1}{\rho}, 0), \quad (3.18)$$

where the proximal center is $\mathbf{p}_{i,j} = \nabla^{g(\cdot)} \mathbf{U}_{i,j}^{k+1} - \frac{\boldsymbol{\mu}_{i,j}^k}{\rho}$, and by convention we set $\text{sign}(0) = 0$.

Algorithm 1 ADMMSolver: ADMM for ultrasound despeckling with VOFTV

Require: Noisy image $\mathbf{F} \in \mathbb{R}^{M \times N}$ with $f_{i,j} > 0$, parameters $\lambda > 0$, $\rho > 0$, $\rho_1 > 0$, $\epsilon > 0$, $K_{\max}$, K_{inner}

- 1: Precompute $g_{i,j} \in (1, 2)$ at each pixel using one of the indicator functions
- 2: Compute GL weights $w_k^{(i,j)}$ using $w_0^{(i,j)} = 1$, $w_k^{(i,j)} = \left(1 - \frac{g_{i,j}+1}{k}\right) w_{k-1}^{(i,j)}$

Initialization

- 3: Initialize $\mathbf{U}^0 = \mathbf{U}^1 = \mathbf{F}$, $\mathbf{d}^0 = \mathbf{0}$, $\boldsymbol{\mu}^0 = \mathbf{0}$, $\mathbf{V}^0 = \mathbf{F}$, $\boldsymbol{\eta}^0 = \mathbf{0}$
- 4: $k \leftarrow 0$
- 5: **while** $k < K_{\max}$ and $\frac{\|\mathbf{U}^k - \mathbf{U}^{k-1}\|_F}{\|\mathbf{U}^{k-1}\|_F} > \epsilon$ **do**

Inner ADMM loop for U-subproblem

- 6: **for** $t = 1$ to K_{inner} **do**
- 7: Solve quadratic subproblem for $\mathbf{U}^{k+1}$: (3.12)
- 8: Solve cubic subproblem for $\mathbf{V}^{k+1}$: (3.14)
- 9: Select the unique positive root $v_{i,j}^{k+1} > 0$.
- 10: Update multiplier $\eta_{i,j}^{k+1}$: (3.10c)
- 11: **end for**

Outer d-subproblem

- 12: Compute (3.17)-(3.18).
- 13: Update multiplier $\mu_{i,j}^{k+1}$: (3.6c).
- 14: $k \leftarrow k + 1$
- 15: **end while**

Output: $\mathbf{U}^{k+1}$

We summarize the above steps of the ADMM based speckle noise removal using VOFTV regularizer in Algorithm 1.

We now establish the convergence of the proposed ADMM scheme (Algorithm 1) to a solution of the discrete constrained problem (3.3).

Theorem 3.4. Assume that $f_{i,j} > 0$ for all pixels and that the variable fractional order satisfies $1 < g_{\min} \leq g_{i,j} \leq g_{\max} < 2$. Let $\rho, \rho_1 > 0$ be fixed penalty parameters. Then the sequence $\{(\mathbf{U}^k, \mathbf{d}^k, \mathbf{V}^k, \boldsymbol{\mu}^k, \boldsymbol{\eta}^k)\}$ generated by Algorithm 1 converges to a saddle point of the augmented Lagrangian associated with the constrained problem (3.3). Consequently, $\mathbf{U}^k$ converges to a solution of the discrete minimization problem (3.2).

For fixed variable order $g_{i,j}$, the discrete operators D_x^{g-} and D_y^{g-} are linear. Hence the discrete VOFTV functional (3.1) is convex as a sum of absolute values of linear mappings. The discrete version of the fidelity term $\Phi(\mathbf{U}) = \sum_{i,j} \frac{(f_{i,j} - u_{i,j})^2}{u_{i,j}}$ is convex for $u_{i,j} > 0$. Extending it by

$$\tilde{\Phi}(\mathbf{U}) = \begin{cases} \Phi(\mathbf{U}), & u_{i,j} > 0, \\ +\infty, & \text{otherwise,} \end{cases}$$

yields a proper, convex, and lower semi-continuous functional. Thus, the constrained formulation (3.3) is a convex optimization problem with linear equality constraints. The outer ADMM iterations therefore correspond to a standard two-block ADMM scheme applied to convex objectives. The inner ADMM loop used to solve the U-subproblem computes the exact minimizer of a convex subproblem, and therefore does not modify the convergence properties of the outer ADMM iterations. Consequently, classical ADMM convergence results for convex problems with linear constraints ensure that the generated sequence converges to a saddle point of the augmented Lagrangian; see [30, 31, 32]. The limit point satisfies the KKT conditions of (3.3), and the corresponding $\mathbf{U}^*$ solves (3.2).

