

Synchronization Analysis of Uncertain Fractional-Order Quaternion-Valued Memristive Neural Networks with Time-Varying Delays

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This manuscript is an extended and significantly improved version of a full paper entitled "Quasi-synchronization of fractional-order quaternion-valued memristive neural networks via event triggered control" presented at ICRAMI 2025 (Tunisia, November 2025). Compared to the conference version, this paper includes the introduction of a novel nonlinear adaptive control strategy for complete synchronization (CS), whereas the conference version focused exclusively on quasi-synchronization (QS).

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Abstract: This paper investigates the synchronization problems for a class of uncertain fractional-order quaternion-valued memristive neural networks (FOQVMNNs) subject to time-varying delays. A key feature of the addressed model is the presence of non-identical connection weights between the drive and response systems, which introduces additional structural mismatches. To address the challenges, two distinct control strategies are developed. First, to realize complete synchronization (CS) in the presence of unknown system perturbations, a nonlinear adaptive control law is formulated. This adaptive scheme ensures the robust convergence of the slave system to the master system by control gains. Second, to achieve quasi-synchronization (QS) while minimizing the communication burden and transmission frequency, an event-triggered control (ETC) mechanism is designed. By utilizing fractional-order Lyapunov stability theory and constructing appropriate triggering conditions, it is proven that the synchronization error converges to a bounded region, effectively reducing resource consumption. Unlike traditional methods that rely on the decomposition of quaternions into real-valued subsystems, the proposed analysis is conducted directly in the quaternion domain, thereby preserving the intrinsic algebraic structure and reducing computational complexity. Finally, numerical simulations are provided to demonstrate the effectiveness and superiority of the proposed event-triggered and adaptive control strategies in ensuring the stable operation of FOQVMNNs under various constraints.

1 Introduction

In 1971, Chua theorized the memristor as the fourth fundamental circuit element, a passive device bridging "memory" and "resistance." The experimental realization of memristive behavior by Hewlett-Packard in 2008 [1] marked a milestone in neuromorphic engineering [2]. Due to their ability to mimic synaptic plasticity, memristors have been extensively integrated into artificial neural networks to represent synaptic weights [3, 4]. However, in memristive neural networks (MNNs), connection weights are state dependent, making them highly sensitive to initial conditions. This sensitivity often leads to parameter mismatches and modeling uncertainties [5, 6], which pose significant challenges for conventional analytical tools in guaranteeing synchronization.

Simultaneously, fractional calculus (FC) has emerged as a superior mathematical framework for modeling complex systems. Unlike classical integer order derivatives, fractional order systems (FOS) possess intrinsic memory and hereditary characteristics [7], providing a more accurate description of the long-term dynamics of neural networks. Consequently, the study of Fractional-Order Memristive Neural Networks (FOMNNs) has become a vibrant research area. To further enhance computational efficiency in multi-dimensional tasks such as color image processing and 3D signal modeling [8] these networks have been extended into the quaternion-valued domain. Quaternion Valued Neural Networks (QVNNs) offer a powerful framework for handling multidimensional data, although their non-commutative multiplication introduces significant mathematical complexity.

In quaternion-valued memristive neural networks (QVNNs), memristive connections function as adaptive coupling mechanisms. Each component of the quaternion weight is physically represented by a memristive state, allowing the network to process multidimensional information dynamically. This plasticity is crucial for synchronization: the memristive connection modulates the interaction strength based on the neurons' activity history, enabling the network to overcome parameter mismatches and converge to a synchronous state. By replacing conventional resistors with memristors, we enhance both the computational capacity and the physical robustness of the synchronization process against modeling uncertainties.

Synchronization phenomena have attracted considerable attention due to their wide applications in pattern recognition, image processing, and other engineering fields [9, 10]. In recent years, the dynamical behaviors of fractional-order memristive neural networks (FOMNNs) have been extensively studied, and numerous works have focused on their synchronization properties [11, 12, 13]. For example, fixed-time synchronization of FONNs was investigated via feedback control in [11], while predefined-time synchronization of Cohen-Grossberg FONNs was achieved using two control strategies in [12]. Moreover, dissipativity of FOMNNs were analyzed in [13]. In [14], *Jebril et al.* studied finite-time synchronization for a discrete reaction-diffusion FitzHugh-Nagumo model with unknown parameters using a Lyapunov-based method. Likewise, in [15], *Bendib et al.* derived a finite-time stability condition for fractional-order reaction-diffusion systems and proposed a new Lyapunov functional. These studies demonstrate that synchronization of fractional-order neural networks is predominantly realized through appropriately designed control schemes, which play a fundamental role in compensating for the inherent complexity of network structures and node dynamics.

In addition, with the increasing scale and complexity of neural networks, continuous time or periodic control strategies may lead to excessive communication and computation burdens. To alleviate this issue, event-triggered control (ETC) has emerged as an effective framework that updates control signals only when certain triggering conditions are satisfied, thereby significantly reducing resource consumption. Although ETC has been successfully applied to synchronization problems in various dynamical systems, its application to fractional-order memristive neural networks especially in the presence of uncertainties, time-varying delays, and structural mismatches remains limited. Consequently, developing efficient event-triggered synchronization strategies for such complex neural network models is still a challenging and meaningful problem.

