

MIXED hp -VERSION DISCONTINUOUS GALERKIN METHOD FOR THE SECOND ORDER CURL-CURL OPERATOR

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This manuscript is an extended and significantly improved version of a full paper entitled "A Mixed Discontinuous Galerkin Scheme for Maxwell's Equations" presented at ICRAMI 2025 (Tunisia, November 2025). Compared to the conference version, this paper includes a complete theoretical analysis of the proposed mixed discontinuous Galerkin scheme, new convergence proofs, additional implementation details, and computational efficiency of the proposed study

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Abstract In this paper, we propose a local discontinuous Galerkin method for the discretization of second-order curl-curl Maxwell's equations. We develop a mixed approach using conforming Nédélec (for $H(\text{curl})$) and Raviart-Thomas (for $H(\text{div})$) finite elements for the electromagnetic field u , coupled with nodal Lagrange elements for the pressure multiplier p . We carry out a complete hp -error analysis of this mixed scheme and introduce a novel technique to derive explicit optimal error estimates in terms of the mesh size h . Numerical experiments are provided to confirm the stability and accuracy of the proposed scheme.

1 Introduction

The discontinuous Galerkin method is one of the many numerical techniques for approximately solving a variety of partial differential equations (PDEs). This method is based on spaces of discontinuous finite elements by considering discontinuities of the solution on the interfaces of the element in mesh discretizations of the domain. The DG method is characterized by approximate differential operators in the formulation (curl, divergence, gradient) by some discrete derivative.

In recent years, many schemes of the discontinuous Galerkin method have been intensively developed and widely studied to solve various boundary value problems arising from physics, mechanics, chemistry, biology, etc. The schemes of the DG method are locally and globally conservative.

The DG method has been used for many mathematical problems, for example, the reaction diffusion problem [1], Navier-Stokes equations [2], Maxwell's equations [3, 4, 5, 6] . . . , etc. For Maxwell's equations, the main feature of the method is to consider a stated scheme via approximations of the unknown u and the flow vector p . For linear problems p can be deleted element by element, from the scheme system. This manuscript is an extended and significantly improved version of a full paper entitled "A Mixed Discontinuous Galerkin Scheme for Maxwell's Equations" presented at ICRAMI 2025 (Tunisia, November 2025). Compared to the conference version, this paper includes a complete theoretical analysis of the proposed mixed discontinuous Galerkin scheme, new convergence proofs, additional implementation details, and computational efficiency of the proposed study [7].

In this paper, we will focus on the following model problem in three dimensions, which is derived from Maxwell's equations.

$$\begin{cases} \sigma u + \nabla \times (\mu^{-1} \nabla \times u) - \varepsilon \nabla p = f & \text{in } \Omega, \\ \nabla \cdot (\varepsilon u) - \alpha p = 0 & \text{in } \Omega, \\ n \times u = 0 \quad \text{and} \quad p = 0 & \text{on } \partial\Omega \end{cases} \quad (1.1)$$

Here, n is the outward normal unit vector to $\partial\Omega$, $f \in [L^2(\Omega)]^3$ is an external source field. σ and α are positive functions in $L^\infty(\Omega)$, μ is the magnetic permeability and ε is the electric permittivity of the medium and when $\sigma(x) = 0$ for every $x \in \Omega$, u is related to the electric field E as $E(x, t)$ is the real part of $u(x)\exp(i\omega t)$, where $\omega \neq 0$ is a given frequency.

We assume that μ and ε are smooth functions and there exist ε_m , ε_M , μ_m , and μ_M such that $0 < \varepsilon_m < \varepsilon(x) < \varepsilon_M$ and $0 < \mu_m < \mu(x) < \mu_M$ for all x in Ω . We introduce the Lagrange multiplier p to control the divergence of the electric field E .

The aim of this paper is to propose and analyze a new DG method for solving Maxwell's equations. The rest of this paper is organized as follows. In Section 2, we introduce notations and give the regularity results. In Section 3, we propose a DG method for the Maxwell model problem (1.1). The stability analysis and convergence analysis for the proposed DG scheme are given in Sections 4. Finally, In Section 5, we perform some numerical experiments to verify the theoretical results.

2 Function spaces

In this section, we introduce some standard notation for Sobolev spaces. Henceforth, Ω is a bounded Lipschitz domain in $\mathbb{R}^d$, $d = 2, 3$, we denote by $L^p(\Omega)$ the usual Sobolev spaces with $p \geq 0$ equipped with the norm $\|\cdot\|_{L^p(\Omega)}$. We denote, as usual, by $H^s(\Omega)$ the Sobolev spaces of real or complex functions with integer or fractional exponent $s \geq 0$ endowed with the norm $\|\cdot\|_{s,\Omega}$ and write $L^2(\Omega) = H^0(\Omega)$. We denote by $(\cdot, \cdot)_\Omega$ the standard inner product in $[L^2(\Omega)]^3$. We also use the following spaces:

$$\begin{aligned} H(\text{curl}, \Omega) &= \{u \in [L^2(\Omega)]^3 : \nabla \times u \in [L^2(\Omega)]^3\}, \\ H_0(\text{curl}, \Omega) &= \{u \in H(\text{curl}, \Omega) : n \times u = 0 \text{ on } \partial\Omega\}, \\ H^s(\text{curl}, \Omega) &= \{u \in [H^s(\Omega)]^3 : \nabla \times u \in [H^s(\Omega)]^3\}, \\ H(\text{div}, \Omega) &= \{u \in [L^2(\Omega)]^3 : \nabla \cdot u \in L^2(\Omega)\}, \\ H_0(\text{div}, \Omega) &= \{u \in H(\text{div}, \Omega) : \nabla \cdot u = 0 \text{ on } \partial\Omega\}, \end{aligned}$$

with $n \times u$ the tangential trace of $u \in H(\text{curl}, \Omega)$, $n \cdot u$ is the normal trace of $u \in H(\text{div}, \Omega)$ and are both belong to $H^{1/2}(\partial\Omega)$.

We note that, for $d = 2$, $\nabla \times u = \begin{pmatrix} \frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y} \end{pmatrix}$ and, for $d = 3$, $\nabla \times u = \begin{pmatrix} \frac{\partial u_3}{\partial y} - \frac{\partial u_2}{\partial z}, \frac{\partial u_1}{\partial z} - \frac{\partial u_3}{\partial x}, \frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y} \end{pmatrix}^T$, nevertheless, $\nabla \cdot u = \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} + \frac{\partial u_3}{\partial z}$.

The norms for $H(\text{curl}, \Omega)$ and $H(\text{div}, \Omega)$ are the graph norms: $\|u\|_{\text{curl}, \Omega} = (\|u\|_{0,\Omega}^2 + \|\nabla \times u\|_{0,\Omega}^2)^{1/2}$ and $\|u\|_{\text{div}, \Omega} = (\|u\|_{0,\Omega}^2 + \|\nabla \cdot u\|_{0,\Omega}^2)^{1/2}$ respectively.

We recall the integration by parts for vector functions below:

$$\left. \begin{aligned} \int_{\Omega} \nabla u \cdot v \, dx &= - \int_{\Omega} u \nabla \cdot v \, dx + \int_{\partial\Omega} nu \cdot v \, ds \\ \int_{\Omega} \nabla \times u \cdot v \, dx &= \int_{\Omega} u \cdot \nabla \times v \, dx + \int_{\partial\Omega} n \times u \cdot v \, ds \\ \int_{\Omega} \nabla \cdot uv \, dx &= - \int_{\Omega} u \cdot \nabla v \, dx + \int_{\partial\Omega} n \cdot uv \, ds \end{aligned} \right\} \quad (2.1)$$

We denote by $V = H_0(\text{curl}, \Omega)$ and $Q = H_0^1(\Omega)$, the mixed weak formulation of the problem (1.1) is to find $u \in V$ and $p \in Q$ satisfies

$$\begin{aligned} a(u, v) + b(v, p) &= (f, v), \quad \text{for all } v \in V, \\ b(u, \psi) - c(p, \psi) &= 0 \quad \text{for all } \psi \in Q, \end{aligned}$$

where the bilinear forms $a(\cdot, \cdot)$, $b(\cdot, \cdot)$ and $c(\cdot, \cdot)$ are defined as follows

$$\begin{aligned} a(u, v) &= (\mu^{-1} \nabla \times u, \nabla \times v), \quad \text{for all } u, v \in V, \\ b(v, p) &= (-\varepsilon p, \nabla \cdot v), \quad \text{for all } v \in V, p \in Q, \\ c(p, \psi) &= (\alpha p, \psi) \quad \text{for all } \psi \in Q. \end{aligned}$$

3 Spaces of discontinuous finite elements

For the convenience of presentation, we first give some notations and functional spaces that will be used throughout the paper. Let h be a positive parameter, let T be a polyhedron in $\mathbb{R}^d$ ($d = 2, 3$) that is a basic finite element, more precisely, T is a triangle or square for $d = 2$, a tetrahedron, cube, or prism for $d = 3$; let $\mathcal{T}_h$ be a triangulation of Ω into finite elements of the maximum diameter h . Denote by $\mathcal{E}_h$ the set of all faces of the subdivision, $\mathcal{T}_h$ then let $\mathcal{E}_h^I$ and $\mathcal{E}_h^D$ be the set of all interior/exterior faces of the subdivision of Ω , respectively. Also, denote by h_T the diameter of T for all fixed $T \in \mathcal{T}_h$, and define as known $h = \max_{T \in \mathcal{T}_h} h_T$.