4 Numerical results

4.1 Adaptive model and parameter updates

In Algorithm 1, the fractional order $g_{i,j}$ is computed once from the noisy image and then kept fixed during all ADMM iterations. However, during the restoration process the estimate $\mathbf{U}^k$ becomes progressively cleaner. As a result, it provides a more reliable description of local structures than the initial noisy data. For this reason, updating the order function during optimization can improve the adaptation of the regularization to image features. To improve adaptation, we update the fractional order function during iterations using the current restored image. At selected outer iterations, the indicator function is recomputed from the current $\mathbf{U}^k$, producing an updated order map $g_{i,j}^k \in (1, 2)$. The corresponding Grünwald–Letnikov weights are then recomputed and used in subsequent iterations.

We adopt this data-driven strategy based on local gradient information. To reduce noise sensitivity, the image is first pre-smoothed using a Gaussian kernel G_σ . The fractional order is then computed from the gradient magnitude of the smoothed image. In the continuous form, the order is defined by (2.2). In the discrete setting and at iteration k , the order at pixel (i, j) is updated using the current restored image $\mathbf{U}^k$ as

$$g_{i,j}^k = 1 + \exp(-\|\nabla(G_\sigma * \mathbf{U}^k)_{i,j}\|), \quad (4.1)$$

where the gradient is computed by standard finite differences. This choice automatically adapts the regularization to local image content:

- Near edges and important contours, the gradient magnitude is large. Consequently, the exponential term becomes small and $g_{i,j}^k$ approaches 1. In this case, the model behaves similarly to classical TV regularization, which preserves sharp transitions and fine structures.
- In smooth regions, the gradient magnitude is small. The exponential term is then close to one, and the order approaches 2. The regularization becomes stronger and promotes smoother solutions, allowing efficient noise reduction in homogeneous areas.

In addition to adapting the fractional order, the convergence speed and numerical stability of ADMM depend strongly on the penalty parameters ρ and ρ_1 appearing in the augmented Lagrangians of the outer and inner ADMM loops. If ρ is too small, the constraint $\mathbf{d} = \nabla^{g(\cdot)} \mathbf{U}$ is enforced slowly, which leads to weak coupling between primal variables and multipliers. If ρ is too large, the $\mathbf{U}$ -subproblem becomes ill-conditioned and the iterations may slow down due to stiffness. A similar behavior is observed for ρ_1 in the inner ADMM loop coupling $\mathbf{U}$ and $\mathbf{V}$. The regularization weight λ also plays a critical role. Small values of λ result in weak regularization and insufficient noise suppression, while large values may oversmooth the image and destroy important structural details. Therefore, the parameters (λ, ρ, ρ_1) jointly influence both restoration quality and convergence behavior.

To avoid manual tuning and to obtain a parameter configuration that is optimal for the full adaptive restoration process, we adopt an automatic parameter selection strategy based on Bayesian optimization (BO). BO is particularly suitable in this context because each parameter evaluation requires running the complete adaptive ADMM solver, which is computationally expensive and does not provide derivative information with respect to the parameters. BO aims to minimize the number of such expensive evaluations while efficiently exploring the parameter space. Let $\Theta = (\lambda, \rho, \rho_1)$ denote the vector of tunable parameters. For a given Θ , the ADMM algorithm produces a restored image $\mathbf{U}_\Theta$. When a reference image $\mathbf{U}_{\text{ref}}$ is available, restoration quality is assessed using the peak signal-to-noise ratio (PSNR). Since BO is formulated as a minimization problem, we define the objective function

$$F(\Theta) = -\text{PSNR}(\mathbf{U}_\Theta, \mathbf{U}_{\text{ref}}). \quad (4.2)$$

Minimizing $F(\Theta)$ is therefore equivalent to maximizing PSNR. The search is performed over bounded intervals chosen to ensure numerical stability and practical restoration performance, $\lambda \in [0.1, 5]$, and $\rho, \rho_1 \in [0.01, 4]$.

Bayesian optimization constructs a probabilistic surrogate model of the unknown objective function $F(\Theta)$. In our implementation, F is modeled as a Gaussian process (GP),

$$F(\Theta) \sim \mathcal{GP}(m(\Theta), \kappa(\Theta, \Theta')),$$

where $m(\Theta)$ is the prior mean function and $\kappa(\Theta, \Theta')$ is a covariance kernel encoding smoothness assumptions on the parameter-response relationship.