It is important to explicitly highlight that the main novelty of this study lies in the direct quaternion-domain framework adopted for fractional-order memristive neural networks, without decomposing quaternion variables into real-valued subsystems. In addition, the proposed model considers non-identical connection weights between the drive and response systems, which introduces a more general and realistic mismatch scenario that has not been sufficiently addressed in existing literature. Furthermore, the integration of both adaptive control and event-triggered control strategies within a unified fractional-order quaternion-valued setting represents a significant extension compared to existing works.

To address these issues, this paper investigates the synchronization analysis of Uncertain Fractional-Order Quaternion-Valued Memristive Neural Networks (FOQVMNNs) with time-varying delays. We focus on two distinct control paradigms. First, to handle communication constraints, an event-triggered mechanism is developed. Unlike continuous control, ETM transmits signals only when a specific triggering condition is met, thereby minimizing resource consumption while ensuring quasi-synchronization. Second, to achieve high-precision complete synchronization despite system uncertainties, a nonlinear adaptive control law is designed. This dual-approach allows for both resource efficiency and robust stability. This manuscript is an ex-

tended and significantly improved version of a full paper entitled [16] "Quasi-synchronization of fractional-order quaternion-valued memristive neural networks via event triggered control" presented at ICRAMI 2025 (Tunisia, November 2025). Compared to the conference version, this paper includes the introduction of a novel nonlinear adaptive control strategy for complete synchronization (CS), whereas the conference version focused exclusively on quasi-synchronization (QS). Additionally, this version incorporates the analysis of parameter uncertainties, provides a rigorous proof for the exclusion of Zeno behavior, and includes new numerical simulations to validate the robustness of the dual control strategies.

The main contributions and highlights of this work are summarized as follows:

- A comprehensive FOQVMNN model is established, incorporating both parameter uncertainties and time-varying delays within the quaternion domain.
- For the first time, a nonlinear adaptive controller is formulated to guarantee the complete synchronization of uncertain FOQVMNNs, effectively compensating for parameter fluctuations.
- An event-triggered control strategy is rigorously designed to achieve quasi-synchronization, significantly reducing the communication frequency and computational costs.
- By utilizing fractional-order Lyapunov stability theory and specialized inequalities, new synchronization criteria are derived that preserve the intrinsic algebraic structure of quaternions.

The remainder of this paper is organized as follows: Section 2 introduces essential definitions and the system model. Section 3 presents the quasi-synchronization results under event-triggered control and the complete synchronization criteria via nonlinear adaptive control. Section 4 provides numerical simulations to validate the theoretical findings, and Section 5 concludes the paper.

2 Preliminaries and Model Description

2.1 Preliminaries

This section presents basic preliminaries and essential concepts of fractional calculus. A quaternion is a type of hypercomplex number consisting of one real component and three imaginary components. Any quaternion $Q \in \mathbb{H}$ can be represented as

$$Q = Q^R + Q^I i + Q^J j + Q^K k,$$

where $Q^R, Q^I, Q^J, Q^K \in \mathbb{R}$ denote the real and imaginary components, respectively. The imaginary units i, j, k satisfy the Hamilton multiplication rules:

	i	j	k
i	$i^2 = -1$	$ij = -ji = k$	$ik = -ki = -j$
j	$ji = -ij = -k$	$j^2 = -1$	$jk = -kj = i$
k	$ki = -ik = j$	$kj = -jk = -i$	$k^2 = -1$

Let us define the set of quaternions as

$$\mathbb{H} \triangleq \{Q = Q^R + Q^I i + Q^J j + Q^K k \mid Q^R, Q^I, Q^J, Q^K \in \mathbb{R}\}.$$

The conjugate of a quaternion Q , denoted by $\overline{Q}$, is given by

$$\overline{Q} = Q^R - Q^I i - Q^J j - Q^K k.$$

Two commonly used norms for quaternions are defined as follows. The 1-norm is

$$|Q|_1 = |Q^R| + |Q^I| + |Q^J| + |Q^K|,$$

and the 2-norm is

$$|Q|_2 = \sqrt{Q\overline{Q}} = \sqrt{(Q^R)^2 + (Q^I)^2 + (Q^J)^2 + (Q^K)^2}.$$

For a quaternion vector $\mathcal{Q} = (\mathcal{Q}_1, \mathcal{Q}_2, \dots, \mathcal{Q}_n)^\top \in \mathbb{H}^n$, the corresponding vector norms are defined as

$$\|\mathcal{Q}\|_1 = \sum_{p=1}^n |\mathcal{Q}_p|_1, \quad \|\mathcal{Q}\|_2 = \left(\sum_{p=1}^n |\mathcal{Q}_p|_2^2 \right)^{1/2}.$$

Definition 2.1. [17] For a given function $x(s)$, the fractional integral of order $\epsilon > 0$ in the sense of Riemann-Liouville is defined by

$${}_{{\dagger}_0}^C I_{{\dagger}}^\epsilon x({\dagger}) = \frac{1}{\Gamma(\epsilon)} \int_{{\dagger}_0}^{{\dagger}} ({\dagger} - u)^{\epsilon-1} x(u) du,$$

where $\Gamma(\cdot)$ denotes the Gamma function.