We use the standard notation $H^s(T)$ for the usual Sobolev space of order s . Now for $s > 0$, let

$$\mathbb{H}^s(\mathcal{T}_h) := \{v \in L^2(\Omega) : v|_T \in H^s(T) \quad \forall T \in \mathcal{T}_h\}$$

and

$$\mathbb{H}^s(\nabla \times, \mathcal{T}_h) := \{v \in L^2(\Omega) : v|_T \in H^s(T)^3 \text{ and } \nabla \times (v|_T) \in H^s(T)^3, \forall T \in \mathcal{T}_h\}.$$

In particular, we note

$$V(h) = \mathbb{H}^1(\nabla \times, \mathcal{T}_h) \quad \text{and} \quad Q(h) = \mathbb{H}^1(\mathcal{T}_h),$$

for

$$\nabla \times u \in \Pi_{T \in \mathcal{T}_h} L^2(\partial T)^3$$

and

$$p \in \Pi_{T \in \mathcal{T}_h} L^2(\partial T).$$

To define the finite element spaces, let $\mathcal{P}_k(T)$ be the set of polynomials up to degree k on each mesh element $T \in \mathcal{T}_h$. Later, we will seek our discrete solution in the piece-wise polynomial denoted by:

$$\mathbb{P}_k(\mathcal{T}_h) = \{v \in L^2(\Omega) : \forall T \in \mathcal{T}_h, v|_T \in \mathcal{P}_k(T)\}.$$

In particular, let $\mathcal{V}_h$ be the finite-dimensional space discretizing $V(h)$. Similarly, we define $\mathcal{Q}_h$ as the finite-dimensional space of $Q(h)$ which contains the discrete solution as follows:

$$\mathcal{V}_h = \mathbb{P}_k(\mathcal{T}_h)^3, \quad \mathcal{Q}_h = \mathbb{P}_{k-1}(\mathcal{T}_h), \text{ for } k \geq 1. \quad (3.1)$$

Note that functions of $\mathbb{P}_k(\mathcal{T}_h)$ are discontinuous across element boundaries.

Now we introduce the definition of the trace operators. For e an element of $\mathcal{E}_h^I$ shared by the two elements T_i and T_j . We denote by n_e the outward unit normal vector on e points from T_i to T_j . Let u in $\mathbb{H}^s(\mathcal{T}_h)^3$ a vector valued function, we denote by u_i, u_j the restrictions of u to T_i and T_j . Then, we define respectively, the tangential, the normal jumps, and the average of u

$$\begin{aligned} \text{if } e \in \mathcal{E}_h^I, e = \partial T_i \cap \partial T_j : \quad & [u]_T = (u_j|_e - u_i|_e) \times n_e, \\ & [u]_N = (u_j|_e - u_i|_e) \cdot n_e, \\ & \{u\} = \omega_1 u_i|_e + \omega_2 u_j|_e, \end{aligned}$$

$$\begin{aligned} \text{if } e \in \mathcal{E}_h^D, e = \partial T_i \cap \partial \Omega : \quad & [u]_T = -u|_e \times n_e, \\ & [u]_N = u|_e \cdot n_e, \\ & \{u\} = u|_e. \end{aligned}$$

By the same way, and for a given p in $\mathbb{H}^s(\mathcal{T}_h)$, we note p_i, p_j the restrictions of p to T_i and T_j . therefore, we define respectively, the normal jump and the average of p as follow

$$\begin{aligned} \text{if } e \in \mathcal{E}_h^I, e = \partial T_i \cap \partial T_j : \quad & [p]_N = (p_j|_e - p_i|_e) n_i, \\ & \{p\} = \omega_1 p_i|_e + \omega_2 p_j|_e, \end{aligned}$$

$$\begin{aligned} \text{if } e \in \mathcal{E}_h^D, e = \partial T_i \cap \partial \Omega : \quad & [p]_N = p|_e n_e, \\ & \{p\} = p|_e. \end{aligned}$$

where $0 \leq \omega_1, \omega_2 \leq 1$ such that $\omega_1 + \omega_2 = 1$.

Finally, we introduce a stabilization parameter η used later in the definition of numerical fluxes. For a given $\kappa > 0$, define η such that

$$\begin{aligned} \kappa &= \eta \min(h_{T_i}, h_{T_j}) \text{ for } T_i \text{ and } T_j \text{ are two interior elements of } \mathcal{T}_h, \\ \kappa &= \eta h_T \text{ for } T \text{ is a boundary element of } \mathcal{T}_h. \end{aligned}$$

For more details about the construction of the stabilization parameter η , we refer to [8].

3.1 Some basic tools

Let us first give the following result.

Lemma 3.1. *for all $t \in \text{Tr}(\mathcal{E}_h^I)$ and $\psi, \Phi \in \text{Tr}(\mathcal{E}_h)^3$ we have the following identities:*

$$\begin{aligned} \sum_{T \in \mathcal{T}_h} \int_{\partial T} \psi \cdot (\Phi \times n_T) ds &= - \int_{\mathcal{E}_h} \{\psi\} [\Phi]_T ds + \int_{\mathcal{E}_h^I} [\psi]_T \{\Phi\} ds, \\ \sum_{T \in \mathcal{T}_h} \int_{\partial T} t(\Phi \cdot n_T) ds &= \int_{\mathcal{E}_h^I} ([\Phi]_N \{t\} + [t]_N \{\Phi\}) ds + \int_{\mathcal{E}_h^D} t(\Phi \cdot n) ds. \end{aligned}$$

The proof is based on the identities: $a \cdot c - b \cdot d = \frac{1}{2}(a+b) \cdot (c-d) + \frac{1}{2}(a-b) \cdot (c+d)$ and $a \cdot (b \times c) = \det(a, b, c) = -\det(b, a, c) \forall a, b, c, d \in \mathbb{R}^3$. We have

$$\sum_{T \in \mathcal{T}_h} \int_{\partial T} \psi \cdot (\Phi \times n_T) ds = \int_{\mathcal{E}_h^I} \psi \cdot (\Phi \times n_T) ds + \int_{\mathcal{E}_h^D} \{\psi\} [\Phi]_T ds,$$

for two elements adjacent T_i and T_j shared by e in $\mathcal{E}_h^I$, we can write

$$\begin{aligned} \psi_i \cdot (\Phi_i \times n_i) + \psi_j \cdot (\Phi_j \times n_j) &= \psi_i \cdot (\Phi_i \times n_i) - \psi_j \cdot (\Phi_j \times n_i) \\ &= \frac{1}{2}(\psi_i + \psi_j) \cdot ((\Phi_i - \Phi_j) \times n_i) + \frac{1}{2}(\psi_i - \psi_j) \cdot ((\Phi_i + \Phi_j) \times n_i) \\ &= -\{\psi\} \cdot [\Phi]_T - \frac{1}{2}(\Phi_i + \Phi_j) \cdot ((\psi_i - \psi_j) \times n_i) \\ &= -\{\psi\} \cdot [\Phi]_T + \{\Phi\} \cdot [\psi]_T, \end{aligned}$$

by using the convention that

$$\sum_{T \in \mathcal{T}_h} \int_{\partial T} \cdot ds = \int_{\mathcal{E}_h} \cdot ds,$$

we deduce

$$\begin{aligned} \sum_{T \in \mathcal{T}_h} \int_{\partial T} \psi \cdot (\Phi \times n_T) ds &= - \int_{\mathcal{E}_h^I} \{\psi\} [\Phi]_T ds + \int_{\mathcal{E}_h^I} \{\Phi\} [\psi]_T ds - \int_{\mathcal{E}_h^D} \{\psi\} [\Phi]_T ds \\ &= - \int_{\mathcal{E}_h} \{\psi\} [\Phi]_T ds + \int_{\mathcal{E}_h^I} \{\Phi\} [\psi]_T ds. \end{aligned}$$

In the same way, we can prove the second identity without difficulty.

