

GENERALISED FORMULA FOR THE INFECTION RATE IN THE DYNAMIC ANALYSIS OF A FRACTIONAL-ORDER HIV/AIDS MODEL

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This manuscript is an extended and significantly improved version of a full paper entitled '[Study of a Fractional- Order Models for HIV/AIDS]' presented at ICRAMI 2025 (Tunisia, November 2025). Compared to the conference version, this paper includes (Extended methodology in existence of equilibrium points and proving the local and global stability of the proposed model, additional figures and results, and new analyses).

Abstract The long-memory nature of HIV/AIDS progression, treatment exposure, and delayed development of advanced disease necessitates a fractional formulation. We analyze a deterministic HIV/AIDS model governed by the Caputo derivative in which HIV-infected individuals are represented through a distinct compartmental structure. An explicit expression for the basic reproductive number $\mathcal{R}_0$ is obtained and used to distinguish disease clearance from persistent transmission. Suitable Lyapunov functionals, together with LaSalle's invariance principle, yield global asymptotic stability of the disease-free and endemic equilibria under the corresponding threshold conditions. The simulations support the analytical thresholds and emphasize that early intervention can reduce HIV transmission, slow progression to AIDS, and strengthen the effectiveness of treatment strategies.

1 Introduction

HIV is the one of the most destructive viruses with widespread impact on human populations. According to estimates from the World Health Organization (WHO) and the joint United Nations Programme on HIV/AIDS (UNAIDS), approximately 630,000 people died from AIDS-related illnesses in 2023, 39.9 million people were living with HIV, and 1.3 million were newly infected with HIV [1]. Several mathematical frameworks have been proposed to analyze the dynamics of HIV transmission in vivo. The field of fractional calculus extends classical differentiation and integration concepts to operators of fractional-order, utilizing fractional-order differential operators offers substantially higher accuracy compared to traditional integer-order formulations. The application of fractional calculus to HIV modeling has yielded diverse and insightful results. Numerous studies have investigated fractional mathematical models of HIV/AIDS infectious dynamics [2, 3, 4, 5]. Fractional-order frameworks have been particularly effective in revealing the rich dynamical behavior embedded within epidemiological systems, including stability properties, bifurcation structures, and the influence of fractional orders on disease persistence [6]. Furthermore, studies such as [7] and [8] have established strong theoretical foundations for adoption of fractional-order derivative methods in biological modeling, underscoring the necessity of fractional-order approaches for accurately capturing real-world processes that involve anomalous diffusion or long-memory effects. The basic reproductive number $\mathcal{R}_0$, a fundamental infection dynamics of disease spread, has been extensively studied in fractional-order models to determine threshold conditions for disease eradication or persistence [9, 10, 11, 12, 13]. Stability analysis, both local and global, using eigenvalue techniques and Lyapunov functions adapted for fractional systems, facilitates a comprehensive analysis of the long term patterns of infectious disease transmission [6, 14, 15, 16, 17]. Despite these advancements, challenges remain in fully characterizing the qualitative behavior of fractional reaction-diffusion models, particularly

with respect to the influence of spatial heterogeneity, fractional orders, and nonlinear infection rates [7, 8, 18, 19, 20]. Existing studies suggest that a deeper investigation of these factors can significantly enhance the predictive power and reliability of mathematical models for infectious diseases. The current study research can be viewed as a generalization of the findings reported in [21]. The remainder of this paper is organized as follows. In Section 2, we present the fundamental results of fractional calculus required for the analysis. Section 3 is devoted to the formulation of the proposed model. In Section 4, we investigate the positivity and boundedness of solutions and existence of equilibria. In section 5, we study the local and global stability of the equilibrium points. Finally, Section 6 provides numerical simulations that illustrate the theoretical findings.

2 Fundamental Fractional Results

Let us review some fundamental definitions related to the Caputo differential operator in fractional calculus.

Definition 2.1. Let the $\lambda > 0, t > a$ with $\lambda, a, t \in \mathbb{R}$. The fractional derivative of Caputo is given by

$${}_a^C D_t^\lambda f(t) = \frac{1}{\Gamma(n-\lambda)} \int_a^t \frac{f^{(n)}(\xi)}{(t-\xi)^{\lambda+1-n}} d\xi, \quad (2.1)$$

$$n-1 < \lambda < n \in \mathbb{N}.$$

where Γ representing the gamma function.

Proposition 2.2. [13] Let $f(t), g(t) : [a, b] \rightarrow \mathbb{R}$ be such that ${}_a^C D_t^\lambda f(t)$ and ${}_a^C D_t^\lambda g(t)$ exist almost everywhere and let $c_1, c_2 \in \mathbb{R}$, then, ${}_a^C D_t^\lambda (c_1 f(t) + c_2 g(t))$ exists almost everywhere and

$${}_a^C D_t^\lambda (c_1 f(t) + c_2 g(t)) = c_1 {}_a^C D_t^\lambda f(t) + c_2 {}_a^C D_t^\lambda g(t). \quad (2.2)$$

Corollary 2.3. [22] Suppose that $g(x) \in C[0, b]$ and ${}_0^C D_x^\lambda g(x) \in C(0, b]$ for $0 < \lambda \leq 1$. If

$${}_0^C D_x^\lambda g(x) \geq 0, \quad \forall x \in (0, b),$$

then the function g is non-decreasing on $(0, b)$. Moreover, if

$${}_0^C D_x^\lambda g(x) \leq 0, \quad \forall x \in (0, b),$$

then the function g is non-increasing on $(0, b)$.

Lemma 2.4. [26] Let $x : [t_0, \infty) \rightarrow \mathbb{R}$ be a continuous function satisfying

$${}_0^C D_t^\lambda x(t) + \mu x(t) \leq \nu, \quad t \geq t_0, 0 < \lambda \leq 1.$$

where $\mu, \nu \in \mathbb{R}$ with $\mu \neq 0$. Then the following inequality holds:

$$x(t) \leq \left(x(t_0) - \frac{\nu}{\mu} \right) E_\alpha(-\mu(t-t_0)^\alpha) + \frac{\nu}{\mu}, \quad \forall t \geq t_0,$$

where $E_\lambda(\cdot)$ is the Mittag-Leffler function defined by

$$E_\lambda(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\lambda k + 1)}.$$

Proposition 2.5. [8] The fractional derivative for a constant function $f(t) = c$ is zero

$${}_a^C D_t^\lambda C = 0.$$

Lemma 2.6. [10] Let $x(t) \in \mathbb{R}$ be a continuous and derivable function. Then, for any time instant $t \geq t_0$

$$\frac{1}{2}({}_a^C D_t^\lambda)x^2(t) \leq {}_a^C D_t^\lambda x(t), \quad \forall \lambda \in (0, 1). \quad (2.3)$$

Remark 2.7. Consider the family of quadratic Lyapunov function

$$L(x_1, x_2, \dots, x_n) = \sum_{i=1}^n \frac{c_i}{2} (x_i(t) - x_i^*)^2, \quad i = 1, 2, \dots, n$$

where $c_i > 0$. By using (2.3), we get

$${}_a^C D_t^\lambda L(x_1, x_2, \dots, x_n) \leq \sum_{i=1}^n c_i {}_a^C D_t^\lambda (x_i(t) - x_i^*), \quad (2.4)$$

$$x_i(t), x_i^* \in \mathbb{R}, \quad 0 < \lambda \leq 1.$$

Lemma 2.8. [10] Suppose that $x(t) \in \mathbb{R}_+$ is a continuous and differentiable function. Then, for all $t \geq t_0$, $\lambda \in (0, 1]$, and $x^* \in \mathbb{R}_+$, we have

$${}_a^C D_t^\lambda \left[x(t) - x^* - x^* \ln \frac{x(t)}{x^*} \right] \leq \left(1 - \frac{x^*}{x(t)} \right) {}_a^C D_t^\lambda x(t). \quad (2.5)$$

3 Formulation of Model

In this section, we extend the model studied by [21]. The population N is divided into the following six compartments:

(U), the susceptible class

(V_1), the asymptomatic latent HIV cases

(V_2), the symptomatic cases

(A), the AIDS patients

(T), the individuals under treatment

(R), the individuals who change their behavior to avoid infection. The groups are explained by the following equations:

$$\begin{cases} {}_0^C D_t^\alpha U = \Lambda - \beta_1 U \varphi(V_1) - \beta_2 U \varphi(V_2) - (\omega + \mu) U, \\ {}_0^C D_t^\alpha V_1 = \beta_1 U \varphi(V_1) + \beta_2 U \varphi(V_2) - (k_1 + k_2 + \mu) V_1, \\ {}_0^C D_t^\alpha V_2 = \alpha_1 T + k_1 V_1 - (\delta_1 + \delta_2 + \mu) V_2, \\ {}_0^C D_t^\alpha T = \delta_1 V_2 - (\alpha_1 + \alpha_2 + \zeta_2 + \mu) T, \\ {}_0^C D_t^\alpha A = k_2 V_1 + \delta_2 V_2 + \alpha_2 T - (\zeta_1 + \mu) A, \\ {}_0^C D_t^\alpha R = \omega U - \mu R, \end{cases} \quad (3.1)$$

where the initial conditions are

$$U(0) \geq 0, V_1(0) \geq 0, V_2(0) \geq 0, T(0) \geq 0, R(0) \geq 0,$$

where ${}_a^C D_t^\alpha$ denotes the Caputo fractional-order derivative with $0 < \alpha \leq 1$. All parameters in model (3.1) are positive constants. The parameters' description, value, units, and reference are given in Table 1.

Table 1. Parameters description, values, units and sources used in model (3.1)

Parameter	Description	Value	Source
Λ	Population recruitment rate	0.55 year^{-1}	[23]
β_1	Transmission rate (asymptomatic individuals)	0.01 year^{-1}	Estimate
β_2	Transmission rate (symptomatic individuals)	0.03 year^{-1}	Estimate
ω	Behavioral transition rate of susceptible individuals	0.03 year^{-1}	[24]
μ	Natural mortality rate	0.0196 year^{-1}	[25]
k_1	Progression rate from V_1 to V_2	0.198 year^{-1}	[25]
k_2	Transfer rate from V_1 to T	0.35 year^{-1}	[24]
α_1	Rate of successful treatment	0.1 year^{-1}	[24]
α_2	Rate of treatment failure	0.01 year^{-1}	[24]
δ_1	Progression rate from V_2 to A	0.4621 year^{-1}	[25]
δ_2	Transfer rate from V_2 to T	0.35 year^{-1}	[24]
ζ_1	Mortality rate in AIDS class	0.0909 year^{-1}	[25]
ζ_2	Mortality rate in treatment class T	0.0667 year^{-1}	[25]

For simplicity, denote $b_1 = k_1 + k_2 + \mu$, $b_2 = \delta_1 + \delta_2 + \mu$, and $b_3 = \alpha_1 + \alpha_2 + \zeta_2 + \mu$, then the model (3.1) becomes. The system becomes as follows:

$$\begin{cases} {}^C_0 D_t^\alpha U = \Lambda - \beta_1 U \varphi(V_1) - \beta_2 U \varphi(V_2) - (\omega + \mu)U, \\ {}^C_0 D_t^\alpha V_1 = \beta_1 U \varphi(V_1) + \beta_2 U \varphi(V_2) - b_1 V_1, \\ {}^C_0 D_t^\alpha V_2 = \alpha_1 T + k_1 V_1 - b_2 V_2, \\ {}^C_0 D_t^\alpha T = \delta_1 V_2 - b_3 T, \\ {}^C_0 D_t^\alpha A = k_2 V_1 + \delta_2 V_2 + \alpha_2 T - (\zeta_1 + \mu) A, \\ {}^C_0 D_t^\alpha R = \omega U - \mu R. \end{cases} \quad (3.2)$$

The nonlinear infection rate $\varphi(V_i)$ is assumed to satisfy:

$$\varphi(0) = 0, \quad (3.3)$$

and

$$0 < V_i \varphi'(V_i) \leq \varphi(V_i), \quad \forall V_i > 0, i = 1, 2 \quad (3.4)$$

Lemma 3.1. Conditions (3.3)-(3.4) implies

$$0 < \frac{\varphi(V_i)}{V_i} \leq \varphi'(0), \quad (3.5)$$

4 Positivity and boundedness

We record the well-posedness properties needed for system (3.2), namely existence, uniqueness, positivity, and boundedness of its solutions.

Theorem 4.1. For any positive initial condition $X(0) = (U(0), V_1(0), V_2(0), T(0), A(0), R(0))^T$, there is always a unique solution for model (3.2) on $[0, +\infty)$.

Proof. Let $X(t) = (U(t), V_1(t), V_2(t), T(t), A(t), R(t))^T$. Model (3.2) is written as

$${}_a^C D_t^\alpha X(t) = F(X(t)), t \geq 0 \quad (4.1)$$

where

$$F(X(t)) = \begin{pmatrix} \Lambda - \beta_1 U \varphi(V_1) - \beta_2 U \varphi(V_2) - (\omega + \mu)U \\ \beta_1 U \varphi(V_1) + \beta_2 U \varphi(V_2) - b_1 V_1 \\ \alpha_1 T + k_1 V_1 - b_2 V_2 \\ \delta_1 V_2 - b_3 T \\ k_2 V_1 + \delta_2 V_2 + \alpha_2 T - (\zeta_1 + \mu) A \\ \omega U - \mu R \end{pmatrix}.$$

By Lemma 3.1 and the Lagrange mean value theorem, we can write

$$F(X) = Q_0 + (Q_1(\xi) + Q_2)X$$

$$Q_0 = \begin{pmatrix} \Lambda \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, Q_1(\xi) = \begin{pmatrix} -(\beta_1 \varphi(\xi_1) + \beta_2 \varphi(\xi_2)) & 0 & 0 & 0 & 0 & 0 \\ (\beta_1 \varphi(\xi_1) + \beta_2 \varphi(\xi_2)) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$Q_2 = \begin{pmatrix} -(\omega + \mu) & 0 & 0 & 0 & 0 & 0 \\ 0 & -b_1 & 0 & 0 & 0 & 0 \\ 0 & k_1 & -b_2 & \alpha_1 & 0 & 0 \\ 0 & 0 & \delta_1 & -b_3 & 0 & 0 \\ 0 & k_2 & \delta_2 & \alpha_2 & -(\zeta_1 + \mu) & 0 \\ \omega & 0 & 0 & 0 & 0 & -\mu \end{pmatrix}.$$

For model (3.2), one obtains $\|F(X(t))\| \leq \|Q_0\| + (\|Q_1\| + \|Q_2\|) \|X\|$. Hence, by Lemma 4 in [18], model (3.2) has a unique solution on $[0, +\infty)$.

It remains to place the solution of model (3.2) in the non-negative region H , where

$$H = \{(U, V_1, V_2, T, A, R) \in \mathbb{R}_+^6 \mid U \geq 0, V_1 \geq 0, V_2 \geq 0, A \geq 0, T \geq 0, R \geq 0\}.$$

Theorem 4.2. *If the initial conditions of model (3.2) belong to H , then the corresponding solutions remain in H .*

Proof. Using Lemma 3.1, we have:

$$\begin{aligned} {}^C_0 D_t^\alpha U|_{U=0} &= \Lambda > 0, \\ {}^C_0 D_t^\alpha V_1|_{V_1=0} &= \beta_2 U \varphi(V_2) \geq 0, \\ {}^C_0 D_t^\alpha V_2|_{V_2=0} &= \alpha_1 T + k_1 V_1 \geq 0, \\ {}^C_0 D_t^\alpha T|_{T=0} &= \delta_1 V_2 \geq 0, \\ {}^C_0 D_t^\alpha A|_{A=0} &= k_2 V_1 + \delta_2 V_2 + \alpha_2 T \geq 0, \\ {}^C_0 D_t^\alpha R|_{R=0} &= \omega U \geq 0. \end{aligned}$$

Lemma 2.4 then implies that the solution of (3.2) is nonnegative.