Definition 2.2. [18] The Caputo's fractional-order derivative of function $x({\dagger})$ with order ϵ is defined as

$${}_{{\dagger}_0}^C D_{{\dagger}}^\epsilon x({\dagger}) = \frac{1}{\Gamma(m-\epsilon)} \int_{{\dagger}_0}^{{\dagger}} ({\dagger} - u)^{m-\epsilon-1} x^{(m)}(u) du,$$

in which $m-1 < \epsilon < m \in \mathbb{Z}^+$.

Throughout the paper, ${}_{{\dagger}_0}^C D_{{\dagger}}^\epsilon$ is simply denoted by D^ϵ .

Definition 2.3. [19] Consider the following fractional-order differential equation of order ϵ with a possibly discontinuous right-hand side:

$$\begin{cases} D^\epsilon v({\dagger}) = g(s, v), & s > s_0, \quad v \in \mathbb{H}^n, \\ v(s_0) = v_0. \end{cases}$$

A function $v({\dagger})$ defined on a set $Z \subset \mathbb{H}^n$ is called a *Filippov solution* of the system above if $v({\dagger})$ is absolutely continuous on every compact subset of Z and continuous on $\mathbb{H}^n$.

In this case, for each $s \in Z$, the solution satisfies the differential inclusion $D^\epsilon v({\dagger}) \in \mathcal{G}(s, v)$, where the set-valued map $\mathcal{G}$ is given by

$$\mathcal{G}(s, v) = \bigcap_{\delta > 0} \bigcap_{\mu(\mathcal{N})=0} \overline{\text{co}}[g(\mathcal{B}(v, \delta) \setminus \mathcal{N})].$$

Here, $\overline{\text{co}}$ denotes the closure of the convex hull of a set, μ is the Lebesgue measure, $\mathcal{N}$ is any set of measure zero ($\mu(\mathcal{N}) = 0$), and $\mathcal{B}(v, \delta)$ represents an open neighborhood centered at v with radius $\delta > 0$.

Lemma 2.4. [18] For $h_1({\dagger}), h_2({\dagger}) \in C^n(\mathbb{R}_+, \mathbb{R}^n)$, $\mu, \nu \in \mathbb{R}$ and $m-1 < \epsilon < m$ we obtain

$$D^\epsilon(\mu h_1({\dagger}) + \nu h_2({\dagger})) = \mu D^\epsilon h_1({\dagger}) + \nu D^\epsilon h_2({\dagger}).$$

Lemma 2.5. [20] Let $w({\dagger}) \in C^1(\mathbb{R}_+, \mathbb{R})$ be a real-valued function that is continuously differentiable on its domain. Then, for almost every $t \geq 0$, the following inequality is satisfied:

$$D^\epsilon |w({\dagger})| \leq \text{sign}(w({\dagger})) D^\epsilon (w({\dagger})).$$

Lemma 2.6. [21] Let $\ell_1({\dagger}), \ell_2({\dagger}) \in \mathbb{H}$, then for any $0 < \alpha < 1$, the properties listed below are satisfied:

- $\overline{\text{sign}(\ell_1({\dagger}))} \text{sign}(\ell_1({\dagger})) + \text{sign}(\ell_1({\dagger})) \overline{\text{sign}(\ell_1({\dagger}))} = 2 \|\text{sign}(\ell_1({\dagger}))\|_1.$
- $\left(\overline{\text{sign}(\ell_1({\dagger}))} (\ell_2({\dagger})) + \overline{(\ell_2({\dagger}))} \text{sign}(\ell_1({\dagger})) \right) \leq 2 \|\ell_2({\dagger})\|_1.$

- $\|\ell_1(\dagger)\|_1 = \frac{1}{2} \left[\overline{\text{sign}(\ell_1(\dagger))}(\ell_1(\dagger)) + \overline{(\ell_1(\dagger))} \text{sign}(\ell_1(\dagger)) \right].$
- $D^\epsilon \left[\overline{\text{sign}(\ell_1(\dagger))}(\ell_1(\dagger)) + \overline{(\ell_1(\dagger))} \text{sign}(\ell_1(\dagger)) \right] \leq \overline{\text{sign}(\ell_1(\dagger))} D^\epsilon [\ell_1(\dagger)] + D^\epsilon [\ell_1(\dagger)] \text{sign}(\ell_1(\dagger)).$