Lemma 3.2. *For T a d -simplex, we have the following inverse trace inequality:*

$$\|p_h\|_{0,\partial T} \leq \frac{C_{inv}}{\sqrt{h_T}} \|p_h\|_{0,T} \quad \text{for all } p_h \in \mathcal{P}_k(T)^d \quad (3.2)$$

Consider the d -simplex

$$T^d = \left\{ (x_1, x_2, \dots, x_d) \text{ s.t. } |x_i| \leq 1, \sum_{i=1}^d x_i \leq 2-d \right\},$$

we have

$$\mathcal{P}_k(T)^d = \text{span} \left\{ x_1^{i_1} x_2^{i_2} \dots x_d^{i_d} \text{ s.t. } i_k \geq 0, \sum_k i_k \leq p \right\}$$

For a regular d -simplex T , we have

$$\text{Vol}_d(T) = \frac{h_T^d}{d!} \sqrt{\frac{d+1}{2^d}}, \quad \text{and } \text{Vol}_{d-1}(\partial T) = (d+1) \frac{h_T^{d-1}}{(d-1)!} \sqrt{\frac{d}{2^{d-1}}}$$

We shall loosely follow the work of T. Warburton [9]. We get

$$\|p_h\|_{0,\partial T} \leq \sqrt{\frac{(k+1)(k+d)}{d}} \sqrt{\frac{\text{Vol}_{d-1}(\partial T)}{\text{Vol}_d(T)}} \|p_h\|_{0,T},$$

we have

$$\sqrt{\frac{\text{Vol}_{d-1}(\partial T)}{\text{Vol}_d(T)}} = \frac{\sqrt{d(d+1)}}{h_T^{\frac{1}{2}}} \left(\frac{2d}{d+1} \right)^{\frac{1}{4}}$$

to obtain the desired inequality with

$$C_{inv} = C(k, d) = \sqrt{(d+1)(k+1)(k+d)} \left(\frac{2d}{d+1} \right)^{\frac{1}{4}}$$

4 Families of DG-Schemes

By employing an integration by parts with Stokes formula (2.1), the variational formulation for the Maxwell's equation (1.1) is to find $(u, p) \in V(h) \times Q(h)$ such that for all test functions v and ψ in $V(h)$, $Q(h)$ respectively, and on each element $T \in \mathcal{T}_h$

$$\begin{aligned} \int_T \sigma uv \, dx + \int_T \mu^{-1} (\nabla \times u) \cdot (\nabla \times v) \, dx + \int_T p \nabla \cdot \varepsilon v \, dx - \int_{\partial T} (\varepsilon v \cdot n_T) p \, ds \\ - \int_{\partial T} v \cdot ((\mu^{-1} \nabla \times u) \times n_T) \, ds = \int_T f \cdot v \, dx \end{aligned} \quad (4.1)$$

and

$$- \int_T \varepsilon u \cdot \nabla \psi \, dx - \int_T \alpha p \psi \, dx + \int_{\partial T} (\varepsilon u \cdot n_T) \psi \, ds = 0. \quad (4.2)$$

As is known for the discontinuous Galerkin method, the solutions u and p are discontinuous on each element $e \in \mathcal{E}_h^I$. Therefore, we introduce some numerical fluxes $\widetilde{\mu^{-1} \nabla \times u}$, $\widetilde{\varepsilon u}$, and $\widetilde{p}$ face by face of the partition, to approximate their traces on the interfaces of triangulation. For the curl-curl operator, we adapt the same reasoning in [10, 8] as follows: Let $e \in \mathcal{E}_h^I$ be an interior face, and using the parameter η , we define the numerical fluxes by

$$\begin{cases} \widetilde{\mu^{-1} \nabla \times u} &= \{\mu^{-1} \nabla \times u\} - \eta[u]_T, \\ \widetilde{\varepsilon u} &= \{\varepsilon u\} - \eta^{-1}[p]_N, \\ \widetilde{p} &= \{p\} - \eta[\varepsilon u]_N \end{cases} \quad (4.3)$$

and for $e \in \mathcal{E}_h^D$, an exterior face of the triangulation by

$$\begin{cases} \widetilde{\mu^{-1} \nabla \times u} &= \{\mu^{-1} \nabla \times u\} - \eta[u]_T, \\ \widetilde{\varepsilon u} &= \varepsilon u - \eta^{-1} p n, \\ \widetilde{p} &= 0. \end{cases} \quad (4.4)$$

The goal of introducing these stabilizing terms is to obtain a consistent scheme of (1.1), and this is deduced from the conservative effect of the given numerical fluxes in the sense of [11].

Now, equations (4.1)-(4.2) can be easily modified to

$$\begin{aligned} \int_T \sigma uv \, dx + \int_T (\mu^{-1} \nabla \times u) \cdot (\nabla \times v) \, dx + \int_T p \nabla \cdot \varepsilon v \, dx - \int_{\partial T} (\varepsilon v \cdot n_T) \tilde{p} \, ds \\ - \int_{\partial T} v \cdot ((\mu^{-1} \nabla \times u) \times n_T) \, ds = \int_T f \cdot v \, dx \end{aligned} \quad (4.5)$$

and

$$- \int_T \varepsilon u \cdot \nabla \psi \, dx - \int_T \alpha p \psi \, dx + \int_{\partial T} (\widetilde{\varepsilon u} \cdot n_T) \psi \, ds = 0.$$

After an integration by parts, the above equation, we get

$$\int_T \psi \nabla \cdot \varepsilon u \, dx - \int_T \alpha p \psi \, dx + \int_{\partial T} ((\widetilde{\varepsilon u} - \varepsilon u) \cdot n_T) \psi \, ds = 0. \quad (4.6)$$

In what follows, summing the equations (4.5), (4.6) over all elements $T \in \mathcal{T}_h$, one gets

$$\begin{aligned} \int_{\Omega} \sigma uv \, dx + \int_{\Omega} (\mu^{-1} \nabla \times u) \cdot (\nabla \times v) \, dx + \int_{\Omega} p \nabla \cdot \varepsilon v \, dx - \sum_{T \in \mathcal{T}_h} \int_{\partial T} (\varepsilon v \cdot n_T) \tilde{p} \, ds \\ - \sum_{T \in \mathcal{T}_h} \int_{\partial T} v \cdot ((\mu^{-1} \nabla \times u) \times n_T) \, ds = \int_{\Omega} f \cdot v \, dx \end{aligned} \quad (4.7)$$

together with

$$\int_{\Omega} \psi \nabla \cdot \varepsilon u \, dx - \int_{\Omega} \alpha p \psi \, dx + \sum_{T \in \mathcal{T}_h} \int_{\partial T} ((\widetilde{\varepsilon u} - \varepsilon u) \cdot n_T) \psi \, ds = 0. \quad (4.8)$$

To deal with the terms on the edges, we apply Lemma (3.1) and with the help of the definition of the numerical fluxes, we obtain

$$\begin{aligned} \sum_{T \in \mathcal{T}_h} \int_{\partial T} \tilde{p} (\varepsilon v \cdot n_T) \, ds &= \int_{\mathcal{E}_h^I} [\varepsilon v]_N \{\tilde{p}\} + [\tilde{p}]_N \{\varepsilon v\} \, ds + \int_{\mathcal{E}_h^D} \tilde{p} (\varepsilon v \cdot n) \, ds \\ &= \int_{\mathcal{E}_h^I} [\varepsilon v]_N (\{p\} - \eta [\varepsilon u]_N) \, ds, \end{aligned}$$

$$\begin{aligned} \sum_{T \in \mathcal{T}_h} \int_{\partial T} v \cdot ((\mu^{-1} \nabla \times u) \times n_T) \, ds &= \int_{\mathcal{E}_h} [\mu^{-1} \nabla \times u]_T \{v\} \, ds - \int_{\mathcal{E}_h^D} \{\mu^{-1} \nabla \times u\} [v]_T \, ds \\ &= \int_{\mathcal{E}_h^I} (\{\mu^{-1} \nabla \times u\} - \eta [u]_T) [v]_T \, ds \end{aligned}$$

and,

$$\begin{aligned} \sum_{T \in \mathcal{T}_h} \int_{\partial T} \psi ((\widetilde{\varepsilon u} - \varepsilon u) \cdot n_T) \, ds \\ &= \int_{\mathcal{E}_h^I} [\widetilde{\varepsilon u} - \varepsilon u]_N \{\psi\} + [\psi]_N \{\widetilde{\varepsilon u} - \varepsilon u\} \, ds + \int_{\mathcal{E}_h^D} \psi (\widetilde{\varepsilon u} - \varepsilon u) \cdot n \, ds \\ &= \int_{\mathcal{E}_h^I} ([\widetilde{\varepsilon u}]_N - [\varepsilon u]_N) \{\psi\} + [\psi]_N (\{\widetilde{\varepsilon u}\} - \{\varepsilon u\}) \, ds - \eta^{-1} \int_{\mathcal{E}_h^D} \psi p \, ds \\ &= - \int_{\mathcal{E}_h^I} [\varepsilon u]_N \{\psi\} \, ds - \eta^{-1} \int_{\mathcal{E}_h^I} [\psi]_N [p]_N \, ds, \end{aligned}$$

since the jump of a continuous function is null and p vanishes on $\mathcal{E}_h^D$.