For boundedness, summing all equations in model (3.2) gives

$${}^C_0 D_t^\alpha N(t) \leq \Lambda - \mu N(t).$$

Applying the Laplace transform gives

$$S^\alpha L[N(t)] - S^{\alpha-1}N(0) \leq \frac{\Lambda}{S} - \mu L[N(t)],$$

then

$$L[N(t)] \leq (S^\alpha + \mu)^{-1} [S^{\alpha-1}N(0) + \frac{\Lambda}{S}].$$

Applying the inverse Laplace transform yields

$$\begin{aligned} N(t) &\leq L^{-1}[(S^\alpha + \mu)^{-1} S^{\alpha-1}N(0)] + L^{-1}[(S^\alpha + \mu)^{-1} \frac{\Lambda}{S}] \\ &= N(0)E_{\alpha,1}(-\mu t^\alpha) + \Lambda t^\alpha E_{\alpha,\alpha+1}(-\mu t^\alpha). \end{aligned}$$

Let $N(0) \leq \frac{\Lambda}{\mu}$, then $N(t) \leq \frac{\Lambda}{\mu} [E_{\alpha,1}(-\mu t^\alpha) + \mu t^\alpha E_{\alpha,\alpha+1}(-\mu t^\alpha)] = \frac{\Lambda}{\mu} \frac{1}{\Gamma(1)}$. Therefore, the solutions of model (3.2) are positive and bounded.

4.1 Equilibria and the basic reproductive number

Because the first four equations do not involve A and R , the equilibrium analysis may be carried out on the subsystem

$$\begin{cases} {}^C_0 D_t^\alpha U = \Lambda - \beta_1 U \varphi(V_1) - \beta_2 U \varphi(V_2) - (\omega + \mu)U, \\ {}^C_0 D_t^\alpha V_1 = \beta_1 U \varphi(V_1) + \beta_2 U \varphi(V_2) - b_1 V_1, \\ {}^C_0 D_t^\alpha V_2 = \alpha_1 T + k_1 V_1 - b_2 V_2, \\ {}^C_0 D_t^\alpha T = \delta_1 V_2 - b_3 T. \end{cases} \quad (4.2)$$

With $U^0 = U(0)$, $V_1^0 = V_1(0)$, $V_2^0 = V_2(0)$, and $T^0 = T(0)$, the disease-free equilibrium of model (4.2) is $E^0 = (U^0, 0, 0, 0)$, where $U^0 = \frac{\Lambda}{\omega + \mu}$. The basic reproduction number of model (4.2) is obtained by the next-generation matrix method [28].

Proposition 4.3. *The basic reproductive number of model (4.2) takes the following form:*

$$\mathfrak{R}_0 = \frac{\Lambda \dot{\varphi}(0)}{(\mu + \omega) b_1} (\beta_1 + \beta_2 \frac{k_1 b_3}{\delta_1 (\alpha_2 + \zeta_2 + \mu) + (\delta_2 + \mu) b_3}).$$

Proof. Model (4.2) can be written as

$${}^C_0 D_t^\alpha Y = F(Y) - G(Y), \quad \text{with } Y = (V_1, V_2, T)^T,$$

where

$$F(Y) = \begin{pmatrix} \frac{\Lambda \beta_1}{\mu + \omega} \dot{\varphi}(0) + \frac{\Lambda \beta_2}{\mu + \omega} \dot{\varphi}(0) \\ 0 \\ 0 \end{pmatrix}, \quad G(Y) = \begin{pmatrix} b_1 V_1 \\ -\alpha_1 T - k_1 V_1 + b_2 V_2, \\ -\delta_1 V_2 + b_3 T \end{pmatrix}.$$

Let $\tilde{F}$ and $\tilde{G}$ be the Jacobian matrices of $F(Y)$ and $G(Y)$ at E^0 , respectively. Then

$$\tilde{F} = \begin{pmatrix} \frac{\Lambda \beta_1}{\mu + \omega} \dot{\varphi}(0) & \frac{\Lambda \beta_2}{\mu + \omega} \dot{\varphi}(0) & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \tilde{G} = \begin{pmatrix} b_1 & 0 & 0 \\ -k_1 & b_2 & -\alpha_1 \\ 0 & -\delta_1 & b_3 \end{pmatrix}.$$

The largest eigenvalue, namely the spectral radius of $\tilde{F} \tilde{G}^{-1}$, gives the basic reproductive number

$$\mathfrak{R}_0 = \frac{\Lambda \dot{\varphi}(0)}{(\mu + \omega) b_1} \left(\beta_1 + \beta_2 \frac{k_1 b_3}{b_2 b_3 - \delta_1 \alpha_1} \right). \quad (4.3)$$

After simplification, this becomes

$$\mathfrak{R}_0 = \frac{\Lambda \dot{\varphi}(0)}{(\mu + \omega) b_1} (\beta_1 + \beta_2 \frac{k_1 b_3}{\delta_1 (\alpha_2 + \zeta_2 + \mu) + (\delta_2 + \mu) b_3}).$$

Theorem 4.4. *If $\mathfrak{R}_0 > 1$, then model (4.2) has a unique Endemic equilibrium point (EEP) $E^* = (U^*, V_1^*, V_2^*, A^*)$.*

Proof. To find the other equilibrium point, let $E^* = (U^*, V_1^*, V_2^*, A^*)$ be any equilibrium point of model (4.2) that satisfies the equations as follows:

$$\begin{cases} 0 = \Lambda - \beta_1 U^* \varphi(V_1^*) - \beta_2 U^* \varphi(V_2^*) - (\omega + \mu) U^*, \\ 0 = \beta_1 U^* \varphi(V_1^*) + \beta_2 U^* \varphi(V_2^*) - b_1 V_1^*, \\ 0 = \alpha_1 T^* + k_1 V_1^* - b_2 V_2^*, \\ 0 = \delta_1 V_2^* - b_3 T^*. \end{cases} \quad (4.4)$$

Calculating the above equalities, we obtain

$$U^* = \frac{\Lambda - b_1 V_1^*}{\omega + \mu}, V_2^* = \frac{k_1 b_3 V_1^*}{b_2 b_3 - \alpha_1 \delta_1}, T^* = \frac{k_1 \delta_1 V_1^*}{b_2 b_3 - \alpha_1 \delta_1}.$$

By the second equation of model (4.4), we define the following:

$$\phi(V_1^*) = \frac{\Lambda - b_1 V_1^*}{\omega + \mu} \left[\beta_1 \varphi(V_1^*) + \beta_2 \varphi\left(\frac{k_1 b_3 V_1^*}{b_2 b_3 - \alpha_1 \delta_1}\right) \right] - b_1 V_1^*. \quad (4.5)$$

Using (3.3), we can deduce that $\phi(0) = 0$ and $\phi\left(\frac{\Lambda}{b_1}\right) = -\Lambda < 0$.