Lemma 2.7. [22] Let $\vartheta(\dagger)$ be bounded on $[\dagger_0 - \tau, \dagger_0]$ and continuous on $[\dagger_0, +\infty)$. Assume that $\vartheta(\dagger)$ and $v(\dagger)$ are nonnegative functions satisfying

$$D^\epsilon(\vartheta(\dagger) + v(\dagger)) \leq -\varsigma\vartheta(\dagger) + \varrho \sup_{-\tau \leq s \leq 0} \vartheta(\dagger + s) + \varphi(\dagger), \quad \dagger \geq \dagger_0,$$

where $0 < \epsilon < 1$, $\varsigma > \varrho \geq 0$, $\tau > 0$ denotes the time delay, and $\varphi(\dagger)$ is a nonnegative bounded function such that

$$\lim_{\dagger \rightarrow +\infty} \varphi(\dagger) = 0.$$

Then,

$$\lim_{\dagger \rightarrow +\infty} \vartheta(\dagger) = 0.$$

Lemma 2.8. [22] Let $\vartheta(\dagger)$ be bounded on $[\dagger_0 - \tau, \dagger_0]$ and continuous on $[\dagger_0, +\infty)$. Assume that $\vartheta(\dagger)$ and $v(\dagger)$ are nonnegative functions satisfying

$$D^\epsilon \vartheta(\dagger) \leq -\varsigma\vartheta(\dagger) + \varrho \sup_{-\tau \leq s \leq 0} \vartheta(\dagger + s) + \Xi, \quad \dagger \geq \dagger_0,$$

where $0 < \epsilon < 1$, $\varsigma > \varrho \geq 0$, $\tau > 0$ denotes the time delay, and $\Xi \geq 0$ is a constant. Then,

$$\lim_{\dagger \rightarrow +\infty} \vartheta(\dagger) = \frac{\Xi}{\varsigma - \varrho}.$$

2.2 Model Description

For $i = 1, \dots, \aleph$, the delayed FOQVMNNs model introduced below is considered:

$$\begin{aligned} D^\epsilon y_i(\dagger) = & -c_i y_i(\dagger) + \sum_{j=1}^{\aleph} [a_{ij}(y_i(\dagger)) + \Delta a_{ij}(\dagger)] f_j(y_j(\dagger)) \\ & + \sum_{j=1}^{\aleph} [b_{ij}(y_i(\dagger)) + \Delta b_{ij}(\dagger)] f_j(y_j(\dagger - \tau(\dagger))) + I_i(\dagger), \end{aligned} \quad (2.1)$$

where $\dagger \geq 0$; $y_i(\dagger)$ is the state variable of the i -th neuron; $c_i > 0$, $a_{ij} > 0$ and b_{ij} are connection weights; $\tau(\dagger)$ is the time varying delay satisfies $0 \leq \tau(\dagger) \leq \tau$ ($\tau = \text{constant}$) and $f_j(\cdot) : \mathbb{H} \rightarrow \mathbb{H}$ indicate the continuous activation functions. $a_{ij}(y_i(\dagger))$ and $b_{ij}(y_i(\dagger))$ represent the memristive feedback connection weights and memristive delayed feedback connection weights, respectively. We define the memristive connection weights as follows:

$$\begin{aligned} a_{ij}(\cdot) = & \begin{cases} a'_{ij} = a_{1ij}^R + a_{1ij}^I i + a_{1ij}^J j + a_{1ij}^K k, & |\cdot| \leq \Im_i \\ a''_{ij} = a_{2ij}^R + a_{2ij}^I i + a_{2ij}^J j + a_{2ij}^K k, & |\cdot| > \Im_i \end{cases} \\ b_{ij}(\cdot) = & \begin{cases} b'_{ij} = b_{1ij}^R + b_{1ij}^I i + b_{1ij}^J j + b_{1ij}^K k, & |\cdot| \leq \Im_i \\ b''_{ij} = b_{2ij}^R + b_{2ij}^I i + b_{2ij}^J j + b_{2ij}^K k, & |\cdot| > \Im_i \end{cases} \end{aligned}$$

The initial condition of (2.1) defined as

$$y_i(u) = \phi_i(u), \quad -\tau \leq u \leq 0, \quad i = 1, \dots, \aleph \quad (2.2)$$

where $\phi_i(\cdot)$ is a quaternion-valued continuous functions on $]-\tau, 0]$.

Assuming that the solution $y(\dagger)$ of system (2.1) with initial condition $y(u) = \phi(u) = (\phi_1(u), \phi_2(u), \dots, \phi_n(u))^T \in C([-\tau, 0], \mathbb{H}^n)$ is well-defined and admits an extension over $\mathbb{R}_+$ in Filippov's sense [19]. Model (2.1) may be rewritten as the differential inclusion:

$$\begin{aligned} D^\epsilon y_i(\dagger) \in & -c_i y_i(\dagger) + \sum_{j=1}^{\aleph} [\text{co}(a_{ij}^m, a_{ij}^M) + \Delta a_{ij}(\dagger)] f_j(y_j(\dagger)) \\ & + \sum_{j=1}^{\aleph} [\text{co}(b_{ij}^m, b_{ij}^M) + \Delta b_{ij}(\dagger)] f_j(y_j(\dagger - \tau(\dagger))) + I_i(\dagger), \end{aligned} \quad (2.3)$$

where

- $a'_{ij} = \min\{a'_{ij}, a''_{ij}\}$, $a''_{ij} = \max\{a'_{ij}, a''_{ij}\}$,
- $b'_{ij} = \min\{b'_{ij}, b''_{ij}\}$, and $b''_{ij} = \max\{b'_{ij}, b''_{ij}\}$.