By using these three results above, we can obtain the following primal formulation of the DG method and the expressions (4.7), (4.8) become

$$\begin{aligned} \int_{\Omega} \sigma uv \, dx + \int_{\Omega} \mu^{-1} \nabla \times u \nabla \times v \, dx - \int_{\mathcal{E}_h^I} [v]_T \{ \mu^{-1} \nabla \times u \} \, ds + \eta \int_{\mathcal{E}_h^I} [v]_T [u]_T \, ds \\ + \int_{\Omega} p \nabla \cdot \varepsilon v \, dx - \int_{\mathcal{E}_h^I} [\varepsilon v]_N \{ p \} \, ds + \eta \int_{\mathcal{E}_h^I} [\varepsilon v]_N [\varepsilon u]_N \, ds = \int_{\Omega} f \cdot v \, dx, \end{aligned} \quad (4.9)$$

together with

$$\int_{\Omega} \psi \nabla \cdot \varepsilon u \, dx - \int_{\Omega} \alpha p \psi \, dx - \int_{\mathcal{E}_h^I} [\varepsilon u]_N \{ \psi \} \, ds - \eta^{-1} \int_{\mathcal{E}_h^I} [p]_N [\psi]_N \, ds = 0. \quad (4.10)$$

In the literature, the primal weak formulation using the DG method is approximated by piecewise polynomials that are discontinuous across each edge, which leads to non-uniqueness of solutions, lack of coercivity in bilinear forms, and in general poor stability. To address these issues, we can introduce additional constraints, such as a penalty term that penalizes the jump in the solution, ensuring numerical stability and optimal convergence. We add the penalization term:

$$-\mathcal{J}(u, v) := - \int_{\mathcal{E}_h^I} [u]_T \{ \mu^{-1} \nabla \times v \} \, ds,$$

since for the exact solution u we have $[u]_T = 0$ in $\mathcal{E}_h^I$ and $n \times u = 0$ in $\mathcal{E}_h^D$. We note that this term is added to symmetrize the principal bilinear form $\mathcal{A}$ of the formulation.

To enhance the coercivity of the bilinear form $\mathcal{A}$, we can add the second term:

$$r \int_{\Omega} (\nabla \cdot (\varepsilon u) - \alpha p)(\nabla \cdot (\varepsilon v) - \alpha p) \, dx \quad \text{with } r \geq 0, \quad (4.11)$$

although it may not be necessary to use it over the entire discrete space. It is sufficient to ensure the coercivity of the bilinear form $\mathcal{A}$ on the kernel of $\mathcal{B}$ only. Since, from (1.1) we have $\nabla \cdot \varepsilon u - \alpha p = 0$ in Ω . And expression (4.11) can be modified to

$$r \int_{\Omega} \nabla \cdot \varepsilon u \nabla \cdot \varepsilon v \, dx - r \alpha \int_{\Omega} p \nabla \cdot \varepsilon v \, dx, \quad (4.12)$$

Remark 4.1. We note that the first term of (4.12) is added to the bilinear form $\mathcal{A}$, whereas the second term is added to $\mathcal{B}$. In this case, and by identifying with (4.10), we have

$$1 - r\alpha = 1, \quad (4.13)$$

and so, we discuss if $\alpha = 0$, we can add the term of penalization ($r \neq 0$). If $\alpha \neq 0$, necessarily we must have $r = 0$.

To lead to our DG scheme, let us define the following bilinear forms

$$\mathcal{A}(u, v) := a(u, v) - \mathcal{J}(u, v) - \mathcal{J}(v, u),$$

with,

$$\begin{aligned} a(u, v) &:= \int_{\Omega} \sigma uv \, dx + \int_{\Omega} \mu^{-1} (\nabla \times u) \cdot (\nabla \times v) \, dx + \eta \int_{\mathcal{E}_h^I} [u]_T [v]_T \, ds \\ &\quad + \eta \int_{\mathcal{E}_h^I} [\varepsilon u]_N [\varepsilon v]_N \, ds + r \int_{\Omega} (\nabla \cdot \varepsilon u)(\nabla \cdot \varepsilon v) \, dx, \\ \mathcal{B}(v, p) &:= \int_{\Omega} p \nabla \cdot \varepsilon v \, dx - \int_{\mathcal{E}_h^I} [\varepsilon v]_N \{ p \} \, ds \end{aligned}$$

and

$$\mathcal{C}(p, \psi) := \int_{\Omega} \alpha p \psi dx + \eta^{-1} \int_{\mathcal{E}_h} [p]_N [\psi]_N ds.$$

Referred to the mixed discontinuous Galerkin method and using these definitions, the Maxwell's problem (1.1) becomes: find $(u, p) \in V(h) \times Q(h)$ such as

$$\begin{cases} \mathcal{A}(u, v) + \mathcal{B}(v, p) = \mathcal{L}(v), \quad \forall v \in V(h), \\ \mathcal{B}(u, \psi) - \mathcal{C}(p, \psi) = 0, \quad \forall \psi \in Q(h). \end{cases} \quad (4.14)$$

4.1 Stability of the discrete scheme

To define the discrete DG finite element scheme for system (1.1), we will approach spaces $Q(h)$ and $V(h)$ by the finite-dimensional spaces $\mathcal{Q}_h$ and $\mathcal{V}_h$, respectively. Let us emphasize that the fact of using this discretization brings us back to rewrite (4.14) as

$$\begin{cases} \mathcal{A}(u_h, v) + \mathcal{B}(v, p_h) = \mathcal{L}(v), \quad \forall v \in \mathcal{V}_h, \\ \mathcal{B}(u_h, \psi) - \mathcal{C}(p_h, \psi) = 0, \quad \forall \psi \in \mathcal{Q}_h \end{cases} \quad (4.15)$$

with $(u_h, p_h) \in \mathcal{V}_h \times \mathcal{Q}_h$ be the discrete solution of the formulation.

We start by introducing two norms dependent on the mesh discretization. Therefore, $\mathcal{V}_h$ and $\mathcal{Q}_h$ are equipped with appropriate norms, which will be defined to ensure stability and convergence of the discrete problem

$$\begin{aligned} \|u\|_h^2 := & \|\sigma^{\frac{1}{2}} u\|_{0,\Omega}^2 + \|\mu^{-\frac{1}{2}} \nabla \times u\|_{0,\Omega}^2 + r \|\nabla \cdot \varepsilon u\|_{0,\Omega}^2 + \eta \|\varepsilon u\|_{0,\mathcal{E}_h^I}^2 \\ & + \eta \| [u]_T \|_{0,\mathcal{E}_h}^2 + \eta^{-1} \| \{ \mu^{-1} \nabla \times u \} \|_{0,\mathcal{E}_h}^2 \quad \text{for all } u \in \mathcal{V}_h, \end{aligned} \quad (4.16)$$

and

$$\|p\|_h^2 := \|p\|_{0,\Omega}^2 + \eta^{-1} \| [p]_N \|_{0,\mathcal{E}_h}^2 + \eta^{-1} \| \{ p \} \|_{0,\mathcal{E}_h}^2 \quad \text{for all } p \in \mathcal{Q}_h. \quad (4.17)$$

When $\sigma \neq 0$, it easy to prove that the above quantities define two norms on the spaces $\mathcal{V}_h$ and $\mathcal{Q}_h$, respectively. the case where $\sigma = 0$ we have $\mu^{-\frac{1}{2}} \nabla \times u = 0$ coupled with $\nabla \cdot \varepsilon u = 0$ admits 0 as a solution. We note that if $\sigma = 0$ and $r = 0$, quantity (4.16) is a semi-norm.

The discontinuous Galerkin finite element scheme (4.15) is well posed and consistent, indeed consistency, can be obtained from the derivation of the DG scheme (4.15) since the exact solution (u, p) is in $H_0(\nabla \times, \Omega) \cap H(\nabla_\varepsilon \cdot, \Omega) \times H_0^1(\Omega)$, and after some integration by parts we can see that the exact solution of (1.1) verify (4.15).

The well-posedness of the DG method follows from the fact that the linear scheme (4.15) in finite-dimensional space admits a unique solution. To establish this, it suffices to show that if $\mathcal{L} = 0$ then $(u_h, p_h) = (0, 0)$.

5 Estimate of the scheme's accuracy

To derive the error estimate for the scheme (4.15), we need at first to show some properties for the three bilinear forms $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$. Let us briefly discuss the continuity of these forms. From the introduced mesh-dependent norm (4.16) and (4.17), the definition of $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ and together with the Cauchy Schwarz inequality. The continuity of the bilinear forms $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ on spaces $\mathcal{V}_h \times \mathcal{V}_h$, $\mathcal{V}_h \times \mathcal{Q}_h$, and $\mathcal{Q}_h \times \mathcal{Q}_h$, respectively, follows.

