

NON-UNIQUENESS OF SOLUTIONS OF THE GAUSE SYSTEM: GENERALIZATION TO THE KOLMOGOROV SYSTEM

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Abstract This paper is devoted to the study of non-uniqueness phenomena arising in a class of predator–prey systems of Kolmogorov type involving singular functional responses near the prey-free axis. We establish sufficient conditions ensuring that every point located on the positive vertical axis generates multiple solutions. The proof combines a suitable reduction of the original system to a scalar differential equation together with Tamarkine’s classical criterion for non-uniqueness of ordinary differential equations. The obtained result extends previous works devoted exclusively to singular Gause-type systems. Several ecological models are revisited within this unified framework, including the Hsu, Bazykin, Cavani–Farkas and Variable-Territory models.

1 Introduction

Predator–prey systems constitute one of the most important classes of nonlinear dynamical systems arising in mathematical biology. Since the pioneering works of Lotka and Volterra, many models have been proposed in order to incorporate more realistic ecological mechanisms such as saturation effects, density dependence, refuge phenomena and variable mortality rates. Among the broad families of ecological models, Kolmogorov systems occupy a central position because they provide a general framework capable of including numerous predator–prey interactions as special cases. In this work, we investigate the qualitative behavior of a Kolmogorov-type predator–prey system exhibiting singular dynamics close to the prey-free axis.

More precisely, we consider the system

$$\begin{cases} \dot{x} = xG(x, y), \\ \dot{y} = yH(x, y), \end{cases} \quad (1.1)$$

where $x(t) \geq 0$ and $y(t) \geq 0$ represent the prey and predator populations, respectively.

System (1.1) contains, as a particular situation, the classical Gause-type model

$$\begin{cases} \dot{x} = xf(x) - yg(x), \\ \dot{y} = y(\mu g(x) - D), \end{cases} \quad (1.2)$$

where $f(x)$ represents the intrinsic growth of the prey population and $g(x)$ describes the predator functional response.

The study of singular predator–prey systems has recently attracted considerable attention. In particular, it was shown in previous work [1] that when $g(x)$ becomes singular at the origin, the associated Gause model may lose uniqueness of solutions on the vertical axis. The present paper aims at extending this phenomenon to a much broader class of Kolmogorov systems.

The main contribution of this article is the establishment of a general non-uniqueness theorem valid for systems of the form (1.1). Under suitable structural and monotonicity assumptions, we

prove that every point $(0, y_0)$, with $y_0 > 0$, gives rise to at least two distinct solutions. The proof relies on a transformation reducing the system to a scalar differential equation together with Tamarkine's classical theorem.

The paper is organized as follows. Section 2 introduces the assumptions and preliminary results. Section 3 contains the main theorem and its proof. In Section 4, several well-known ecological models are analyzed within the proposed framework. Finally, Section 5 discusses qualitative dynamical aspects and possible future developments.

2 Hypotheses and Preliminary Results

Throughout this paper, the functions $G(x, y)$ and $H(x, y)$ are assumed to satisfy the following structural hypotheses (see [2] for the background on the Kolmogorov system (1.1)):

- (K1) G and H are C^1 on $[0, \infty)^2$, except possibly on the vertical axis $x = 0$.
- (K2) $\partial_y G(x, y) < 0$ for all $x \geq 0$.
- (K3) $\partial_y H(x, y) \leq 0$ for all $x \geq 0$.
- (K4) $\partial_x H(x, y) > 0$ for all $y \geq 0$.
- (K5) $G(0, 0) > 0$.
- (K6) There exists $K > 0$ such that $G(K, 0) = 0$ and $(x - K)G(x, 0) < 0$ for $x > 0$, $x \neq K$.
- (K7) $H(0, 0) < 0$, $H(K, 0) > 0$, and there exists $x_1 \in (0, K)$ such that $H(x_1, 0) = 0$.

In addition, the singular nature of the system near the prey-free axis is expressed through the following assumptions.