Based on the measurable selection theorem [23], for any solution y of system (2.1), there exist corresponding measurable selections $a_{ij}^*(\dagger) \in \text{co}(a'_{ij}, a''_{ij})$ and $b_{ij}^*(\dagger) \in \text{co}(b'_{ij}, b''_{ij})$ so that for all $t \in [-\tau, 0]$, we obtain

$$\begin{aligned} D^\epsilon y_i(\dagger) = & -c_i y_i(\dagger) + \sum_{j=1}^{\aleph} [a_{ij}^*(\dagger) + \Delta a_{ij}(\dagger)] f_j(y_j(\dagger)) \\ & + \sum_{j=1}^{\aleph} [b_{ij}^*(\dagger) + \Delta b_{ij}(\dagger)] f_j(y_j(\dagger - \tau(\dagger))) + I_i(\dagger). \end{aligned} \quad (2.4)$$

We treat system (2.4) as the drive system, and the associated response system can be expressed as:

$$\begin{aligned} D^\epsilon z_i(\dagger) = & -c_i z_i(\dagger) + \sum_{j=1}^{\aleph} [a_{ij}^{**}(\dagger) + \Delta a_{ij}(\dagger)] f_j(z_j(\dagger)) \\ & + \sum_{j=1}^{\aleph} [b_{ij}^{**}(\dagger) + \Delta b_{ij}(\dagger)] f_j(z_j(\dagger - \tau(\dagger))) + I_i(\dagger) + u_i(\dagger), \end{aligned} \quad (2.5)$$

with $u_i(\dagger)$ is the controller.

The initial condition of system (2.5) is given by

$$z_i(u) = \varphi_i(u), \quad -\tau \leq u \leq 0, \quad i = 1, \dots, \aleph \quad (2.6)$$

where $\varphi_i(\cdot)$ is a quaternion-valued continuous functions on $]-\tau, 0]$.

Remark 2.9. In this paper, the time-varying delays are assumed to be bounded, i.e., $0 \leq \tau(\dagger) \leq \tau$, where τ is a positive constant. This assumption ensures that the delayed states are confined to a finite interval $[-\tau, 0]$, which is essential for the stability analysis. In particular, it allows the use of supremum-based estimates such as $\sup_{-\tau \leq s \leq 0} \|x(s)\|$ in the derivation of the synchronization criteria. Without this boundedness condition, the influence of delayed states could extend over an unbounded interval, making the analysis intractable.

Remark 2.10. In this study, the Caputo fractional derivative is adopted to model the dynamics of the considered neural network. This choice is justified by its clear physical interpretation and its appropriateness for initial value problems, as it allows the incorporation of classical initial conditions defined in terms of integer-order derivatives.

The synchronization error is defined $\nu_i(\dagger) = z_i(\dagger) - y_i(\dagger)$, then we get

$$\begin{aligned} D^\epsilon \nu_i(\dagger) = & -c_i \nu_i(\dagger) + \sum_{j=1}^N [a_{ij}^{**}(\dagger) + \Delta a_{ij}(\dagger)] (f_j(z_j(\dagger)) - f_j(y_j(\dagger))) \\ & + \sum_{j=1}^N [b_{ij}^{**}(\dagger) + \Delta b_{ij}(\dagger)] (f_j(z_j(\dagger - \tau(\dagger))) - f_j(y_j(\dagger - \tau(\dagger)))) \\ & + \sum_{j=1}^N [a_{ij}^{**}(\dagger) - a_{ij}^*(\dagger)] f_j(y_j(\dagger)) + \sum_{j=1}^N [b_{ij}^{**}(\dagger) - b_{ij}^*(\dagger)] f_j(y_j(\dagger - \tau(\dagger))) + u_i(\dagger). \end{aligned} \quad (2.7)$$

Assumption 2.11. There exist M_j , N_j , F_j and F_j positives constants satisfying

$$\|f_j(z_j(\dagger)) - f_j(y_j(\dagger))\|_1 \leq M_j \|z_j(\dagger) - y_j(\dagger)\|_1, \quad \|f_j(z_j(\dagger))\|_1 \leq F_j.$$

Assumption 2.12. Uncertain parameters meet:

$$|\Delta a_{ij}(\dagger)| \leq \Delta a_{ij}, \quad |\Delta b_{ij}(\dagger)| \leq \Delta b_{ij}.$$