The coercivity of the bilinear form $\mathcal{A}$ on the space $\mathcal{V}_h \times \mathcal{V}_h$ is given in the following proposition.

Proposition 5.1 (Discrete coercivity). *The bilinear form $\mathcal{A}$ satisfies.*

$$\mathcal{A}(u, u) \geq \alpha_0 \|u\|_h^2, \quad \text{for all } u \in \mathcal{V}_h$$

provided $\kappa > C_{inv}(1 + \frac{1}{\mu_m})$, with $\alpha_0 < 1$ is a constant independent of h .

We have

$$\mathcal{A}(u, u) = a(u, u) - 2\mathcal{J}(u, u) = \|u\|_h^2 - \eta^{-1} \|\{\mu^{-1} \nabla \times u\}\|_{0, \mathcal{E}_h}^2 - 2\mathcal{J}(u, u)$$

Let us first estimate the term $\mathcal{J}(u, u)$ by applying the Cauchy-Schwarz inequality followed by Young's inequality

$$\begin{aligned} \mathcal{J}(u, u) &= \int_{\mathcal{E}_h^I} [u]_T \{\mu^{-1} \nabla \times u\} ds \leq \| [u]_T \|_{0, \mathcal{E}_h^I} \| \{\mu^{-1} \nabla \times u\} \|_{0, \mathcal{E}_h^I} \\ &\leq \frac{\delta}{2} \eta \| [u]_T \|_{0, \mathcal{E}_h^I}^2 + \frac{1}{2\delta} \eta^{-1} \| \{\mu^{-1} \nabla \times u\} \|_{0, \mathcal{E}_h^I}^2, \quad \forall \delta > 0 \end{aligned}$$

and the fact that μ is bounded, we obtain

$$\mathcal{J}(u, u) \leq \frac{\delta}{2} \eta \| [u]_T \|_{0, \mathcal{E}_h^I}^2 + \frac{1}{2\delta \mu_m} \eta^{-1} \| \{\mu^{-\frac{1}{2}} \nabla \times u\} \|_{0, \mathcal{E}_h^I}^2,$$

it follows by using the inverse inequality (3.2) and the fact that $\nabla \times \mathcal{V}_h$ is included in $\mathcal{V}_h$

$$\mathcal{J}(u, u) \leq \frac{\delta}{2} \eta \| [u]_T \|_{0, \mathcal{E}_h^I}^2 + \frac{C_{inv}}{2h\delta \mu_m} \eta^{-1} \| \mu^{-\frac{1}{2}} \nabla \times u \|_{0, \Omega}^2, \text{ for every } \delta > 0$$

therefore

$$\begin{aligned} \mathcal{A}(u, u) - \alpha \|u\|_h^2 &\geq (1 - \alpha) \|u\|_h^2 - \delta \eta \| [u]_T \|_{0, \mathcal{E}_h^I}^2 - \frac{C_{inv} \eta^{-1}}{h} (1 + \frac{1}{\delta \mu_m}) \| \mu^{-\frac{1}{2}} \nabla \times u \|_{0, \Omega}^2 \\ &\geq (1 - \alpha) \| \sigma^{\frac{1}{2}} u \|_{0, \Omega}^2 + (1 - \alpha - \frac{C_{inv} \eta^{-1}}{h} (1 + \frac{1}{\delta \mu_m})) \| \mu^{-\frac{1}{2}} \nabla \times u \|_{0, \Omega}^2 \\ &\quad + r(1 - \alpha) \| \nabla \cdot \varepsilon u \|_{0, \Omega}^2 + (1 - \alpha) \eta \| [\varepsilon u]_N \|_{0, \mathcal{E}_h^I}^2 + (1 - \alpha - \delta) \eta \| [u]_T \|_{0, \mathcal{E}_h}^2 \\ &\quad + (1 - \alpha) \eta^{-1} \| \{\mu^{-\frac{1}{2}} \nabla \times u\} \|_{0, \mathcal{E}_h}^2 \end{aligned}$$

We note that $\eta h = \kappa$ and the term $1 - \alpha - \frac{C_{inv} \eta^{-1}}{h} (1 + \frac{1}{\delta \mu_m}) = 1 - \alpha - \frac{C_{inv}}{\kappa} (1 + \frac{1}{\delta \mu_m})$ is independent of h . If $\kappa > C$, there exists $\delta > 0$ such that $0 < \frac{1}{\delta \mu_m} < \frac{\kappa}{C_{inv}} - 1$. In this case, we have $1 - \frac{C_{inv}}{\kappa} (1 + \frac{1}{\delta \mu_m}) > 0$, and there exists $\alpha < 1 - \frac{C_{inv}}{\kappa} (1 + \frac{1}{\delta \mu_m})$, we take $\alpha_0 \leq \alpha$.

To ensure the existence of $\delta > 0$, we have $\delta > \frac{C_{inv}}{(\kappa - C_{inv})\mu_m}$ with $\delta < 1 - \alpha$. This is true if $\frac{C_{inv}}{(\kappa - C_{inv})\mu_m} < 1 - \alpha$. This means $1 - \frac{C_{inv}}{(\kappa - C_{inv})\mu_m} > 0$ yields to $\kappa > C_{inv}(1 + \frac{1}{\mu_m})$. This ends the proof.

5.1 Discrete Inf-sup condition

Now, we will give a discrete inf-sup condition satisfied by the second bilinear form B on $Q_h \times V_h$. To do this, we need to consider an operator R_h presented in the following lemma.

Lemma 5.2. *For all $v \in H^1(\mathcal{T}_h)^3$, there exists an interpolant $R_h \in \mathcal{Q}_h$ such that*

$$\begin{aligned} \int_T q_h \nabla \cdot (\varepsilon v - R_h(v)) dx &= 0, \quad \forall q_h \in P_1(T), \quad \forall T \in \mathcal{T}_h, \\ \int_e (\varepsilon v - R_h(v)) ds &= 0, \quad \forall e \in \mathcal{E}_h. \end{aligned}$$

To prove this lemma, we follow the steps of Theorem 1 in [12]. Such that, for all $v \in H^1(\mathcal{T}_h)^3$ and for a given ε sufficiently smooth, we have εv is in $H^1(\mathcal{T}_h)^3$. Straightforward, we give the following inf-sup condition for $\mathcal{B}$.

Proposition 5.3. *The following stability result of $\mathcal{B}$ holds*

$$\inf_{q \in Q_h \setminus \{0\}} \sup_{v \in V_h \setminus \{0\}} \frac{\mathcal{B}(v, q)}{\|v\|_h \|q\|_h} \geq \beta.$$

with β is a positive constant, independent of h .

Using the inverse inequality (3.2) and the fact that $h = \frac{\kappa}{\eta}$, we get for all $q \in Q_h$

$$\begin{aligned} \|q\|_{0,\Omega}^2 &\geq \frac{1}{3}\|q\|_{0,\Omega}^2 + \frac{1}{3}\|q\|_{0,\Omega}^2 + \frac{1}{3}\|q\|_{0,\Omega}^2 \\ &\geq \frac{1}{3}\|q\|_{0,\Omega}^2 + C\|\frac{1}{\sqrt{\eta}}[q]\|_{0,\varepsilon_h}^2 + C\|\frac{1}{\sqrt{\eta}}\{q\}\|_{0,\varepsilon_h}^2 \\ &\geq \frac{1}{3}\|q\|_{0,\Omega}^2 + C\eta^{-1}\|[q]\|_{0,\varepsilon_h}^2 + C\eta^{-1}\|\{q\}\|_{0,\varepsilon_h}^2 \\ &\geq C\|q\|_h^2. \end{aligned}$$

Let q_h in Q_h , we can find $\tilde{v}_h$ in $H^1(\Omega)^3$ satisfying $\nabla \cdot \varepsilon \tilde{v}_h = q_h$ with

$$\|\tilde{v}_h\|_{1,\Omega} \leq C\|q_h\|_{0,\Omega}.$$

By setting $v_h = R_h(\tilde{v}_h)$ and using Lemma 5.2, we write $\mathcal{B}(v_h, q_h)$ as follow

$$\mathcal{B}(v_h, q_h) = \sum_{T \in \mathcal{T}_h} \int_T q_h \nabla \cdot R_h(\tilde{v}_h) = \sum_{T \in \mathcal{T}_h} \int_T q_h \nabla \cdot \varepsilon \tilde{v}_h = \|q_h\|_{0,\Omega}^2 \geq C\|q_h\|_h^2.$$

Since the operator R_h is continuous, we have $\|v_h\|_h = \|R_h(\tilde{v}_h)\|_h \leq C\|\tilde{v}_h\|_{1,\Omega} \leq C\|q_h\|_{0,\Omega} \leq C\|q_h\|_h$. This completes the proof.