- (H1) For fixed $y > 0$, $x \mapsto x \partial_y G(x, y)$ is not smooth at $x = 0$, i.e., $\partial_y G(0^+, y) = +\infty$.
- (H2) For all $(x, y) \in \Omega$, $\partial_y \left(\frac{G}{H} \right) (x, y) \geq 0$.
- (H3) For fixed $y > \varphi(x)$, the integral $\int_\varepsilon^\alpha \frac{dx}{x \partial_y G(x, y)}$ converges as $\varepsilon \rightarrow 0$, for some $\alpha > 0$.

The equilibria of system (1.1) correspond to the intersections of the prey and predator nullclines. The origin $E_0 = (0, 0)$ is always an equilibrium point, whereas assumption (K6) guarantees the existence of the boundary equilibrium $E_1 = (K, 0)$.

The prey nullcline is given by $\dot{x} = 0$ that is, either $x = 0$ or $G(x, y) = 0$. Under assumptions (K2), (K5) and (H1), the implicit equation $G(x, y) = 0$ defines a differentiable function $y = \phi(x)$. More precisely, the following lemma holds.

Lemma 2.1. *Assume that (K2), (K5) and (H1) hold. Then there exists a C^1 function $\phi : [0, K] \rightarrow \mathbb{R}_+$ such that $G(x, \phi(x)) = 0$ and $\phi(0) = 0$. Moreover,*

$$\phi'(x) = -\frac{G_x(x, \phi(x))}{G_y(x, \phi(x))}. \quad (2.1)$$

Proof. The monotonicity assumption (K2) allows the application of the Implicit Function Theorem. Consequently, the equation $G(x, y) = 0$ determines locally a unique differentiable function $y = \phi(x)$. Differentiating the identity $G(x, \phi(x)) = 0$ with respect to x immediately yields the derivative formula (2.1).

To establish the relation $\phi(0) = 0$, we use the mean value theorem. Since $\phi(x) > 0$ for $x > 0$, there exists some intermediate value $y_1 \in (0, \phi(x))$ satisfying

$$G(x, \phi(x)) - G(x, 0) = \phi(x)G_y(x, y_1).$$

Because $G(x, \phi(x)) = 0$, we obtain

$$\phi(x) = -\frac{G(x, 0)}{G_y(x, y_1)}.$$

Passing to the limit as $x \rightarrow 0^+$ and using assumptions (K5) and (H1) gives the desired conclusion.

The predator nullcline is defined by $\dot{y} = 0$, that is, $y = 0$ or $H(x, y) = 0$. Assumptions (K3), (K4) and (K7) imply the existence of an increasing function $x = \psi(y)$. We have the following more precise result.

Lemma 2.2. *Suppose that assumptions (K3), (K4) and (K7) are satisfied. Then there exists a differentiable function $\psi : [0, +\infty) \rightarrow \mathbb{R}_+$ such that $H(\psi(y), y) = 0$, with $\psi(0) = x_1$, and*

$$\psi'(y) = -\frac{H_y(\psi(y), y)}{H_x(\psi(y), y)}. \quad (2.2)$$

Proof. The proof again follows from the Implicit Function Theorem. Assumption (K4) guarantees that the derivative with respect to x never vanishes along the predator nullcline. Therefore, the equation $H(x, y) = 0$ uniquely determines a differentiable function $x = \psi(y)$. From (K3) and (K4), we deduce that $\psi'(y) \geq 0$. The relation $\psi(0) = x_1$ follows directly from assumption (K7).

3 Main Result

We study the initial value problem

$$\begin{cases} \dot{x} = xG(x, y), & x(0) = 0 \\ \dot{y} = yH(x, y), & y(0) = y_0 \end{cases} \quad (3.1)$$

where $y_0 > 0$. Define the region

$$\Omega = \{(x, y) : 0 \leq x < x_1, y > \phi(x)\}.$$

In the following lemma, we prove that G and H are negative on Ω .

Lemma 3.1. *For every point $(x, y) \in \Omega$, one has $G(x, y) < 0$ and $H(x, y) < 0$.*

Proof. Let $(x, y) \in \Omega$. Since $y > \phi(x)$ and $G_y < 0$, we obtain

$$G(x, y) < G(x, \phi(x)) = 0.$$

Similarly, because $x < x_1$, assumptions (K3), (K4) and (K7) imply

$$H(x, y) < H(x_1, y) < H(x_1, 0) = 0.$$

We now state the principal theorem of this paper.