Definition 2.13. [22] The delayed FOQVMNNs (2.4) and (2.5) with uncertain parameters are said to achieve quasi-synchronization (QS) if there exists a sufficiently small constant $F > 0$ such that

$$\lim_{\dagger \rightarrow +\infty} \|z_i(\dagger) - y_i(\dagger)\|_1 \leq F.$$

Moreover, the delayed FOQVMNNs (2.4) and (2.5) with uncertain parameters are said to realize complete synchronization (CS) if

$$\lim_{\dagger \rightarrow +\infty} \|z_i(\dagger) - y_i(\dagger)\|_1 = 0.$$

3 Main Results

The main theoretical contributions of this paper are presented in the following section. We begin by establishing sufficient conditions for achieving complete synchronization of the considered FOQVMNNs under the proposed adaptive control scheme.

3.1 Complete Synchronization

An adaptive control scheme is developed, and a set of sufficient conditions is derived to rigorously establish that the proposed approach ensures complete synchronization of the system.

$$\begin{cases} u_i(\dagger) = -\frac{k_i^2(\dagger) \|\nu_i(\dagger)\|_1 \nu_i(\dagger)}{k_i(\dagger) \|\nu_i(\dagger)\|_1 + \lambda(\dagger)} - \tilde{k}_i \text{sign}(\nu_i(\dagger)) \\ D^\epsilon k_i(\dagger) = \alpha_i \|\nu_i(\dagger)\|_1, \end{cases} \quad (3.1)$$

where $\lambda(\dagger)$ satisfies $\lim_{\dagger \rightarrow +\infty} \lambda(\dagger) = 0$.

Theorem 3.1. Suppose that Assumptions 2.11–2.12 hold. If

$$\sum_{j=1}^N [(a_{ij}'' - a_{ij}') + (b_{ij}'' - b_{ij}')] F_j \leq N \tilde{k}_i,$$

then the delayed FOQVMNNs (2.4) and (2.5) with uncertain parameters achieve CS under the adaptive controller (3.1).

A function is introduced via

$$\Lambda(\dagger) = \Lambda_1(\dagger) + \Lambda_2(\dagger)$$

where

$$\begin{aligned} \bullet \Lambda_1(\dagger) &= 2\|\nu_i(\dagger)\|_1 = \sum_{i=1}^N \left[\overline{\text{sign}(\nu_i(\dagger))}(\nu_i(\dagger)) + (\overline{\nu_i(\dagger)}) \text{sign}(\nu_i(\dagger)) \right]. \\ \bullet \Lambda_2(\dagger) &= \sum_{i=1}^N \frac{1}{\alpha_i} (k_i(\dagger) - k^*)^2. \end{aligned}$$

Then,

$$\begin{aligned} D^\epsilon \Lambda(\dagger) &\leq \sum_{i=1}^N \overline{\text{sign}(\nu_i(\dagger))} D^\epsilon[\nu_i(\dagger)] + D^\epsilon(\overline{\nu_i(\dagger)}) \text{sign}(\nu_i(\dagger)) \\ &\leq \sum_{i=1}^N \overline{\text{sign}(\nu_i(\dagger))} \left\{ -c_i \nu_i(\dagger) + \sum_{j=1}^N [a_{ij}^{**}(\dagger) + \Delta a_{ij}(\dagger)] (f_j(z_j(\dagger)) - f_j(y_j(\dagger))) \right. \\ &\quad + \sum_{j=1}^N [b_{ij}^{**}(\dagger) + \Delta b_{ij}(\dagger)] (f_j(z_j(\dagger - \tau(\dagger))) - f_j(y_j(\dagger - \tau(\dagger)))) \\ &\quad \left. + \sum_{j=1}^N [a_{ij}^{**}(\dagger) - a_{ij}^*(\dagger)] f_j(y_j(\dagger)) + \sum_{j=1}^N [b_{ij}^{**}(\dagger) - b_{ij}^*(\dagger)] f_j(y_j(\dagger - \tau(\dagger))) - \tilde{k}_i \text{sign}(\nu_i(\dagger)) \right\} \\ &\quad + \sum_{i=1}^N \overline{\left\{ -c_i \nu_i(\dagger) + \sum_{j=1}^N [a_{ij}^{**}(\dagger) + \Delta a_{ij}(\dagger)] (f_j(y_j(\dagger)) - f_j(x_j(\dagger))) \right.} \\ &\quad \left. + \sum_{j=1}^N [b_{ij}^{**}(\dagger) + \Delta b_{ij}(\dagger)] (f_j(y_j(\dagger - \tau(\dagger))) - f_j(x_j(\dagger - \tau(\dagger)))) \right.} \\ &\quad \left. + \sum_{j=1}^N [a_{ij}^{**}(\dagger) - a_{ij}^*(\dagger)] f_j(y_j(\dagger)) \right. \\ &\quad \left. + \sum_{j=1}^N [b_{ij}^{**}(\dagger) - b_{ij}^*(\dagger)] f_j(y_j(\dagger - \tau(\dagger))) - \tilde{k}_i \text{sign}(\nu_i(\dagger)) \right\} \text{sign}(\nu_i(\dagger)) \\ &\quad - \sum_{i=1}^N \frac{k_i^2(\dagger) \|\nu_i(\dagger)\|_1 \{\nu_i(\dagger)\}}{k_i(\dagger) \|\nu_i(\dagger)\|_1 + \lambda(\dagger)} + 2 \sum_{i=1}^N (k_i(\dagger) - k^*) \|\nu_i(\dagger)\|_1 \end{aligned} \quad (3.2)$$