The function $\phi(E)$ is continuously differentiable with respect to the variable V_1^* in the interval $[0, \frac{\Lambda}{b_1}]$, we obtained

$$\begin{aligned} \phi'(V_1^*) &= \frac{-b_1}{\omega + \mu} \left[\beta_1 \varphi(V_1^*) + \beta_2 \varphi\left(\frac{k_1 b_3 V_1^*}{b_2 b_3 - \alpha_1 \delta_1}\right) \right] \\ &\quad + \frac{\Lambda - b_1 V_1^*}{\omega + \mu} \left[\beta_1 \phi'(V_1^*) + \beta_2 \frac{k_1 b_3}{b_2 b_3 - \alpha_1 \delta_1} \phi'\left(\frac{k_1 b_3 V_1^*}{b_2 b_3 - \alpha_1 \delta_1}\right) \right] - b_1. \end{aligned}$$

If $\phi'(0) > 0$, then $\phi(V_1^*) = 0$, the function ϕ admits at least one positive root V_1^* in the interval $(0, \frac{\Lambda}{b_1}]$. By (3.3) and (3.4), we obtain

$$\begin{aligned} \phi'(0) &= \frac{-b_1}{\omega + \mu} [\beta_1 \varphi(0) + \beta_2 \varphi(0)] + \frac{\Lambda}{\omega + \mu} \left[\beta_1 \phi'(0) + \beta_2 \frac{k_1 b_3}{b_2 b_3 - \alpha_1 \delta_1} \phi'(0) \right] - b_1. \\ &= b_1(\mathfrak{R}_0 - 1) \end{aligned}$$

Therefore, $\mathfrak{R}_0 > 1$ leads to $\phi'(0) > 0$. Then, the model (3.4) has at least an unique Endemic equilibrium point $E^* = (U^*, V_1^*, V_2^*, A^*)$, where

$$U^* = \frac{\Lambda - b_1 V_1^*}{\omega + \mu}, V_2^* = \frac{k_1 b_3 V_1^*}{b_2 b_3 - \alpha_1 \delta_1}, T^* = \frac{k_1 \delta_1 V_1^*}{b_2 b_3 - \alpha_1 \delta_1},$$

we prove the uniqueness of the Endemic equilibrium point. Substituting E^* into the second equation of model (4.4) yields the following:

$$\frac{\beta_1 U^* \varphi(V_1^*)}{V_1^*} + \frac{\beta_2 U^* \varphi(V_2^*)}{V_1^*} = b_1,$$

Using the above equality and (3.5), we obtain $\phi'(V_1^*) < 0$. We utilize the contradiction argument to prove the uniqueness of the Endemic equilibrium point $E^* = (U^*, V_1^*, V_2^*, A^*)$. Suppose there is another the Endemic equilibrium point $E^{**} = (U^{**}, V_1^{**}, V_2^{**}, A^{**})$, after some simple calculations, it is easy to arrive that $\phi(E^{**}) > 0$, which is a contradiction.

5 The local and global stability of the equilibrium points

We first analyze the local stability of the equilibrium points E^0 and E^* of model (4.2).

Theorem 5.1. *For $0 < \alpha \leq 1$, the disease-free equilibrium E^0 is locally asymptotically stable if $\mathfrak{R}_0 \leq 1$.*

Proof. The Jacobian matrix of model (4.2) is

$$J(U, V_1, V_2, T) = \begin{pmatrix} -(\omega + \mu) - \beta_1 U \varphi(V_1) - \beta_2 U \varphi(V_2) & -\beta_1 U \dot{\varphi}(V_1) & -\beta_2 U \dot{\varphi}(V_2) & 0 \\ \beta_1 U \varphi(V_1) + \beta_2 U \varphi(V_2) & \beta_1 U \dot{\varphi}(V_1) - b_1 & \beta_2 U \dot{\varphi}(V_2) & 0 \\ 0 & k_1 & -b_2 & \alpha_1 \\ 0 & 0 & \delta_1 & -b_3 \end{pmatrix}.$$

Evaluating $J(U, V_1, V_2, T)$ at E^0 gives

$$J(E^0) = \begin{pmatrix} -(\omega + \mu) & -\frac{\Lambda\beta_1}{\mu+\omega}\dot{\varphi}(0) & -\frac{\Lambda\beta_2}{\mu+\omega}\dot{\varphi}(0) & 0 \\ 0 & \frac{\Lambda\beta_1}{\mu+\omega}\dot{\varphi}(0) - b_1 & \frac{\Lambda\beta_2}{\mu+\omega}\dot{\varphi}(0) & 0 \\ 0 & k_1 & -b_2 & \alpha_1 \\ 0 & 0 & \delta_1 & -b_3 \end{pmatrix},$$

The characteristic equation is $|\lambda I - J(E^0)| = (\lambda + \omega + \mu)P(\lambda)$, where

$$P(\lambda) = \lambda^3 + x_1\lambda^2 + x_2\lambda + x_3,$$

and

$$\begin{aligned} x_1 &= b_2 + b_3 + (b_1 - \frac{\Lambda\beta_1}{\mu+\omega}\dot{\varphi}(0)), \\ x_2 &= (b_2 + b_3)(b_1 - \frac{\Lambda\beta_1}{\mu+\omega}\dot{\varphi}(0)) + b_2b_3 - \alpha_1\delta_1 - \frac{\Lambda\beta_2}{\mu+\omega}\dot{\varphi}(0)k_1, \\ x_3 &= b_2b_3(b_1 - \frac{\Lambda\beta_1}{\mu+\omega}\dot{\varphi}(0)) - \frac{\Lambda\beta_2}{\mu+\omega}\dot{\varphi}(0)k_1b_3 - (b_1 - \frac{\Lambda\beta_1}{\mu+\omega}\dot{\varphi}(0))\alpha_1\delta_1 \\ &= [\delta_1(\alpha_2 + \zeta_2 + \mu) + (\delta_2 + \mu)b_3]b_1(1 - \mathfrak{R}_0). \end{aligned}$$

If $\mathfrak{R}_0 \leq 1$, then one can verify that $x_1 > 0$, $x_2 > 0$, $x_3 > 0$, and $x_1x_2 > x_3$.

Hence, all eigenvalues satisfy $|\arg(\lambda_i)| > \frac{\alpha\pi}{2}$. By the Routh-Hurwitz criteria (see [27]), these conditions ensure the local asymptotic stability of E^0 of the fractional-order model (4.2) whenever $\mathfrak{R}_0 \leq 1$.

Theorem 5.2. *If $\mathfrak{R}_0 > 1$, then E^* is locally asymptotically stable.*

Proof. Evaluating $J(U, V_1, V_2, T)$ at E^* gives

$$J(U^*, V_1^*, V_2^*, T^*) = \begin{pmatrix} a_{11} & a_{12} & a_{13} & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & 0 & a_{43} & a_{44} \end{pmatrix},$$

where

$$\begin{aligned} a_{11} &= -(\omega + \mu) - \beta_1 U^* \varphi(V_1^*) - \beta_2 U^* \varphi(V_2^*), a_{12} = -\beta_1 U^* \dot{\varphi}(V_1^*), a_{13} = -\beta_2 U^* \dot{\varphi}(V_2^*), \\ a_{21} &= \beta_1 U^* \varphi(V_1^*) + \beta_2 U^* \varphi(V_2^*), a_{22} = \beta_1 U^* \dot{\varphi}(V_1^*) - b_1, a_{23} = \beta_2 U^* \dot{\varphi}(V_2^*), \\ a_{32} &= k_1, a_{33} = -b_2, a_{34} = \alpha_1, \\ a_{43} &= \delta_1, a_{44} = -b_3, \end{aligned}$$

the characteristic polynomial of $J(E^*)$ is

$$P(\lambda) = \lambda^4 + y_1\lambda^3 + y_2\lambda^2 + y_3\lambda + y_4, \quad (5.1)$$

where

$$y_1 = -(a_{11} + a_{22} + a_{33} + a_{44}),$$

$$y_2 = a_{11}a_{22} - a_{12}^2 - a_{23}a_{32} - a_{34}a_{43} + (a_{11} + a_{22})(a_{33} + a_{44}),$$

$$y_3 = a_{23}a_{32}(a_{11} + a_{44}) + a_{34}a_{43}(a_{11} + a_{22}) - a_{13}a_{21}a_{32} - (a_{33} + a_{44})(a_{11}a_{22} - a_{12}^2),$$

$$y_4 = a_{13}a_{21}a_{32}a_{44} - a_{23}a_{32}a_{11}a_{44} - a_{34}a_{43}(a_{11}a_{22} - a_{12}^2).$$

For $\Re_0 > 1$, all coefficients satisfy $y_i > 0, i = 1, 2, 3, 4$, and the Routh-Hurwitz criterion inequality $y_1 y_2 y_3 > y_1^2 y_4 + y_3^2$ holds, implying that all eigenvalues of $J(E^*)$ have negative real parts [27]. By Matignon's stability theorem for fractional-order systems with $0 < \alpha \leq 1$, this guarantees that $|\arg(\lambda_i)| > \frac{\alpha\pi}{2}$ for all eigenvalues. Therefore, the endemic equilibrium E^* is locally asymptotically stable.