After k evaluations, we obtain the dataset

$$D_k = \{(\Theta_i, F(\Theta_i))\}_{i=1}^k.$$

Conditioning the GP prior on D_k yields a posterior predictive distribution at any candidate Θ , characterized by a mean $\mu_k(\Theta)$ and variance $s_k^2(\Theta)$. The mean represents the predicted restoration performance, while the variance quantifies the uncertainty of this prediction. The next candidate parameter is selected by maximizing an acquisition function that balances exploitation of promising regions, low predicted objective value, and exploration of uncertain regions (high predictive variance). In this work, we employ the Expected Improvement (EI) acquisition function. Let

$$F_k^{\min} = \min_{1 \leq i \leq k} F(\Theta_i)$$

denote the best objective value observed so far. The EI at Θ measures the expected decrease relative to $F_k^{\min}$ under the GP posterior. The next parameter configuration is therefore chosen as

$$\Theta_{k+1} = \arg \max_{\Theta} \text{EI}_k(\Theta).$$

We summarize this adaptive strategy in Algorithm 2.

Algorithm 2 integrates two adaptive mechanisms. First, the fractional order $g_{i,j}$ is updated periodically to reflect the evolving local image structure. Second, the penalty parameters (λ, ρ, ρ_1) are selected automatically through Bayesian optimization to balance regularization strength and convergence efficiency. The fractional order is not updated inside the inner ADMM loop. Re-computing $g_{i,j}$ requires recalculation of the Grünwald–Letnikov weights and the associated fractional operators, which is computationally expensive. To maintain adaptability, the order is updated only every K_g outer iterations. Each evaluation within the Bayesian optimization loop corresponds to running the adaptive ADMM solver with a reduced iteration budget $K_{\max}^{\text{BO}}$. Therefore, the objective function used by BO reflects the true adaptive restoration mechanism rather than a simplified or static approximation of the model.

In our experiments, the number of BO evaluations is set to $T_{\max} = 20$, with $K_{\max}^{\text{BO}} = 40$ reduced ADMM iterations per evaluation, K_{inner} inner iterations, and periodic order update $K_g = 5$. After determining the optimal parameter configuration Θ^* , the adaptive ADMM solver is executed once more using the full iteration budget $K_{\max}$ to produce the final restored image.

4.2 Results and comparison

In this subsection, we present the restoration performance of the proposed VOFTV model on images corrupted by multiplicative noise. The restoration quality is assessed using several quantitative measures, namely SNR, PSNR, relative error, and SSIM. These metrics allow comparison between the original, noisy, and reconstructed images.

We first evaluate the two proposed models, the classical VOFTV model (Algorithm 1) and the adaptive VOFTV model (Algorithm 2), on the synthetic image shown in Figure 1. The image is degraded by multiplicative speckle noise with variance 0.06. The noisy observation suffers from severe quality degradation compared to the reference image, with $SNR = 12.539$, $PSNR = 21.527$, and $SSIM = 0.243$. Figure 1 presents the clean image, the noisy observation, and the restoration results obtained with both models. Speckle noise strongly corrupts homogeneous regions and blurs structural details. The classical VOFTV model significantly reduces the noise while preserving the main geometric structures. However, since the fractional order is fixed, the regularization strength remains spatially uniform, which may slightly limit its ability to adapt to varying local features. In contrast, the adaptive VOFTV model updates the fractional order map periodically according to the evolving image structure. This adaptive mechanism allows

Algorithm 2 AdaptADMMSolver: Adaptive ADMM for ultrasound despeckling using VOFTV

Require: Noisy image $\mathbf{F}$, reference image $\mathbf{U}_{\text{ref}}$, parameter domain $\Theta = (\lambda, \rho, \rho_1)$, maximum BO evaluations T_{max} , initial BO samples n_0 , ADMM iterations for BO $K_{\text{max}}^{\text{BO}}$, maximum ADMM iterations K_{max} , inner ADMM iterations K_{inner} , order update period K_g , tolerance ϵ .

Generate initial parameter samples $\{\Theta^{(i)}\}_{i=1}^{n_0}$.

Initialize dataset $\mathcal{D} = \emptyset$.

for $i = 1, \dots, T_{\text{max}}$ **do**

if $i \leq n_0$ **then**

 Select $\Theta^{(i)}$ from initial samples.

else

 Select $\Theta^{(i)} = \arg \max_{\Theta} \text{EI}(\Theta)$.

end if

 Set parameters $(\lambda, \rho, \rho_1) = \Theta^{(i)}$.