5.2 Error equations

Now we are ready to start the error estimate for our scheme in the next section. For $k \geq 1$, let us consider the Nédélec-Raviart-Thomas (NRT) interpolant $\mathcal{I}_h^{NRT}$ for the velocity u onto the discrete space $\mathcal{V}_h = \mathcal{P}_k^3 \cap H_0(\text{curl}) \cap H(\text{div})$ and $\mathcal{I}_h$ the Lagrange interpolation of p onto $\mathcal{Q}_h = \mathcal{P}_{k+1} \cap H_0^1(\Omega)$.

We point out that the regularity exponent $s \geq 1$ stems from the embedding

$$H_0(\text{curl}, \Omega) \cap H(\text{div}, \Omega) \hookrightarrow H^s(\Omega)^3$$

The lowest order $H(\text{curl}, \Omega)$ -conforming edge elements of the first family due to Nédélec, that is

$$\mathcal{V}_h = \{u_h \in H_0(\text{curl}, \Omega) \text{ s.t. } u|_T(x) = a_T + b_T \times x, \ a_T, b_T \in \mathbb{R}^3, x \in T \ \forall T \in \mathcal{T}_h\}$$

It is well established that the finite element interpolation operators $\mathcal{I}_u$ and $\mathcal{I}_p$ are well-defined and have the following approximation properties, similar to results given in Lemma 3.1 in [13]

Lemma 5.4. *Let $\mathcal{T}_h$ be a quasi-uniform partition of Ω , if $u \in H^{s+1}(\mathcal{T}_h)^3$ and $p \in H^s(\mathcal{T}_h)$ with $s, k \geq 1$. Then the interpolant $\mathcal{I}_u$ and $\mathcal{I}_p$ possess the optimal approximation properties:*

- (a) $\|u - \mathcal{I}_h u\|_0 + h\|\nabla \times (u - \mathcal{I}_h u)\|_0 \leq Ch^{\min(k+1, s+1)}\|u\|_{s+1},$
- (b) $\|u - \mathcal{I}_h u\|_0 + h\|\nabla \cdot (u - \mathcal{I}_h u)\|_0 \leq Ch^{\min(k+1, s+1)}\|u\|_{s+1},$

$$\text{and } \|n \times (u - \mathcal{I}_h u)\|_{0,\partial\Omega} = 0,$$

- (c) $\|p - \mathcal{I}_h p\|_0 + h\|\nabla(p - \mathcal{I}_h p)\|_0 \leq Ch^{\min(k, s)}\|p\|_s$

$$\text{and } \mathcal{I}_h p = 0 \text{ on } \partial\Omega.$$

Now, we decompose the errors as follows:

$$\begin{aligned} e &:= u - u_h = u - \mathcal{I}_h u - (u_h - \mathcal{I}_h u) = \theta - \eta \\ e' &:= p - p_h = p - \mathcal{I}_h p - (p_h - \mathcal{I}_h p) = \theta' - \eta' \end{aligned}$$

with

$$\begin{cases} \theta = u - \mathcal{I}_h u \\ \eta = u_h - \mathcal{I}_h u \end{cases} \quad \text{and} \quad \begin{cases} \theta' = p - \mathcal{I}_h p \\ \eta' = p_h - \mathcal{I}_h p \end{cases}$$

By subtracting the discrete problem (4.15) from the continuous one (4.14), we get

$$\begin{cases} \mathcal{A}(e, v) + \mathcal{B}(v, e') = 0, \ \forall v \in \mathcal{V}_h, \\ \mathcal{B}(e, \psi) - \mathcal{C}(e', \psi) = 0, \ \forall \psi \in \mathcal{Q}_h \end{cases}$$

Proposition 5.5. *We have the error equations*

$$\begin{cases} \mathcal{A}(\eta, v) + \mathcal{B}(v, \eta') = \mathcal{A}(\theta, v) + \mathcal{B}(v, \theta') = \mathcal{L}_0(v), \quad \forall v \in \mathcal{V}_h, \\ \mathcal{B}(\eta, \psi) - \mathcal{C}(\eta', \psi) = \mathcal{B}(\theta, \psi) - \mathcal{C}(\theta', \psi) = \mathcal{L}_1(\psi), \quad \forall \psi \in \mathcal{Q}_h \end{cases} \quad (5.1)$$

with $\mathcal{L}_0(v) = \mathcal{A}(\theta, v) + \mathcal{B}(v, \theta')$ and $\mathcal{L}_1(\psi) = \mathcal{B}(\theta, \psi) - \mathcal{C}(\theta', \psi)$

We claim to obtain the bound $\|\eta\|_h + \|\eta'\|_h \leq C(\|\theta\|_h + \|\theta'\|_h)$. The system (5.1) is a saddle-point problem for (η, η') , to bound its solution, we use the standard Brezzi-Babuška theory, which guarantees the stability of the discrete problem (5.1)

$$\|\eta\|_h + \|\eta'\|_h \leq C(\|\mathcal{L}_0\|_{\mathcal{V}'_h} + \|\mathcal{L}_1\|_{\mathcal{Q}'_h}), \quad (5.2)$$

where $\|\cdot\|_{\mathcal{V}'_h}$ denotes the dual norm.

We now need to bound $\|\mathcal{L}_0\|_{\mathcal{V}'_h}$ and $\|\mathcal{L}_1\|_{\mathcal{Q}'_h}$. By definition

$$\|\mathcal{L}_0\|_{\mathcal{V}'_h} = \sup_{v \in \mathcal{V}_h \setminus \{0\}} \frac{|\mathcal{L}_0(v)|}{\|v\|_h}, \quad \text{and} \quad \|\mathcal{L}_1\|_{\mathcal{Q}'_h} = \sup_{\psi \in \mathcal{Q}_h \setminus \{0\}} \frac{|\mathcal{L}_1(\psi)|}{\|\psi\|_h}.$$

From the definition of $\mathcal{L}_0$ and $\mathcal{L}_1$, and by the continuity of $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$, applying the triangle inequality to obtain

$$\|\mathcal{L}_0\|_{\mathcal{V}'_h} \leq C(\|\theta\|_h + \|\theta'\|_h), \quad \text{and} \quad \|\mathcal{L}_1\|_{\mathcal{Q}'_h} \leq C(\|\theta\|_h + \|\theta'\|_h)$$

substituting into (5.2) gives $\|\eta\|_h + \|\eta'\|_h \leq C(\|\theta\|_h + \|\theta'\|_h)$.

Lemma 5.6. (Trace Inequality) *Let $\mathcal{T}_h$ be a partition of the domain Ω into polygons in 2d or polyhedra in 3d. Then there exists a constant C such that for any $T \in \mathcal{T}_h$ and edge or face $e \in \partial T$, we have the inverse inequality*

$$\|u\|_{0,e}^2 \leq C(h_T^{-1}\|u\|_{0,T}^2 + h_T\|\nabla u\|_{0,T}^2) \quad \text{for all } u \in H^1(T) \quad (5.3)$$

See identities (A 18) in [14].

In the following, we give an apriori error estimation result.

Theorem 5.7. *Let (u, p) be the weak solution of (1.1) and (u_h, p_h) be the solution of (4.15), we assume that $(u, p) \in H^{s+1}(\mathcal{T}_h)^3 \times H^s(\mathcal{T}_h)$ for some $s \geq 1$ and $(u_h, p_h) \in \mathcal{P}_k(\mathcal{T}_h)^3 \times \mathcal{P}_{k+1}(\mathcal{T}_h)$ for $k \geq 1$ then, for all small h we have the following error estimates:*

$$\|u - u_h\|_h + \|p - p_h\|_h \leq Ch^{\min(k,s)} (\|u\|_{s+1} + \|p\|_s),$$

with a positive constant C independent of h .