Theorem 3.2. *Let $y_0 > 0$. Under assumptions (K1)–(K7) and (H1)–(H3), the point $(0, y_0)$ is a non-uniqueness point for system (3.1). More precisely, there exist at least two different solutions sharing the same initial condition. One solution remains on the vertical axis, whereas another positive solution can be continued backward in time.*

Proof. We introduce the transformation $y(t) = y_0 e^{u(t)}$. Substituting this expression into system (3.1) yields

$$\begin{cases} \dot{x} = xG(x, y_0 e^u), \\ \dot{u} = H(x, y_0 e^u), \end{cases} \quad (3.2)$$

with initial condition $x(0) = 0$ and $u(0) = 0$.

Because $H < 0$ in the considered region, the function $u(t)$ is strictly monotone and therefore invertible. Using u as a new independent variable, we derive from (3.2), the scalar equation

$$\frac{dx}{du} = F(u, x, y_0), \quad (3.3)$$

where

$$F(u, x, y_0) = \frac{xG(x, y_0 e^u)}{H(x, y_0 e^u)}.$$

The trivial solution $x(u) \equiv 0$ is clearly a solution of equation (3.3).

Next, we verify the assumptions of Tamarkine's theorem. By assumption (H2), the quantity

$$F(u, x, y_0) - F(u, 0, y_0)$$

can be bounded from below by the increasing function

$$R(x) = \frac{xG(x, y_0)}{H(x, y_0)}.$$

Moreover, using the mean value theorem and the singular behavior imposed by (H1), one obtains the asymptotic estimate

$$\frac{1}{R(x)} \sim \frac{H(0, y_0)}{y_0 x G_y(x, \bar{y})}$$

for small values of x .

Assumption (H3) then guarantees the convergence of the integral

$$\int_{\varepsilon}^{\alpha} \frac{dx}{R(x)}.$$

Consequently, Tamarkine's non-uniqueness criterion (see [3]) applies and equation (3.3) admits at least one nontrivial solution in addition to the trivial one.

Returning to the original variables, we conclude that the initial value problem (3.1) possesses multiple solutions starting from the same point $(0, y_0)$. One of these solutions remains on the vertical axis and satisfies

$$\int_{y_0}^{y(t)} \frac{dv}{vH(0, v)} = t.$$

Another solution is positive for negative times and reaches the point $(0, y_0)$ in finite time. This completes the proof.

4 Application to particular ecological models

The general theorem established above applies to a wide range of predator–prey systems. In this section, we briefly examine several classical models (see [4]).

4.1 The Gause model

The classical Gause system corresponds to

$$\begin{cases} \dot{x} = x f(x) - y g(x), \\ \dot{y} = y(\mu g(x) - D). \end{cases}$$

When the functional response satisfies $\lim_{x \rightarrow 0^+} \frac{g(x)}{x} = +\infty$ (see [1]), all assumptions of Theorem 3.2 are fulfilled. Therefore, every point $(0, y_0)$, with $y_0 > 0$, is a point of non-uniqueness. The solution evolving on the vertical axis is explicitly given by

$$y(t) = y_0 e^{-Dt} \quad \text{for } t \geq 0.$$

4.2 The Hsu model

We now consider the system introduced by Hsu (see [5]),

$$\begin{cases} \dot{x} = xf(x) - yg(x), \\ \dot{y} = y(\mu g(x) - D(x)), \end{cases} \quad (4.1)$$

where the mortality function $D(x)$ is positive and decreasing.

Since $H(x, y) = \mu g(x) - D(x)$ satisfies the monotonicity assumptions required in Section 2, the non-uniqueness theorem remains valid. Thus, every point on the positive vertical axis generates multiple trajectories.