Theorem 5.3. *If $\Re_0 \leq 1$, then E^0 is globally asymptotically stable.*

Proof. Consider the Lyapunov function

$$L_0(t) = V_1 + \phi_1 V_2 + \phi_2 T, \quad (5.2)$$

where

$$\phi_1 = \frac{b_1 \beta_2 b_3}{\beta_1 \tilde{C} + \beta_2 (k_1 b_3)},$$

$$\phi_2 = \frac{\alpha_1 b_1 \beta_2}{\beta_2 \tilde{C} + \beta_1 (k_1 b_3)},$$

and

$$\begin{aligned} \tilde{C} &= \delta_1 (\alpha_2 + \zeta_2 + \mu) + (\delta_2 + \mu) b_3 \\ &= b_2 b_3 - \alpha_1 \delta_1. \end{aligned}$$

Clearly, $L_0(t) \geq 0$ for all t , and $L_0(t) = 0$ if and only if $V_1 = V_2 = T = 0$.

Applying the Caputo fractional derivative to $L_0(t)$ gives

$${}_0^C D_t^\alpha L_0(t) = {}_0^C D_t^\alpha V_1 + \phi_1 {}_0^C D_t^\alpha V_2 + \phi_2 {}_0^C D_t^\alpha T, \quad (5.3)$$

replacing ${}_0^C D_t^\alpha V_1, {}_0^C D_t^\alpha V_2$, and ${}_0^C D_t^\alpha T$ by the corresponding values yields

$$\begin{aligned} {}_0^C D_t^\alpha L_0(t) &= \beta_1 U \varphi(V_1) + \beta_2 U \varphi(V_2) - b_1 V_1 \\ &\quad + \phi_1 [\alpha_1 T + k_1 V_1 - b_2 V_2] + \phi_2 [\delta_1 V_2 - b_3 T]. \end{aligned}$$

Simplifying gives

$$\begin{aligned} {}_0^C D_t^\alpha L_0(t) &= \beta_1 U \varphi(V_1) + \beta_2 U \varphi(V_2) - b_1 V_1 \\ &\quad + \frac{b_1 \beta_2 (k_1 b_3)}{\beta_1 \tilde{C} + \beta_2 (k_1 b_3)} \left[V_1 + \frac{\alpha_1}{k_1} T - \frac{b_2}{k_1} V_2 \right] + \frac{b_1 \beta_2 (k_1 b_3)}{\beta_1 \tilde{C} + \beta_2 (k_1 b_3)} \left[\frac{\alpha_1 \delta_1}{k_1 b_3} V_2 - \frac{\alpha_1}{k_1} T \right] \\ &= \beta_1 U \varphi(V_1) + \beta_2 U \varphi(V_2) - b_1 V_1 + \frac{b_1 \beta_2 (k_1 b_3)}{\beta_1 \tilde{C} + \beta_2 (k_1 b_3)} \left[V_1 - \frac{b_2 b_3 - \alpha_1 \delta_1}{k_1 b_3} V_2 \right] \\ &= \beta_1 U \varphi(V_1) + \beta_2 U \varphi(V_2) - b_1 V_1 + \frac{b_1 \beta_2 (k_1 b_3)}{\beta_1 \tilde{C} + \beta_2 (k_1 b_3)} \left[V_1 - \frac{\tilde{C}}{k_1 b_3} V_2 \right] \\ &= \beta_1 U \varphi(V_1) + \beta_2 U \varphi(V_2) + \frac{b_1 \tilde{C}}{\beta_1 \tilde{C} + \beta_2 (k_1 b_3)} [\beta_1 V_1 + \beta_2 V_2] \\ &= \beta_1 U \varphi(V_1) + \beta_2 U \varphi(V_2) + \frac{b_1 \tilde{C} U_0 \varphi(0)}{\beta_1 U_0 \varphi(0) \tilde{C} + \beta_2 U_0 \varphi(0) (k_1 b_3)} [\beta_1 V_1 + \beta_2 V_2], \end{aligned}$$

where $U^0 = \frac{\Lambda}{\omega + \mu}$.

According to Lemma (3.1), we obtain the following:

$$\begin{aligned} {}_0^C D_t^\alpha L_0(t) &\leq \beta_1 U_0 \varphi(0) V_1 + \beta_2 U_0 \varphi(0) V_2 \\ &\quad + \frac{b_1 \tilde{C}}{\beta_1 U_0 \varphi(0) \tilde{C} + \beta_2 U_0 \varphi(0) (k_1 b_3)} [\beta_1 U_0 \varphi(0) V_1 + \beta_2 U_0 \varphi(0) V_2] \end{aligned}$$

Using the definition of $\mathfrak{R}_0$, we obtain

$${}_0^C D_t^\alpha L_0(t) \leq [\beta_1 U_0 \varphi(0) V_1 + \beta_2 U_0 \varphi(0) V_2] \left(1 - \frac{1}{\mathfrak{R}_0}\right)$$

Hence, ${}_0^C D_t^\alpha L_0(t) \leq 0$. From LaSalle's principle of invariance [34], E^0 is globally asymptotically stable.

For the global stability of the second point E^* , consider

$$L(x) = x - 1 - \ln x, \quad \text{for all } x > 0. \quad (5.4)$$

Lemma 5.4. *If φ satisfies criterion (3.4), then*

$$\left(1 - \frac{\varphi(V_i^*)}{\varphi(V_i)}\right) \left(\frac{\varphi(V_i)}{\varphi(V_i^*)} - \frac{V_i}{V_i^*}\right) \leq 0, \quad i = 1, 2.$$

Proof. Let $V_i > 0$ and assume that conditions (3.3)–(3.4) hold. Hence,

$$\frac{d}{dV_i} \left(\frac{\varphi(V_i)}{V_i}\right) = \frac{V_i \varphi'(V_i) - \varphi(V_i)}{V_i^2} \leq 0,$$

which implies that the function $\frac{\varphi(V_i)}{V_i}$ is non-increasing on $(0, +\infty)$. Therefore, for any $V_i, V_i^* > 0$, we have

$$\frac{\varphi(V_i)}{V_i} \leq \frac{\varphi(V_i^*)}{V_i^*} \quad \text{if } V_i > V_i^*,$$

and

$$\frac{\varphi(V_i)}{V_i} \geq \frac{\varphi(V_i^*)}{V_i^*} \quad \text{if } V_i < V_i^*.$$

This implies that

$$\frac{\varphi(V_i)}{\varphi(V_i^*)} - \frac{V_i}{V_i^*} \leq 0 \quad \text{if } V_i > V_i^*,$$

and

$$\frac{\varphi(V_i)}{\varphi(V_i^*)} - \frac{V_i}{V_i^*} \geq 0 \quad \text{if } V_i < V_i^*.$$

On the other hand, since φ is increasing and $\varphi(0) = 0$, we have

$$1 - \frac{\varphi(V_i^*)}{\varphi(V_i)} > 0 \quad \text{if } V_i > V_i^*,$$

and

$$1 - \frac{\varphi(V_i^*)}{\varphi(V_i)} < 0 \quad \text{if } V_i < V_i^*.$$

Thus, the two factors $\left(1 - \frac{\varphi(V_i^*)}{\varphi(V_i)}\right)$ and $\left(\frac{\varphi(V_i)}{\varphi(V_i^*)} - \frac{V_i}{V_i^*}\right)$ have opposite signs, and hence

$$\left(1 - \frac{\varphi(V_i^*)}{\varphi(V_i)}\right) \left(\frac{\varphi(V_i)}{\varphi(V_i^*)} - \frac{V_i}{V_i^*}\right) \leq 0.$$

Moreover, equality holds if and only if $V_i = V_i^*$.