Using the triangle inequality, we have

$$\begin{aligned} \|u - u_h\|_h + \|p - p_h\|_h &= \|\theta - \eta\|_h + \|\theta' - \eta'\|_h \\ &\leq \|\theta\|_h + \|\eta\|_h + \|\theta'\|_h + \|\eta'\|_h \\ &\leq C(\|\theta\|_h + \|\theta'\|_h) \end{aligned}$$

Let us bound the interpolation errors $\|\theta\|_h$ and $\|\theta'\|_h$. To estimate $\|\theta\|_h = \|u - I_h u\|_h$, we must bound each term in the definition of the energy norm (4.16):

$$\begin{aligned} \|\theta\|_h^2 &= \|\sigma^{\frac{1}{2}}\theta\|_{0,\Omega}^2 + \|\mu^{-\frac{1}{2}}\nabla \times \theta\|_{0,\Omega}^2 + r\|\nabla \cdot \varepsilon\theta\|_{0,\Omega}^2 + \eta\|[\varepsilon\theta]_N\|_{0,\varepsilon_h^I}^2 \\ &\quad + \eta\|[\theta]_T\|_{0,\varepsilon_h}^2 + \eta^{-1}\|\{\mu^{-1}\nabla \times \theta\}\|_{0,\varepsilon_h}^2 \end{aligned}$$

The volume terms $\|\sigma^{\frac{1}{2}}\theta\|_{0,\Omega}$, $\|\mu^{-\frac{1}{2}}\nabla \times \theta\|_{0,\Omega}$ and $r\|\nabla \cdot \varepsilon\theta\|_{0,\Omega}$ are bounded directly by the interpolation estimates (a) and (b) from lemma 5.4, yielding a factor of $h^{\min(k,s)}\|u\|_{s+1}$, more precisely we have:

$$\sigma\|\theta\|_{0,\Omega}^2 \leq Ch^{2\min(k,s)+2}\|u\|_{s+1}^2 \leq Ch^{2\min(k,s)}\|u\|_{s+1}^2 \quad \text{for } h < 1,$$

$$\mu^{-1} \|\nabla \times \theta\|_{0,\Omega}^2 \leq Ch^{2\min(k,s)} \|\nabla \times u\|_s^2 \leq Ch^{2\min(k,s)} \|u\|_{s+1}^2, \text{ and } r \|\nabla \cdot \varepsilon \theta\|^2 \leq Ch^{2\min(k,s)} \|u\|_{s+1}^2.$$

The boundary terms are bounded using trace inequality (5.3), since $\eta = \kappa h^{-1}$, we get

$$\begin{aligned} \eta \|[\varepsilon \theta]_N\|_{0,\varepsilon_h^I}^2 &\leq C (h^{-2} \|\theta\|_{0,\Omega}^2 + \|\nabla \theta\|_{0,\Omega}) \\ &\leq C (h^{2\min(k,s)} \|u\|_{s+1}^2 + h^{2\min(k,s)} \|\nabla u\|_s^2) \\ &\leq Ch^{2\min(k,s)} (\|u\|_{s+1}^2 + \|u\|_{s+1}^2) \\ &\leq Ch^{2\min(k,s)} \|u\|_{s+1}^2 \end{aligned}$$

Since $u \in H^{s+1}(\mathcal{T}_h)^3$ implies $\nabla \times u \in H^s(\mathcal{T}_h)^3$. Nédélec interpolant commutes with the curl operator via Raviart-Thomas interpolant I_h^{RT} , such as:

$$\nabla \times I_h^N u = I_h^{RT} \nabla \times u,$$

the last term is bounded as follows

$$\begin{aligned} \eta^{-1} \|\{\mu^{-1} \nabla \times \theta\}\|_{0,\varepsilon_h}^2 &\leq C (\|\nabla \times \theta\|_{0,\Omega}^2 + h^2 \|\nabla(\nabla \times \theta)\|_{0,\Omega}^2) \\ &\leq C (h^{2\min(k,s)} \|\nabla \times u\|_s^2 + h^2 \|\nabla(\nabla \times u - I_h^{RT} \nabla \times u)\|_{0,\Omega}^2) \\ &\leq C (h^{2\min(k,s)} \|\nabla \times u\|_s^2 + h^2 h^{2\min(k,s)-2} \|\nabla \times u\|_s^2) \\ &\leq Ch^{2\min(k,s)} \|u\|_{s+1}^2 \end{aligned}$$

this ends the proof.

6 Numerical experiments

In this section, we present numerical examples to demonstrate our theoretical results obtained for the problem (1.1). All numerical tests are programmed in Python 3.11. The resulting system consists of only the unknowns u_h and p_h in scheme (4.15). Note that the numerical scheme (4.15) contains two parameters κ and r . The constant r may be chosen from every non negative real number without restriction. For the constant κ , it should be considered large to ensure the coercivity of A , but it should not be too large, otherwise the principal matrix associated with the bilinear form $\mathcal{A}$ becomes ill-conditioned. From proposition 5.1 we have:

$$\kappa > \sqrt{(d+1)(k+1)(k+d)} \left(\frac{2d}{d+1}\right)^{\frac{1}{4}} \left(1 + \frac{1}{\mu_m}\right) \quad (6.1)$$

where d is the dimension of the space and k is the polynomial degree.

For simplicity, we restrict ourselves to two-dimensional model problems, additionally assume that ε and μ are constant and equal to one, we also assume that $\sigma(x) = 1$, $\alpha(x)$, and $r \in \{0, 1\}$ in Ω . Using (6.1) with $d = 2$ and $\mu_m = \frac{1}{2}$ we have:

$$\text{For } k = 1 : \quad \kappa > 13.68,$$

$$\text{For } k = 2 : \quad \kappa > 19.34,$$

$$\text{For } k = 3 : \quad \kappa > 24.97.$$

Following Remark 4.13, we consider two illustrative examples depending on the penalization term given by (4.12):

- (i) **Example 1:** We take $(\alpha, r) = (0, 1)$; in this case, the penalization term (4.12) is added to the bilinear form $\mathcal{A}$.
- (ii) **Example 2:** We take $(\alpha, r) = (1, 0)$; here, no penalization term is added to the bilinear form $\mathcal{A}$.

Finally, the domain Ω is the unit square, defined as $\Omega =]0, 1[^2 \subset \mathbb{R}^2$, with boundary $\partial\Omega = [0, 1] \times \{0, 1\} \cup \{0, 1\} \times [0, 1]$, see Figure 1.

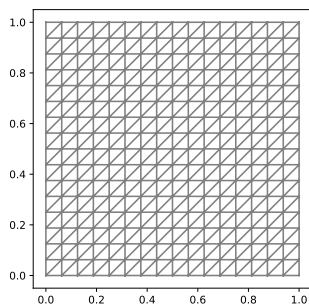


Figure 1. The mesh with $h = \frac{\sqrt{2}}{16}$.

6.1 Example 1.

In this first example, we take $(\alpha, r) = (0, 1)$ so that the analytical solution to the two-dimensional system (1.1) is given by the field $\mathbf{u}^{\text{ex}} = (u_1^{\text{ex}}, u_2^{\text{ex}})^T$ and p^{ex} are given by:

$$\mathbf{u}^{\text{ex}} = (y(y-1)e^y, x(x-1)e^x)^T, \quad p^{\text{ex}} = xy(x-1)(y-1)e^{xy}, \quad (6.2)$$

and the force term $\mathbf{f}^{\text{ex}} = (f_1^{\text{ex}}, f_2^{\text{ex}})^T$, is given by:

$$\mathbf{f}(x, y) = \begin{pmatrix} (\sigma y(y-1) - (y^2 + 3y)) e^y - y(y-1)(2x-1 + x(x-1)y) e^{xy} \\ (\sigma x(x-1) - (x^2 + 3x)) e^x - x(x-1)(2y-1 + xy(y-1)) e^{xy} \end{pmatrix}$$

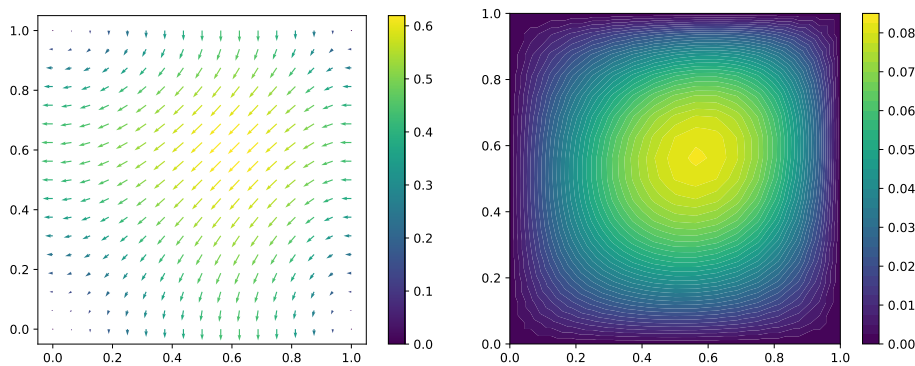


Figure 2. The computed solution $(\mathbf{u}_h, p_h)$, example 1.

In table 1, we obtain the following numerical results related to the computed solution $(\mathbf{u}_h, p_h)$ for example 1, illustrated in Figure 2.