The axis solution is

$$y(t) = y_0 e^{-D(0)t} \quad \text{for } t \geq 0.$$

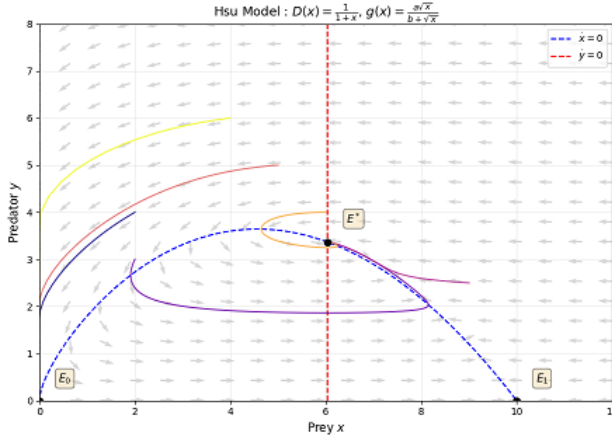


Figure 1. The y -axis points (except the origin) are non-uniqueness points; E^* is LAS.

4.3 The Bazykin model

The Bazykin model (see [6, 7]) incorporates density-dependent mortality:

$$\begin{cases} \dot{x} = xf(x) - yg(x), \\ \dot{y} = y(\mu g(x) - m - \alpha y), \end{cases} \quad (4.2)$$

with positive constants m and α .

The assumptions of Theorem 3.2 are again satisfied. Consequently, non-uniqueness occurs for every initial condition of the form $(0, y_0)$.

The corresponding axis trajectory is given by

$$y(t) = \frac{my_0 e^{-mt}}{\alpha y_0 (1 - e^{-mt}) + m} \quad \text{for } t \geq 0.$$

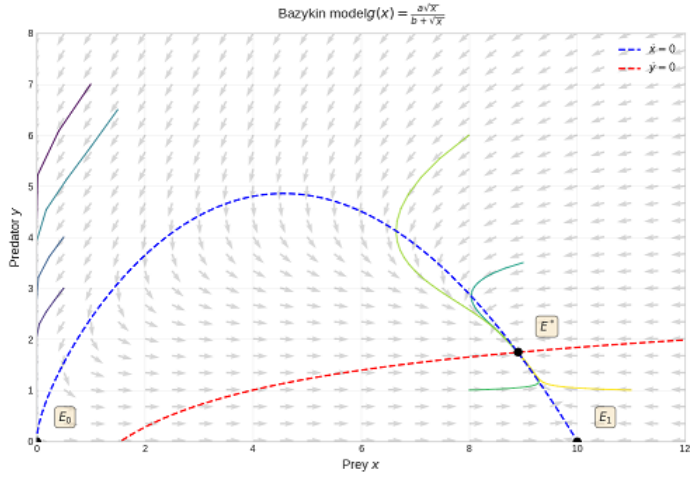


Figure 2. The y -axis points (except the origin) are non-uniqueness points; E^* is LAS.

4.4 The Cavani-Farkas model

Consider now the system (see [8])

$$\begin{cases} \dot{x} = xf(x) - yg(x), \\ \dot{y} = y \left(\mu g(x) - m - \frac{\alpha y}{1+y} \right). \end{cases} \quad (4.3)$$

The nonlinear mortality term preserves the structural properties required in our analysis. Hence, the conclusions of Theorem 3.2 remain valid.

The vertical-axis solution is implicitly characterized by

$$\frac{y(t)}{[(m + \alpha)y(t) + m]^{\frac{\alpha}{m+\alpha}}} = \frac{y_0}{[(m + \alpha)y_0 + m]^{\frac{\alpha}{m+\alpha}}} e^{-mt} \quad \text{for } t \geq 0.$$

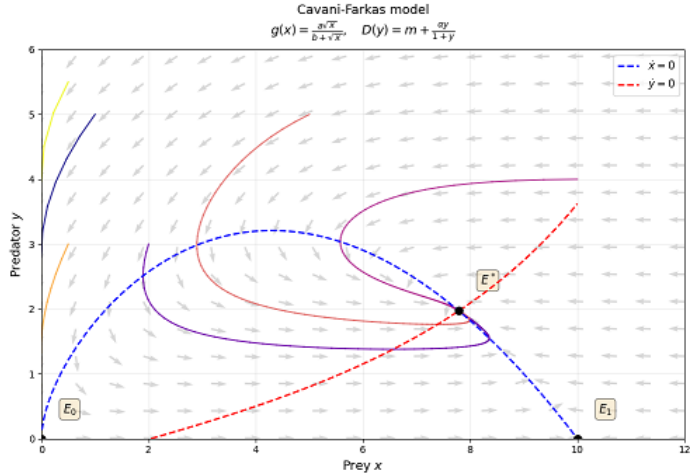


Figure 3. The y -axis points (except the origin) are non-uniqueness points; E^* is LAS.