where $\{\nu_i(\dagger)\} = \overline{\text{sign}(\nu_i(\dagger))} \nu_i(\dagger) + \text{sign}(\nu_i(\dagger)) \overline{\nu_i(\dagger)}$.

By Lemma 2.6 and Assumptions 2.11 and 2.12, we obtain

$$- \sum_{i=1}^N \overline{\text{sign}(\nu_i(\dagger))} c_i \nu_i(\dagger) - \sum_{i=1}^N \overline{c_i \nu_i(\dagger)} \text{sign}(\nu_i(\dagger)) \leq -2 \sum_{i=1}^N c_i \|\nu_i(\dagger)\|_1 \quad (3.3)$$

and

$$\begin{aligned} \bullet \sum_{i=1}^N \sum_{j=1}^N \overline{\text{sign}(\nu_i(\dagger))} [a_{ij}^{**}(\dagger) + \Delta a_{ij}(\dagger)] (f_j(z_j(\dagger)) - f_j(y_j(\dagger))) \\ + \sum_{i=1}^N \sum_{j=1}^N \left[\overline{[a_{ij}^{**}(\dagger) + \Delta a_{ij}(\dagger)] (f_j(z_j(\dagger)) - f_j(y_j(\dagger)))} \right] \text{sign}(\nu_i(\dagger)) \\ \leq 2 \sum_{i=1}^N \sum_{j=1}^N (a_{ij}'' + \Delta a_{ij}) M_j \|\nu_j(\dagger)\|_1; \end{aligned} \quad (3.4)$$

$$\begin{aligned}
& \bullet \sum_{i=1}^{\aleph} \sum_{j=1}^{\aleph} \overline{\text{sign}(\nu_i(\dagger))} [b_{ij}^{**}(\dagger) + \Delta b_{ij}(\dagger)] (f_j(z_j(\dagger - \tau(\dagger))) - f_j(y_j(\dagger - \tau(\dagger)))) \\
& + \sum_{i=1}^{\aleph} \sum_{j=1}^{\aleph} \left[\overline{[b_{ij}^{**}(\dagger) + \Delta b_{ij}(\dagger)] (f_j(z_j(\dagger - \tau(\dagger))) - f_j(y_j(\dagger - \tau(\dagger))))} \right] \text{sign}(\nu_i(\dagger)) \\
& \leq 2 \sum_{i=1}^{\aleph} \sum_{j=1}^{\aleph} (b_{ij}'' + \Delta b_{ij}) M_j \|\nu_j(\dagger - \tau(\dagger))\|_1;
\end{aligned} \tag{3.5}$$

$$\begin{aligned}
& \bullet \sum_{i=1}^{\aleph} \sum_{j=1}^{\aleph} \overline{\text{sign}(\nu_i(\dagger))} [a_{ij}^{**}(\dagger) - a_{ij}^*(\dagger)] f_j(y_j(\dagger)) + \sum_{i=1}^{\aleph} \sum_{j=1}^{\aleph} \left[\overline{[a_{ij}^{**}(\dagger) - a_{ij}^*(\dagger)] f_j(y_j(\dagger))} \right] \text{sign}(\nu_i(\dagger)) \\
& \leq 2 \sum_{i=1}^{\aleph} \sum_{j=1}^{\aleph} (a_{ij}'' - a_{ij}') F_j;
\end{aligned} \tag{3.6}$$

$$\begin{aligned}
& \bullet \sum_{i=1}^{\aleph} \sum_{j=1}^{\aleph} \overline{\text{sign}(\nu_i(\dagger))} [b_{ij}^{**}(\dagger) - b_{ij}^*(\dagger)] f_j(y_j(\dagger - \tau(\dagger))) \\
& + \sum_{i=1}^{\aleph} \sum_{j=1}^{\aleph} \left[\overline{[b_{ij}^{**}(\dagger) - b_{ij}^*(\dagger)] f_j(y_j(\dagger - \tau(\dagger)))} \right] \text{sign}(\nu_i(\dagger)) \\
& \leq 2 \sum_{i=1}^{\aleph} \sum_{j=1}^{\aleph} (b_{ij}'' - b_{ij}') F_j;
\end{aligned} \tag{3.7}$$

$$\bullet \tilde{k}_i \left(\overline{\text{sign}(\nu_i(\dagger))} \text{sign}(\nu_i(\dagger)) + \text{sign}(\nu_i(\dagger)) \overline{\text{sign}(\nu_i(\dagger))} \right) = 2\tilde{k}_i |\text{sign}(\nu_i(\dagger))|_1 \geq 2\tilde{k}_i$$

equivalently,

$$-\sum_{i=1}^{\aleph} \tilde{k}_i \left(\overline{\text{sign}(\nu_i(\dagger))} \text{sign}(\nu_i(\dagger)) + \text{sign}(\nu_i(\dagger)) \overline{\text{sign}(\nu_i(\dagger))} \right) \leq -2\aleph \tilde{k}_i. \tag{3.8}$$