Adaptive ADMM Solver

 Initialize $\mathbf{U}^0 = \mathbf{F}$, $\mathbf{V}^0 = \mathbf{F}$, $\boldsymbol{\eta}^0 = \mathbf{0}$, $\boldsymbol{\mu}^0 = \mathbf{0}$.

 Compute $g_{i,j}^0$ from $\mathbf{U}^0$ using (4.1).

 Compute GL weights and set $\mathbf{d}^0 = \nabla^{g^0(\cdot)} \mathbf{U}^0$.

$k \leftarrow 0$.

while $k < K_{\text{max}}^{\text{BO}}$ **do**

if $k \bmod K_g = 0$ **then**

 Update order $g_{i,j}^k$ from $\mathbf{U}^k$ using (4.1)

 Update GL weights $w_k^{(i,j)}$

end if

for $t = 1, \dots, K_{\text{inner}}$ **do**

 Solve $\mathbf{U}$ -subproblem using (3.12).

 Solve $\mathbf{V}$ -subproblem using (3.14).

 Update multiplier $\boldsymbol{\eta}$ using (3.10c).

end for

 Compute $\mathbf{d}^{k+1}$ via (3.17)–(3.18).

 Update multiplier $\boldsymbol{\mu}^{k+1}$ using (3.6c).

$k \leftarrow k + 1$.

end while

 Compute objective value $F^{(i)} = -\text{PSNR}(\mathbf{U}^k, \mathbf{U}_{\text{ref}})$.

 Update dataset $\mathcal{D} \leftarrow \mathcal{D} \cup \{(\Theta^{(i)}, F^{(i)})\}$.

 Update Gaussian process surrogate model using $\mathcal{D}$.

end for

Select optimal parameter $\Theta^* = \arg \min_{\Theta^{(i)}} F^{(i)}$.

Run adaptive ADMM solver with Θ^* using the full iteration budget K_{max} and stopping criteria $\frac{\|\mathbf{U}^k - \mathbf{U}^{k-1}\|_F}{\|\mathbf{U}^{k-1}\|_F} > \epsilon$ (Steps 14–27).

return restored image $\mathbf{U}$ and optimal parameters Θ^* .

stronger smoothing in flat regions and weaker smoothing near edges. As a result, edges and fine structures are better preserved while noise is effectively suppressed. Quantitative results confirm this improvement. In particular, SSIM increases from 0.243 (noisy image) to 0.940 for the classical VOFTV model and to 0.953 for the adaptive VOFTV model, indicating a closer structural similarity to the reference image.

Figure 2 presents a zoomed region corresponding to the red rectangles in Figure 1. In the noisy image, small structures are distorted by random fluctuations. After restoration, homogeneous regions become smoother while edges remain well defined. This close-up view confirms that the proposed method efficiently removes noise without oversmoothing important image boundaries.

Table 1 summarizes the quantitative comparison between the two models. Both approaches substantially improve image quality compared to the noisy observation. However, the adaptive

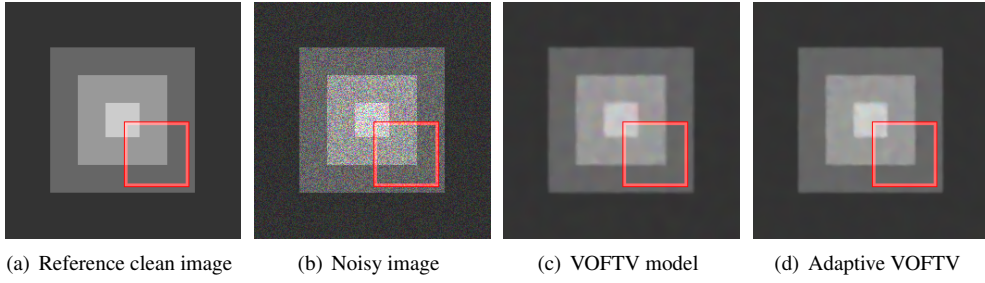


Figure 1. (a) Reference clean image. (b) Image degraded by speckle noise with variance 0.06, resulting in $\text{SNR} = 12.539$, $\text{PSNR} = 21.527$, and $\text{SSIM} = 0.243$. (c) Image restored using the proposed VOFTV (Algorithm 1), and adaptive VOFTV models (Algorithm 2). The red rectangles show zoomed areas for more details.