4.5 The Variable-Territory model

Finally, we examine the Variable-Territory system (see [9, 10])

$$\begin{cases} \dot{x} = xf(x) - yg(x), \\ \dot{y} = y \left(\mu g(x) - m - \frac{\alpha y}{\delta + x} \right), \end{cases} \quad (4.4)$$

where m , α and δ are positive constants.

The dependence of the mortality term on both prey and predator densities does not affect the validity of the hypotheses introduced in Section 2. Therefore, the same non-uniqueness phenomenon persists.

The solution constrained to the vertical axis takes the form

$$y(t) = \frac{m\delta y_0 e^{-mt}}{m\delta + \alpha y_0(1 - e^{-mt})} \quad \text{for } t \geq 0.$$

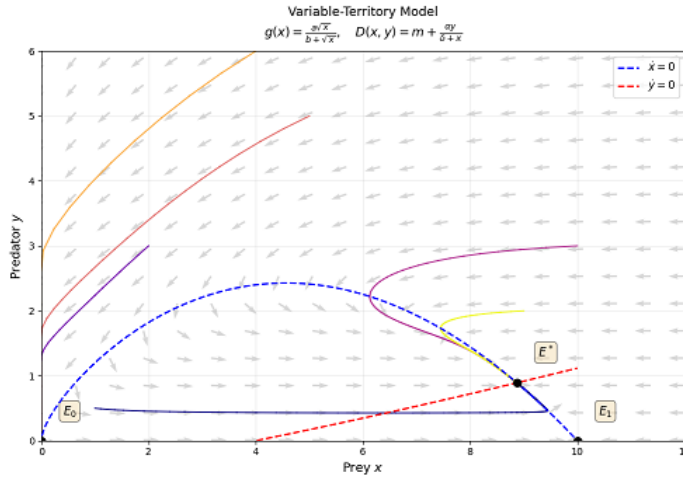


Figure 4. The y -axis points (except the origin) are non-uniqueness points; E^* is LAS.

5 Qualitative Dynamical Aspects

The systems considered in this paper possess two boundary equilibria: $E_0 = (0, 0)$ and $E_1 = (K, 0)$.

The equilibrium E_0 is nonhyperbolic because of the singularity located on the prey-free axis, whereas E_1 typically behaves as a saddle point.

When a unique positive equilibrium $E^* = (x^*, y^*)$ exists, its coordinates satisfy $y^* = \phi(x^*)$ and $x^* = \psi(y^*)$.

The global organization of the phase portrait strongly depends on the interaction between the stable manifold of E_0 and the unstable manifold of E_1 . The existence of multiple positive equilibria may generate more complicated dynamical scenarios involving coexistence regions, bistability or oscillatory behavior.

6 Conclusion

In this article, we established a general non-uniqueness theorem for singular Kolmogorov predator-prey systems. The obtained result considerably extends previously known results restricted to singular Gause models.

The main mechanism responsible for the loss of uniqueness is the singular behavior of the functional response near the prey-free axis. By combining a suitable change of variables with Tamarkine's classical criterion, we proved that every point on the positive vertical axis generates multiple solutions.

Several important ecological models were shown to fit naturally into the proposed framework, demonstrating the robustness of the approach.

Future investigations will focus on a deeper qualitative analysis of the considered systems. In particular, it would be interesting to study the global phase portraits, bifurcation structures and stability properties associated with the non-uniqueness phenomenon. Extensions involving delays, diffusion effects or stochastic perturbations also appear to be promising research directions.

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