Theorem 5.5. *If $\mathfrak{R}_0 > 1$, then E^* is globally asymptotically stable.*

Proof. We consider a Lyapunov function

$$\begin{aligned} \mathcal{L}(t) = & \left(U - U^* - \int_{U^*}^U \frac{U^*}{S} dS \right) + \left(V_1 - V_1^* - \int_{V_1^*}^{V_1} \frac{V_1^*}{S} dS \right) \\ & + \frac{\beta_2 U^* \varphi(V_2^*)}{k_1 V_1^*} \left(V_2 - V_2^* - \int_{V_2^*}^{V_2} \frac{V_2^*}{S} dS \right) \\ & + \frac{\alpha_1 \beta_2 U^* \varphi(V_2^*)}{k_1 b_3 V_1^*} \left(T - T^* - \int_{T^*}^T \frac{T^*}{S} dS \right). \end{aligned}$$

The fractional derivative of $\mathcal{L}(t)$ is evaluated as follows:

$$\begin{aligned} {}_0^C D_t^\alpha \mathcal{L}(t) \leq & \left(1 - \frac{U^*}{U}\right) {}_0^C D_t^\alpha U + \left(1 - \frac{V_1^*}{V_1}\right) {}_0^C D_t^\alpha V_1 \\ & + \frac{\beta_2 U^* \varphi(V_2^*)}{k_1 V_1^*} \left(1 - \frac{V_2^*}{V_2}\right) {}_0^C D_t^\alpha V_2 + \frac{\alpha_1 \beta_2 U^* \varphi(V_2^*)}{k_1 b_3 V_1^*} \left(1 - \frac{T^*}{T}\right) {}_0^C D_t^\alpha T. \end{aligned}$$

Using the first equation of model (4.2), we obtain:

$$\left(1 - \frac{U^*}{U}\right) {}_0^C D_t^\alpha U = \left(1 - \frac{U^*}{U}\right) (\Lambda - \beta_1 U \varphi(V_1) - \beta_2 U \varphi(V_2) - (\omega + \mu)U).$$

By substituting $\Lambda = \beta_1 U^* \varphi(V_1^*) + \beta_2 U^* \varphi(V_2^*) + (\omega + \mu)U^*$, we get

$$\begin{aligned} \left(1 - \frac{U^*}{U}\right) {}_0^C D_t^\alpha U &= (\omega + \mu) \left(1 - \frac{U^*}{U}\right) (U^* - U) \\ &+ \beta_1 U^* \varphi(V_1^*) \left(1 - \frac{U \varphi(V_1)}{U^* \varphi(V_1^*)} - \frac{U^*}{U} + \frac{\varphi(V_1)}{\varphi(V_1^*)}\right) \\ &+ \beta_2 U^* \varphi(V_2^*) \left(1 - \frac{U \varphi(V_2)}{U^* \varphi(V_2^*)} - \frac{U^*}{U} + \frac{\varphi(V_2)}{\varphi(V_2^*)}\right). \end{aligned} \quad (5.5)$$

From the second equation of model (4.2), we have

$$\left(1 - \frac{V_1^*}{V_1}\right) {}_0^C D_t^\alpha V_1 = \left(1 - \frac{V_1^*}{V_1}\right) (\beta_1 U \varphi(V_1) + \beta_2 U \varphi(V_2) - b_1 V_1).$$

By substituting $b_1 = \frac{\beta_1 U^* \varphi(V_1^*)}{V_1^*} + \frac{\beta_2 U^* \varphi(V_2^*)}{V_1^*}$, we obtain

$$\begin{aligned} \left(1 - \frac{V_1^*}{V_1}\right) {}_0^C D_t^\alpha V_1 &= \beta_1 U^* \varphi(V_1^*) \left(1 + \frac{U \varphi(V_1)}{U^* \varphi(V_1^*)} - \frac{V_1^* U \varphi(V_1)}{V_1 U^* \varphi(V_1^*)} - \frac{V_1}{V_1^*}\right) \\ &+ \beta_2 U^* \varphi(V_2^*) \left(1 + \frac{U \varphi(V_2)}{U^* \varphi(V_2^*)} - \frac{V_1^* U \varphi(V_2)}{V_1 U^* \varphi(V_2^*)} - \frac{V_1}{V_1^*}\right). \end{aligned} \quad (5.6)$$

By the third equation of model (4.2), we obtain

$$\frac{\beta_2 U^* \varphi(V_2^*)}{k_1 V_1^*} \left(1 - \frac{V_2^*}{V_2}\right) {}_0^C D_t^\alpha V_2 = \frac{\beta_2 U^* \varphi(V_2^*)}{k_1 V_1^*} \left(1 - \frac{V_2^*}{V_2}\right) (\alpha_1 T + k_1 V_1 - b_2 V_2).$$

By substituting $V_2^* = \frac{\alpha_1 T^* + k_1 V_1^*}{b_2}$, and $V_2^* = \frac{b_3 T^*}{\delta_1}$, we obtain

$$\begin{aligned} \frac{\beta_2 U^* \varphi(V_2^*)}{k_1 V_1^*} \left(1 - \frac{V_2^*}{V_2}\right) {}_0^C D_t^\alpha V_2 &= \beta_2 U^* \varphi(V_2^*) \left(\frac{\alpha_1 T^*}{k_1 V_1^*} + \frac{\alpha_1 T}{k_1 V_1^*} - \frac{\alpha_1 b_3 T T^*}{\delta_1 k_1 V_1^* V_2} - \frac{\alpha_1 \delta_1 V_2}{k_1 b_3 V_1^*}\right) \\ &+ \beta_2 U^* \varphi(V_2^*) \left(1 + \frac{V_1}{V_1^*} - \frac{V_1 V_2^*}{V_1^* V_2} - \frac{V_2}{V_2^*}\right). \end{aligned} \quad (5.7)$$

Using the fourth equation of model (4.2), we have

$$\frac{\alpha_1 \beta_2 U^* \varphi(V_2^*)}{k_1 b_3 V_1^*} \left(1 - \frac{T^*}{T}\right) {}_0^C D_t^\alpha T = \frac{\alpha_1 \beta_2 U^* \varphi(V_2^*)}{k_1 b_3 V_1^*} \left(1 - \frac{T^*}{T}\right) (\delta_1 V_2 - b_3 T).$$

By substituting $b_3 = \frac{\delta_1 V_2^*}{T^*}$, we obtain

$$\begin{aligned} \frac{\alpha_1 \beta_2 U^* \varphi(V_2^*)}{k_1 b_3 V_1^*} \left(1 - \frac{T^*}{T}\right) {}_0^C D_t^\alpha T &= \beta_2 U^* \varphi(V_2^*) \frac{\alpha_1 \delta_1 V_2}{k_1 b_3 V_1^*} \\ &+ \beta_2 U^* \varphi(V_2^*) \left(\frac{\alpha_1 T^*}{k_1 V_1^*} - \frac{\alpha_1 T}{k_1 V_1^*} - \frac{\alpha_1 \delta_1 T^* V_2}{k_1 b_3 T V_1^*}\right) \end{aligned} \quad (5.8)$$

Adding Equations (5.5)–(5.8), we obtain

$$\begin{aligned} {}^C_0 D_t^\alpha \mathcal{L}(t) &\leq (\omega + \mu) \left(1 - \frac{U^*}{U}\right) (U^* - U) \\ &\quad + \beta_1 U^* \varphi(V_1^*) \left(2 - \frac{U^*}{U} + \frac{\varphi(V_1)}{\varphi(V_1^*)} - \frac{V_1}{V_1^*} - \frac{UV_1^* \varphi(V_1)}{U^* V_1 \varphi(V_1^*)}\right) \\ &\quad + \beta_2 U^* \varphi(V_2^*) \left(3 - \frac{U^*}{U} + \frac{\varphi(V_2)}{\varphi(V_2^*)} - \frac{V_2}{V_2^*} - \frac{UV_1^* \varphi(V_2)}{U^* V_1 \varphi(V_2^*)} - \frac{V_1 V_2^*}{V_1^* V_2}\right) \\ &\quad + \beta_2 U^* \varphi(V_2^*) \left(2 \frac{\alpha_1 T^*}{k_1 V_1^*} - \frac{\alpha_1 \delta_1 T^* V_2}{k_1 b_3 T V_1^*} - \frac{\alpha_1 b_3 T T^*}{\delta_1 k_1 V_1^* V_2}\right). \end{aligned}$$