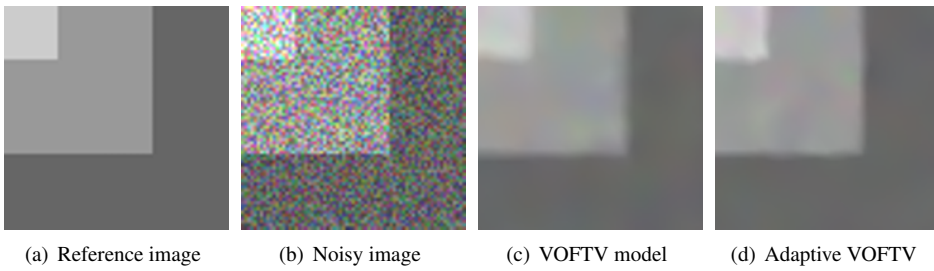


Figure 2. Zoomed region extracted from Figure 1: (a) original image, (b) noisy image, (c) image restored with the VOFTV method, and (d) image restored with adaptive VOFTV method.

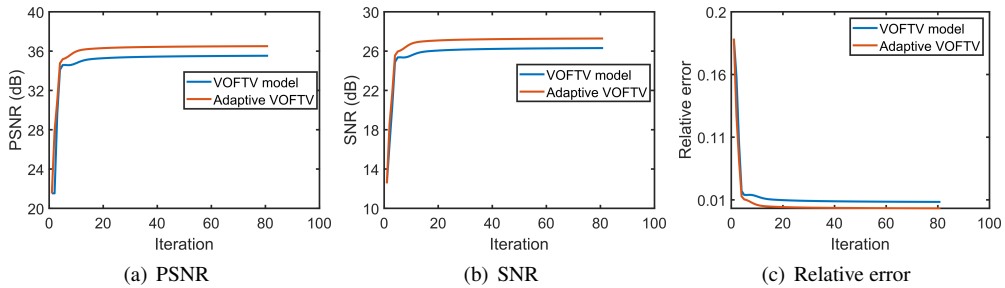


Figure 3. Evolution of restoration metrics along the iterations of the VOFTV algorithm: (a) PSNR, (b) SNR, and (c) relative error.

VOFTV model consistently yields higher SNR, PSNR, and SSIM values. The improvement, although moderate in absolute terms, demonstrates that incorporating structural adaptivity into the fractional regularization enhances the trade-off between noise removal and detail preservation.

We evaluate the proposed adaptive VOFTV method on images corrupted by multiplicative speckle noise. The results obtained with our approach are compared with three existing methods: the FOTV model [19] using a constant fractional order $\alpha \in (1, 2)$, the L^2 -based model [25] and the TV-based model [10]. For fairness, all models are solved using the same ADMM optimization framework. Both synthetic and medical ultrasound images are considered in the experiments.

Figures 4, 5, and 6 present visual comparisons on three medical images: ultrasound ankle, brain MRI, and heart CT images. Each figure includes the reference image, the noisy observation, and the restorations produced by the competing methods. Red rectangles highlight regions of interest that are enlarged to better illustrate local restoration quality.

In the ankle image results shown in Figure 4, speckle noise significantly degrades contrast and

Table 1. Comparisons of the two Algorithms (Algorithms 1 and 2) for denoising image in Figure 1, using PSNR, SNR and SSIM.

Metrics	Noisy image	VOFTV model	Adaptive VOF-TV model
SNR	12.539	25.866	26.911
PSNR	21.527	35.118	36.149
SSIM	0.243	0.940	0.953

masks fine anatomical structures. The TV method partially removes noise but introduces noticeable smoothing near edges. The L2 model produces stronger smoothing, leading to loss of small structures. The FOTV model improves edge preservation, although some textures remain unclear. In contrast, the proposed adaptive VOF-TV method achieves a better balance between noise reduction and detail preservation, maintaining clearer bone boundaries and soft tissue structures, as confirmed by the zoomed regions.

A similar behavior is observed in the MRI example in Figure 5. While classical methods reduce noise, they tend to blur important tissue transitions. The adaptive VOF-TV model better preserves structural details and produces clearer intensity transitions, resulting in visually improved anatomical representation compared with competing approaches.

Figure 6 shows restoration results for the cardiac image, where strong noise fluctuations obscure internal structures. Although TV and L2 models attenuate noise, they also oversmooth organ boundaries. FOTV improves structural preservation but still exhibits residual blur. The adaptive VOF-TV reconstruction provides cleaner homogeneous regions while maintaining sharper contours, making cardiac chambers and surrounding tissues more distinguishable.