6.2 Example 2.

Here, we considered $(\alpha, r) = (1, 0)$ and a current density f so that the exact solution of (1.1) together with the boundary conditions $\mathbf{u}^{\text{ex}} = (u_1^{\text{ex}}, u_2^{\text{ex}})^T$ and p^{ex} are given by:

$$\mathbf{u}^{\text{ex}} = (y(y-1) \cos(x(x-1)), x(x-1) \cos(y(y-1)))^T, \quad p^{\text{ex}} = \nabla \cdot \mathbf{u}^{\text{ex}} \quad (6.3)$$

	h	$\ u - u_h\ _h$		$\ p - p_h\ _h$	
		Error	Rate	Error	Rate
$\mathbb{P}_1 \backslash \mathbb{P}_2$	3.54e-01	5.11e-01	-	4.61e-02	-
	1.77e-01	2.57e-01	0.99316	2.30e-02	1.00389
	8.84e-02	1.28e-01	1.00529	1.15e-02	0.99440
	4.42e-02	6.37e-02	1.00520	5.79e-03	0.99432
	2.21e-02	3.18e-02	1.00330	2.90e-03	0.99640
$\mathbb{P}_2 \backslash \mathbb{P}_3$	3.54e-01	4.34e-02	-	3.01e-03	-
	1.77e-01	1.11e-02	1.96942	7.17e-04	2.07105
	8.84e-02	2.80e-03	1.98707	1.73e-04	2.05024
	4.42e-02	7.02e-04	1.99412	4.24e-05	2.02998
	2.21e-02	1.76e-04	1.99721	1.05e-05	2.01642
$\mathbb{P}_3 \backslash \mathbb{P}_4$	3.54e-01	2.15e-03	-	1.12e-04	-
	1.77e-01	2.73e-04	2.97800	1.36e-05	3.03154
	8.84e-02	3.44e-05	2.99170	1.67e-06	3.03279
	4.42e-02	4.31e-06	2.99652	2.05e-07	3.02213
	2.21e-02	5.39e-07	2.99843	2.54e-08	3.01281

Table 1. History of errors and rate of convergence, for example 1.

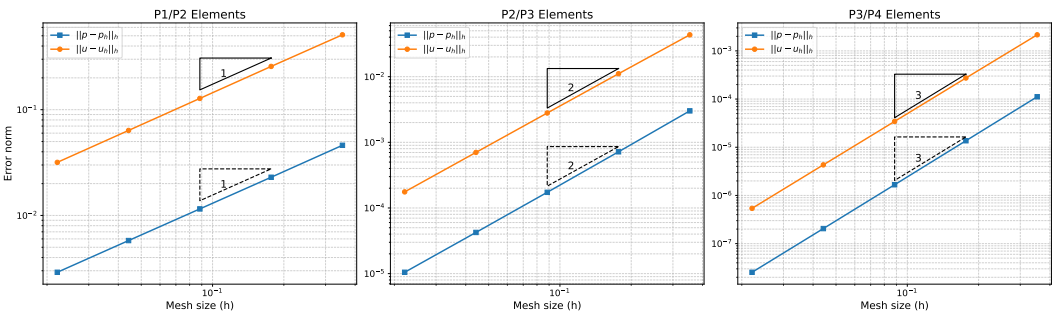


Figure 3. : h-convergence of the errors in (4.16)-(4.17) with $\mathbb{P}_k \backslash \mathbb{P}_{k+1}$ finite elements for $k = 1, 2, 3$.

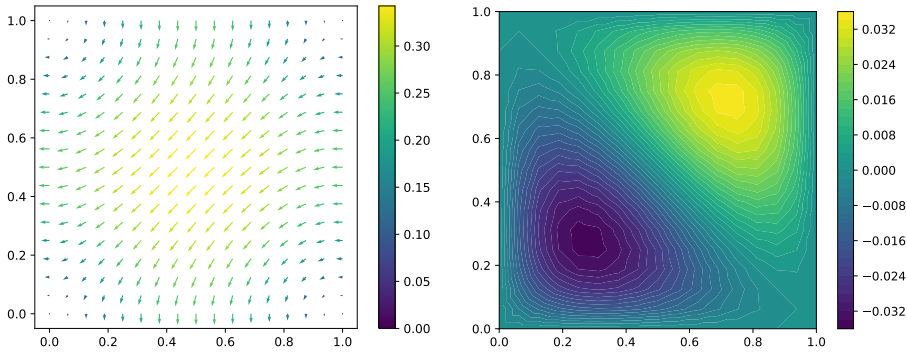


Figure 4. The computed solution (u_h, p_h) , example 2.

Table 2 presents the numerical results associated with the computed solution (u_h, p_h) for Example 2, as illustrated in Figure 4.

	h	$\ u - u_h\ _h$		$\ p - p_h\ _h$	
		Error	Rate	Error	Rate
$\mathbb{P}_1 \setminus \mathbb{P}_2$	3.54e-01	2.85e-02	-	9.17e-03	-
	1.77e-01	1.45e-02	0.97491	4.85e-03	0.91894
	8.84e-02	7.70e-03	0.91164	2.54e-03	0.93164
	4.42e-02	3.95e-03	0.96301	1.29e-03	0.97746
	2.21e-02	1.98e-03	0.99635	6.48e-04	0.99331
$\mathbb{P}_2 \setminus \mathbb{P}_3$	3.54e-01	4.10e-03	-	2.10e-03	-
	1.77e-01	1.15e-03	1.83399	5.90e-04	1.83160
	8.84e-02	3.05e-04	1.91163	1.55e-04	1.92531
	4.42e-02	7.95e-05	1.93978	3.95e-05	1.97234
	2.21e-02	2.05e-05	1.95533	9.90e-06	1.99635
$\mathbb{P}_3 \setminus \mathbb{P}_4$	3.54e-01	4.10e-04	-	3.06e-04	-
	1.77e-01	5.85e-05	2.80912	4.35e-05	2.81444
	8.84e-02	7.65e-06	2.93013	5.65e-06	2.93990
	4.42e-02	9.80e-07	2.96461	7.16e-07	2.98022
	2.21e-02	1.25e-07	2.97085	9.02e-08	2.98876

Table 2. DG errors and convergence rates in the discrete norm $\|\cdot\|_h$ for $k = 1$, for example 2.

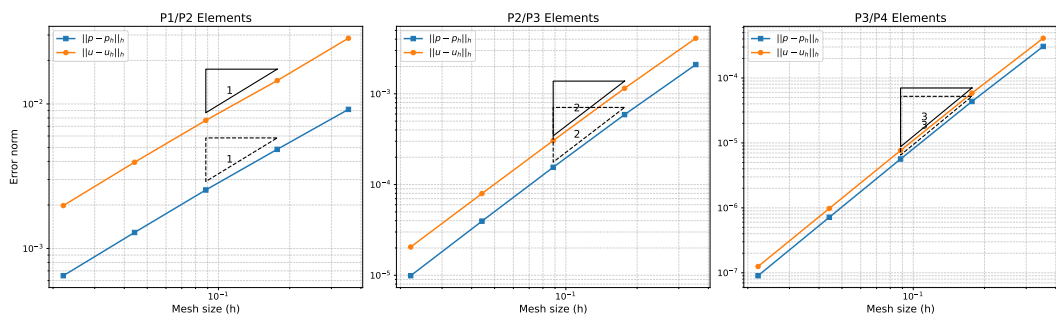


Figure 5. : h -convergence of the errors in (4.16)-(4.17) with $\mathbb{P}_1 \setminus \mathbb{P}_2$ finite elements for example 2.

It can be observed from tables 1 and table 2 that the convergence rates of $\|u - u^h\|_h$ goes to k . It confirms the theoretical results given in Theorem 5.7. Moreover, the convergence rate of $\|p - p^h\|_h$ is around k , these results are also shown in Figures 3 and 5, and our theoretical results are numerically confirmed.

7 Conclusion remarks

In this work, we have developed and analyzed a mixed discontinuous Galerkin method for the second-order curl-curl Maxwell problem. The proposed scheme combines conforming Nédélec and Raviart-Thomas elements for the electromagnetic field with nodal Lagrange elements for the Lagrange multiplier, achieving a stable and accurate discretization.

The main contributions of this paper are threefold:

- (i) We introduced a novel mixed DG formulation that properly handles the curl-curl operator while ensuring divergence control through a Lagrange multiplier approach.
- (ii) We established rigorous *a priori* error estimates in mesh-dependent norms, proving optimal convergence rates under appropriate regularity assumptions. Our analysis demonstrates that the scheme achieves $O(h^k)$ convergence for both the field variable and the multiplier when

using polynomial degrees k and $k + 1$, respectively.

- (iii) Numerical experiments confirmed the theoretical predictions, showing the expected convergence rates and demonstrating the practical effectiveness of the method for both smooth and non-smooth solutions.

The mixed finite element framework presented here offers several advantages: it naturally enforces the divergence condition, provides better stability properties compared to non-mixed approaches, and allows for flexible hp -adaptivity. Future work will focus on extending this approach to time-dependent Maxwell problems, developing efficient preconditioners for the resulting linear systems, and exploring applications in computational electromagnetics with complex geometries and material interfaces.

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