Straightforward algebra gives

$$\begin{aligned} {}^C_0 D_t^\alpha \mathcal{L}(t) &\leq (\omega + \mu) \left(1 - \frac{U^*}{U}\right) (U^* - U) \\ &\quad + \beta_1 U^* \varphi(V_1^*) \left(1 - \frac{\varphi(V_1^*)}{\varphi(V_1)}\right) \left(\frac{\varphi(V_1)}{\varphi(V_1^*)} - \frac{V_1}{V_1^*}\right) \\ &\quad + \beta_1 U^* \varphi(V_1^*) \left(3 - \frac{U^*}{U} - \frac{V_1 \varphi(V_1^*)}{V_1^* \varphi(V_1)} - \frac{UV_1^* \varphi(V_1)}{U^* V_1 \varphi(V_1^*)}\right) \\ &\quad + \beta_2 U^* \varphi(V_2^*) \left(1 - \frac{\varphi(V_2^*)}{\varphi(V_2)}\right) \left(\frac{\varphi(V_2)}{\varphi(V_2^*)} - \frac{V_2}{V_2^*}\right) \\ &\quad + \beta_2 U^* \varphi(V_2^*) \left(4 - \frac{U^*}{U} - \frac{V_2 \varphi(V_2^*)}{V_2^* \varphi(V_2)} - \frac{UV_1^* \varphi(V_2)}{U^* V_1 \varphi(V_2^*)} - \frac{V_1 V_2^*}{V_1^* V_2}\right) \\ &\quad + \beta_2 U^* \varphi(V_2^*) \left(2 \frac{\alpha_1 T^*}{k_1 V_1^*} - \frac{\alpha_1 \delta_1 T^* V_2}{k_1 b_3 T V_1^*} - \frac{\alpha_1 b_3 T T^*}{\delta_1 k_1 V_1^* V_2}\right). \end{aligned}$$

Using Lemma 5.4, we find

$${}^C_0 D_t^\alpha \mathcal{L}(t) \leq 0.$$

(See the appendix on the last page of the article). Hence, by LaSalle's invariance principle, the endemic equilibrium E^* is globally asymptotically stable whenever $\mathcal{R}_0 > 1$.

6 Numerical simulations

The computations in this section link the threshold analysis to representative solution trajectories of the fractional epidemiological model (3.1). Throughout, take $\varphi(V_i) = \frac{V_i}{1+V_i}$, $i = 1, 2$. This choice is admissible under conditions (3.3) and (3.4), since

$$\begin{cases} \varphi(0) = 0 \\ \dot{\varphi}(V_i) = \frac{1}{(1+V_i)^2} > 0 \\ 0 < \frac{\varphi(V_i)}{V_i} \leq \dot{\varphi}(0) = 1 \\ V_i \dot{\varphi}(V_i) \leq \varphi(V_i). \end{cases}$$

To validate the theoretical results, we perform numerical simulations of the fractional-order model for $\alpha \in \{0.85, 0.9, 0.95, 1\}$. Using the parameter values found in table 1. We utilized Adams-Bashforth Moulton predictive correction method to solve model (3.1), the initial conditions as specified in table 2, Where the first case is taken from [24].

Table 2. Initial conditions for the model (3.1)

Case	$U(0)$	$V_1(0)$	$V_2(0)$	$T(0)$	$A(0)$	$R(0)$
1	30	25	23	15	8	2
2	25	18	16	11	6	2
3	20	12	10	7	4	1
4	15	8	6	4	2	1
5	10	4	3	2	1	0

Case 1: Infection-free equilibrium. Setting $\beta_1 = 0.01, \beta_2 = 0.03$ yields $\mathfrak{R}_0 = 0.3899 < 1$. According to Theorem 5.3, the infection-free equilibrium $E^0 = (11.0887, 0, 0, 0, 0, 16.9725)$ is globally asymptotically stable. Figure 1 illustrates the shifts in the proportions of susceptible individuals (U), the combined group ($V_1 + V_2 + T + A$), and low-risk individuals (R) converge toward the approximate value of E^0 associated with the value of $\alpha = 0.95$ over time. In figure 2, the individual trajectory plots further reveal that larger values of α accelerate convergence toward E^0 : at $\alpha = 1$ the variables reach their equilibrium values considerably faster than at $\alpha = 0.85$, which is a direct consequence of the long memory effect inherent in fractional-order systems.

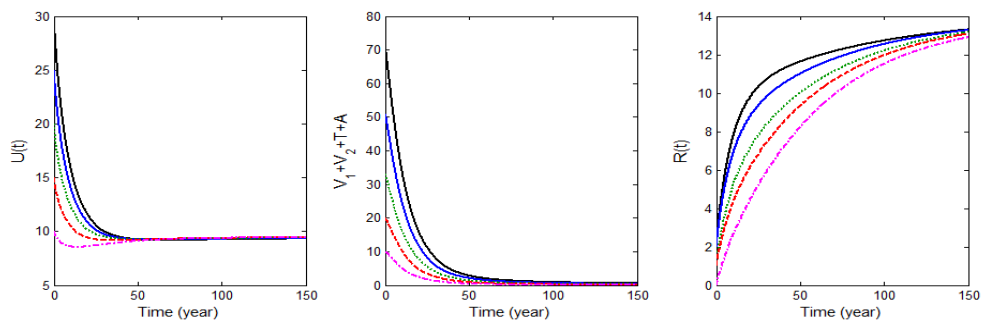


Figure 1. Variation of the propotion of $U, V_1 + V_2 + T + A$, and R over time with different initial values, $\alpha = 0.95, \mathfrak{R}_0 = 0.3899 < 1$

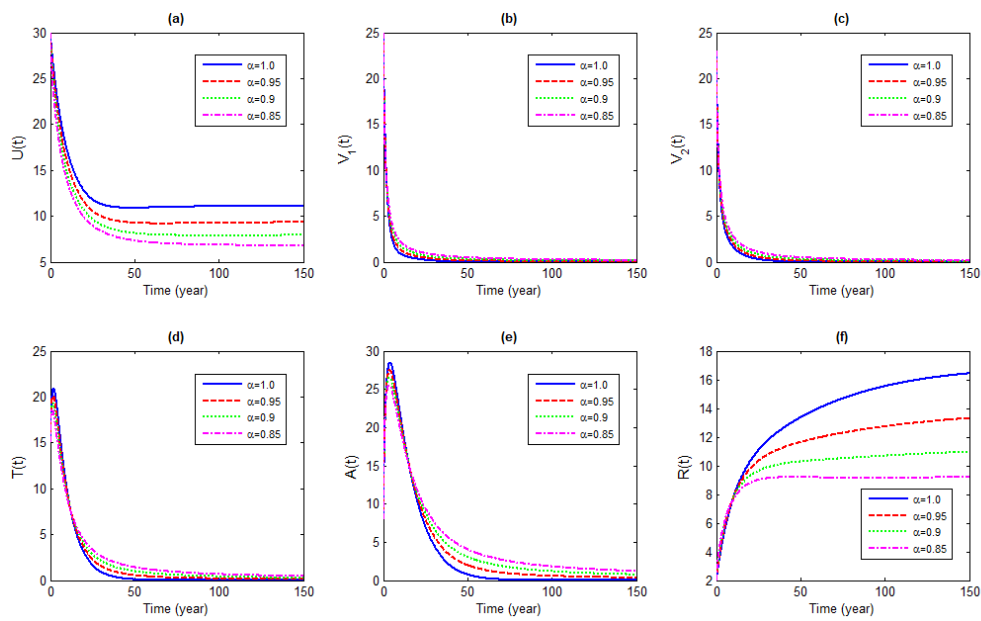


Figure 2. Behavior of $(U^0, 0, 0, 0, 0, R^0)$ with different values of α

Case 2: Endemic equilibrium. Setting $\beta_1 = 0.06$ and $\beta_2 = 0.03$ gives $\mathcal{R}_0 = 1.3667 > 1$. In this regime, the numerical findings coincide with the theoretical outcomes of Theorem 5.5, confirming that the endemic equilibrium $E^* = (U^*, V_2^*, T^*, A^*, R^*)$. Figure 3 indicate that the system's stability demonstrates global convergence, meaning that the system's behavior is independent of the initial conditions and, in all cases, it ultimately converges to the EEP.