Quantitative comparisons are reported in Table 2 using SNR, PSNR, SSIM, and iteration counts required for convergence. Across all tested images, adaptive VOF-TV consistently achieves the highest restoration quality in terms of SNR, PSNR, and SSIM. Moreover, the number of iterations required for convergence remains comparable to or lower than those of competing models, demonstrating that the proposed method improves reconstruction quality without increasing computational cost. These numerical results support the visual observations obtained in the restored images.

Table 2. Quantitative comparison of denoising performance on the medical and natural gray-scale images from Figures 4–6. Results are shown for noisy images and the denoising models at specific variance levels as function as SNR (dB), PSNR (dB), SSIM metrics. Best results are in **bold**, second-best are underlined.

Image	Metrics	Noisy	Models			
			TV [10]	L2 [25]	FOTV [19]	Adaptive VOF-TV
Ankle (309×309) variance = 0.06	SNR	12.526	20.054	20.354	<u>21.150</u>	22.444
	PSNR	22.442	30.310	30.603	<u>31.384</u>	32.657
	SSIM	0.752	0.867	0.873	<u>0.885</u>	0.905
	Iterations	-	120	90	<u>110</u>	<u>110</u>
MRI (512×512) variance = 0.08	SNR	11.647	20.981	20.925	<u>21.572</u>	22.121
	PSNR	23.371	33.038	32.991	<u>33.594</u>	34.104
	SSIM	0.737	0.928	0.928	<u>0.935</u>	0.942
	Iterations	-	160	<u>110</u>	120	90
Heart (128×152) variance = 0.08	SNR	13.323	22.713	<u>23.217</u>	23.00	24.181
	PSNR	20.168	29.796	<u>30.299</u>	30.080	31.242
	SSIM	0.431	0.687	<u>0.714</u>	0.699	0.739
	Iterations	-	70	50	<u>60</u>	<u>60</u>

Figure 7 illustrates the spatial distribution of the adaptive fractional order $g_{i,j}$ after its peri-

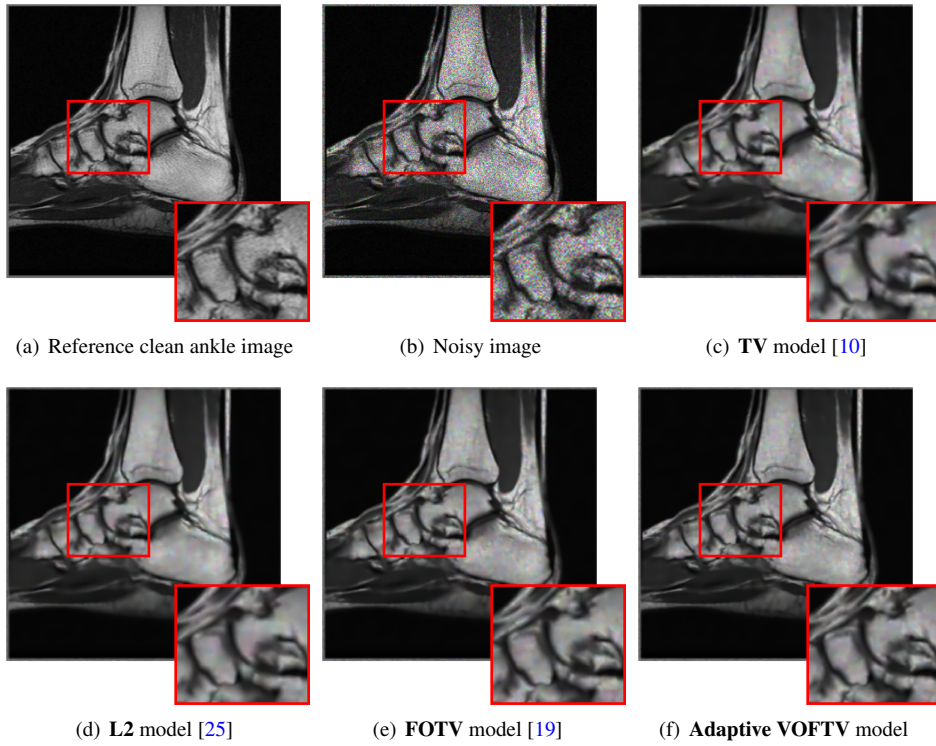


Figure 4. (a) Reference clean image. (b) Image degraded by speckle noise with variance 0.06, resulting in $\text{SNR} = 12.536$, $\text{PSNR} = 22.442$, and $\text{SSIM} = 0.752$. (c) Image restored using the TV, (d) L2, (e) FOTV and (f) the proposed adaptive VOFTV model. The red rectangles show zoomed areas for more details.

odic updates throughout the iterative restoration process. It can be observed that the indicator effectively captures the intrinsic geometric structures of the images: lower-order values are concentrated along edges, contours, and texture transitions, whereas higher-order values are predominantly assigned to smooth and homogeneous regions. This behavior confirms that the adaptive mechanism actively guides the regularization during the iterations, promoting edge preservation while enforcing stronger smoothing in uniform areas. Furthermore, the consistent distribution of $g_{i,j}$ across both synthetic and real medical images highlights the robustness and reliability of the proposed adaptive strategy.

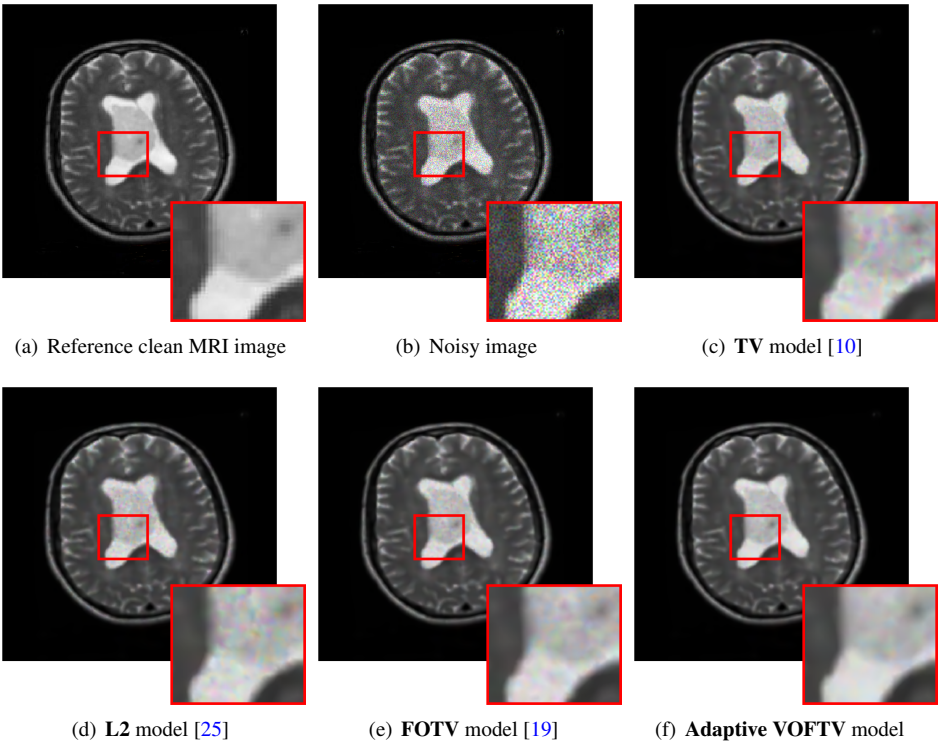


Figure 5. (a) Reference clean image. (b) Image degraded by speckle noise with variance 0.06, resulting in $SNR = 11.647$, $PSNR = 23.371$, and $SSIM = 0.737$. (c) Image restored using the TV, (d) L2, (e) FOTV and (f) the proposed adaptive VOFTV model. The red rectangles show zoomed areas for more details.