This persistent coexistence of virus and immune response is consistent with a chronic infection state, higher values of α produce faster convergence to E^* , while lower values of α slow the transient dynamics without altering the long-term behavior.

Figure 4 separates the six components for $\alpha = 0.85, 0.9, 0.95$ and 1 , with $\alpha = 1$ giving the classical integer-order case. Across these panels, the fractional order mainly reshapes the transient response through the memory index. The susceptible component is shown in Figure 4(a), and Figures 4(b) and 4(c) give the asymptomatic and symptomatic infected classes for the selected values of α . The symptomatic infected class approaches a nonzero equilibrium for both integer and non-integer orders. Figure 4(d) displays the AIDS class, Figure 4(e) tracks the treated class as α varies, and Figure 4(f) shows the individuals who modify their behavior. Taken together, the curves show an early increase in the behavior-change compartment followed by convergence to the equilibrium point.

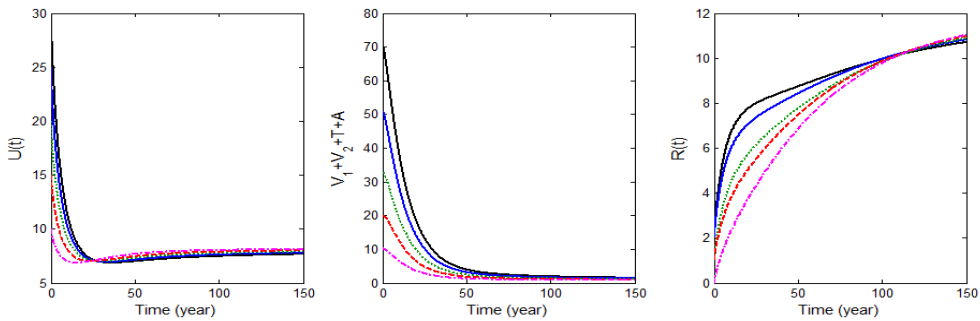


Figure 3. Variation of the propotion of $U, V_1 + V_2 + T + A$, and R over time with different initial values: $\alpha = 0.95, \mathcal{R}_0 = 1.3667 > 1$

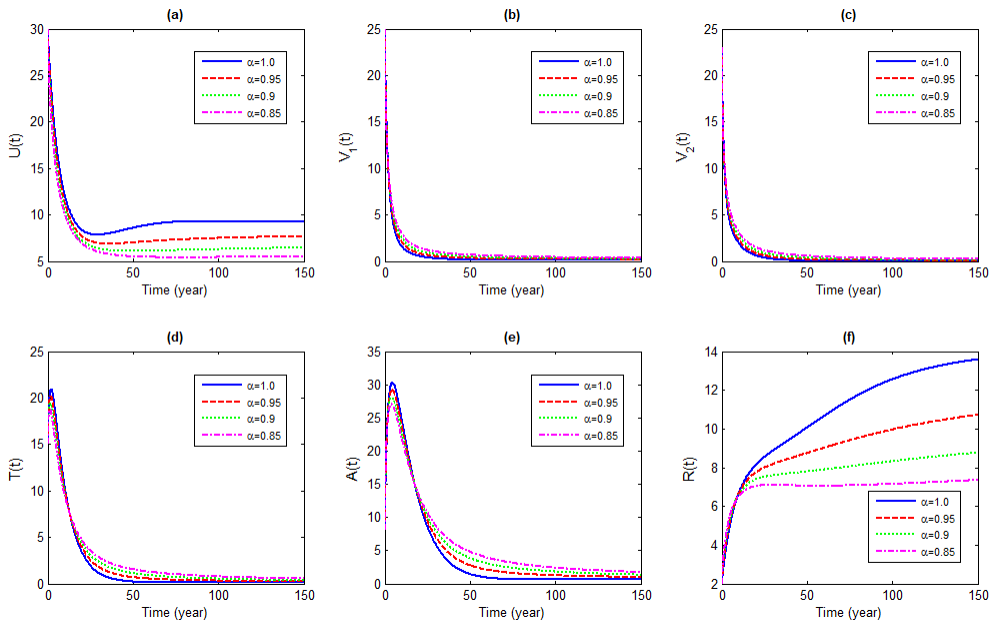


Figure 4. Behavior of $(U^*, V_1^*, V_2^*, T^*, A^*, R^*)$ with different values of α

Conclusion : This work represents a threshold analysis for a Caputo fractional-order HIV/AIDS model with six epidemiological classes: susceptible individuals, asymptomatic latent cases, symptomatic cases, treated individuals, AIDS patients, and individuals who modify their behavior. By retaining these compartments separately, the model keeps treatment, disease progression, AIDS-related mortality, and behavioral change explicit in the dynamics. The basic reproductive number $\mathcal{R}_0$ was derived in closed form, and Lyapunov methods adapted to fractional systems were used to study the two equilibria. When $\mathcal{R}_0 \leq 1$, every admissible trajectory is attracted to the disease-free equilibrium and the infection vanishes. When $\mathcal{R}_0 > 1$, the endemic equilibrium is globally asymptotically stable, so the disease persists at a positive steady state. The numerical results mirror this analytical dichotomy. Varying the fractional order α changes the pace of convergence and the shape of the transient curves, but it does not change the attracting equilibrium selected by the threshold. Smaller α values encode stronger memory and therefore a slower approach to equilibrium, while $\alpha = 1$ recovers the classical integer-order response. Thus the global stability conclusions remain valid for all $\alpha \in (0, 1]$.

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7 Appendix

We show that each term in the fractional derivative of the Lyapunov functional is non-positive (Theorem 5.5).

Term 1. For $U > 0$, let $x = \frac{U}{U^*}$. Then

$$\left(1 - \frac{U^*}{U}\right)(U^* - U) = U^* \left(1 - \frac{1}{x}\right)(1 - x) \leq 0,$$

which implies that the first term is non-positive.

Terms 2 and 4. Using Lemma 5.4, both corresponding terms are non-positive.

Third term. Define

$$x = \frac{U}{U^*}, \quad y = \frac{V_1}{V_1^*}, \quad z = \frac{\varphi(V_1)}{\varphi(V_1^*)}.$$

Then the expression becomes

$$3 - \frac{1}{x} - \frac{y}{z} - \frac{xz}{y}.$$

Since

$$\frac{1}{x} \cdot \frac{y}{z} \cdot \frac{xz}{y} = 1,$$

the arithmetic–geometric mean inequality implies

$$\frac{1}{x} + \frac{y}{z} + \frac{xz}{y} \geq 3,$$

and therefore this term is non-positive.

Fifth term. Let

$$x = \frac{U}{U^*}, \quad y = \frac{V_2}{V_2^*}, \quad z = \frac{V_1}{V_1^*}, \quad w = \frac{\varphi(V_2)}{\varphi(V_2^*)}.$$

Then the expression becomes

$$4 - \frac{1}{x} - \frac{y}{w} - \frac{xw}{z} - \frac{z}{y}.$$

Noting that

$$\frac{1}{x} \cdot \frac{y}{w} \cdot \frac{xw}{z} \cdot \frac{z}{y} = 1,$$

we apply the arithmetic–geometric mean inequality to obtain

$$\frac{1}{x} + \frac{y}{w} + \frac{xw}{z} + \frac{z}{y} \geq 4,$$

which implies that this term is non-positive.

Last term. Factoring out a positive constant, we obtain

$$2 - x - \frac{1}{x}, \quad x = \frac{\delta_1 V_2}{b_3 T} > 0.$$

Since $x + \frac{1}{x} \geq 2$, the expression is non-positive.

Conclusion. All terms are non-positive; hence,

$${}_0^C D_t^\alpha L_1(t) \leq 0.$$

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