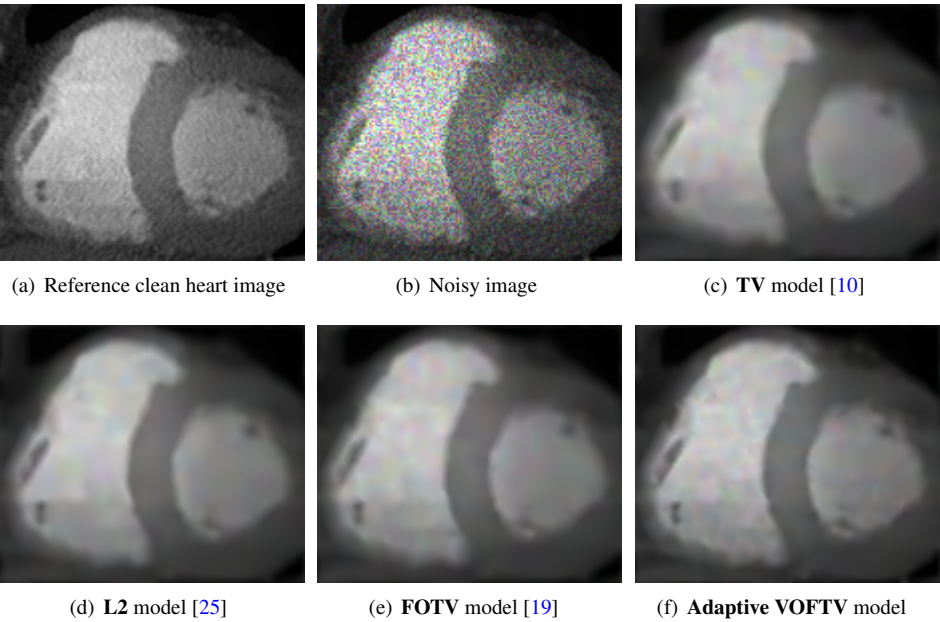


Figure 6. (a) Reference clean image. (b) Image degraded by speckle noise with variance 0.05, resulting in $SNR = 13.323$, $PSNR = 20.168$, and $SSIM = 0.431$. (c) Image restored using the TV, (d) L2, (e) FOTV and (f) the proposed adaptive VOFTV model. The red rectangles show zoomed areas for more details.

5 Conclusions

In this paper, we proposed a novel VOFTV model specifically designed for speckle noise removal in ultrasound images. The method relies on a spatially adaptive fractional order that adjusts to local image structures, allowing strong noise reduction in homogeneous regions while preserving edges and fine details. We proved the existence and uniqueness of the solution in the variable-order bounded variation framework, and used an efficient ADMM-based algorithm with Grünwald–Letnikov discretization for numerical implementation. Experiments on synthetic and real medical images show that the proposed model produces cleaner images with better edge and texture preservation compared with classical approaches. Quantitative results consistently demonstrate improvements in SNR, PSNR, and SSIM while maintaining efficient convergence. Future work will focus on extending the proposed VOFTV framework to joint reconstruction–segmentation tasks, incorporating learning-based priors for data-driven order adaptation for large-scale 3D ultrasound imaging problems.

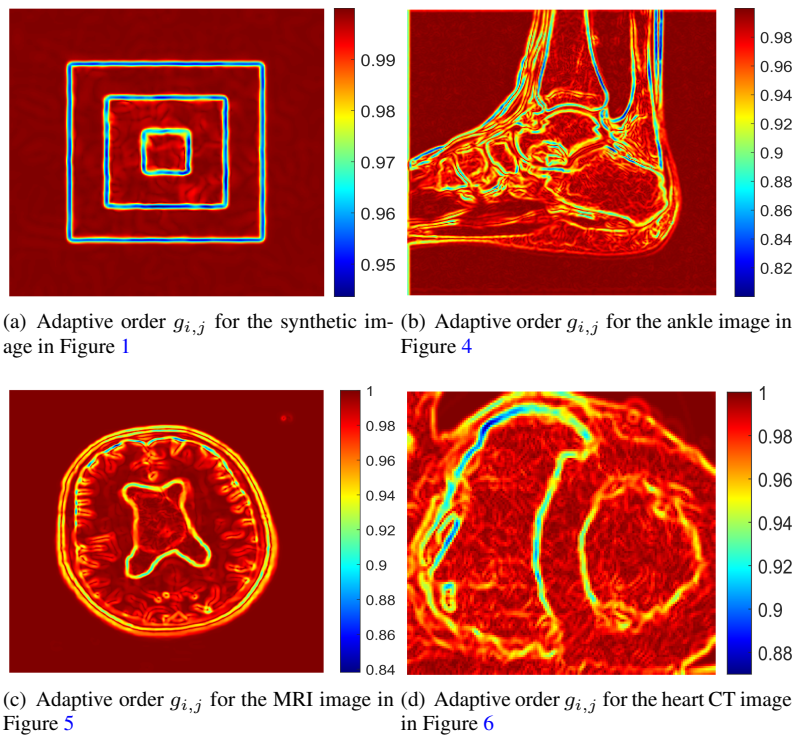


Figure 7. Spatial maps of the adaptive fractional order $g_{i,j}$ during the restoration process. The order is updated periodically from intermediate estimates $\mathbf{U}^k$ and guides the evolution of the solution.

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