By applying Lemma 2.6 and the inequality $0 \leq \frac{fg}{f+g} \leq g$ for all $f, g > 0$, we obtain

$$\begin{aligned}
& 2 \sum_{i=1}^{\aleph} k_i(\dagger) \|\nu_i(\dagger)\|_1 - \sum_{i=1}^{\aleph} \frac{k_i^2(\dagger) \|\nu_i(\dagger)\|_1 \{\nu_i(\dagger)\}}{k_i(\dagger) \|\nu_i(\dagger)\|_1 + \lambda(\dagger)} \\
& = 2 \sum_{i=1}^{\aleph} k_i(\dagger) \|\nu_i(\dagger)\|_1 - 2 \sum_{i=1}^{\aleph} \frac{k_i^2(\dagger) \|\nu_i(\dagger)\|_1^2}{k_i(\dagger) \|\nu_i(\dagger)\|_1 + \lambda(\dagger)} \\
& = 2 \sum_{i=1}^{\aleph} \frac{k_i(\dagger) \|\nu_i(\dagger)\|_1 \lambda(\dagger)}{k_i(\dagger) \|\nu_i(\dagger)\|_1 + \lambda(\dagger)} \\
& \leq 2\aleph \lambda(\dagger).
\end{aligned} \tag{3.9}$$

Taking into account (3.3)-(3.9) with (3.2) we derive

$$\begin{aligned}
 D^\epsilon \Lambda(\dagger) &\leq 2 \sum_{i=1}^N \left(-c_i - k_i^* + \sum_{j=1}^N (a_{ji}'' + \Delta a_{ji}) M_i \right) \|\nu_i(\dagger)\|_1 \\
 &\quad + 2 \sum_{i=1}^N \sum_{j=1}^N (b_{ji}'' + \Delta b_{ji}) M_i \|\nu_i(\dagger - \tau(\dagger))\|_1 \\
 &\quad + 2 \sum_{i=1}^N \left(\sum_{j=1}^N \left[(a_{ij}'' - a_{ij}') + (b_{ij}'' - b_{ij}') \right] F_j - \aleph \tilde{k}_i \right) + 2\aleph \lambda(\dagger) \\
 &\leq -\varrho_1 \Lambda_1(\dagger) + \varrho_2 \Lambda_1(\dagger - \tau(\dagger)) + 2\aleph \lambda(\dagger) \\
 &\leq -\varrho_1 \Lambda_1(\dagger) + \varrho_2 \sup_{-\tau \leq \sigma \leq 0} \Lambda_1(\dagger + \sigma) + 2\aleph \lambda(\dagger),
 \end{aligned} \tag{3.10}$$

where

$$\varrho_1 = \min_{1 \leq i \leq N} \left\{ c_i + k_i^* - \sum_{j=1}^N (a_{ji}'' + \Delta a_{ji}) M_i \right\} > 0, \quad \text{and} \quad \varrho_2 = \sum_{j=1}^N (b_{ji}'' + \Delta b_{ji}) M_i.$$

Since $\lambda(\dagger) \rightarrow 0$ as $\dagger \rightarrow +\infty$, it follows from Lemma 2.7 and inequality (3.10) that

$$\lim_{\dagger \rightarrow +\infty} \Lambda_1(\dagger) = 0,$$

which further implies

$$\lim_{\dagger \rightarrow +\infty} \|z_i(\dagger) - y_i(\dagger)\|_1 = 0.$$

Therefore, according to Definition 2.13, the delayed FOQVMNNs (2.4) and (2.5) with uncertain parameters are said to realize CS under the adaptive controller (3.1).

3.2 Quasi synchronization

This section presents the design of controllers to achieve QS between response (2.4) system and drive system (2.5). Event triggered control is designed as follows

$$U_i(\dagger) = -\chi_i \nu_i(\dagger_\ell), \quad t \in [\dagger_\ell, \dagger_{\ell+1}) \tag{3.11}$$

in which:

- The constant χ_i represents the feedback gain.
- The monotonically increasing times sequence $\dagger_\ell$ denotes the event-triggered instant.

Let $\varepsilon_i(\dagger) = \nu_i(\dagger_\ell) - \nu_i(\dagger)$: the measurement error function.

Constructing the following event-triggered strategy

$$\|\varepsilon_i(\dagger)\|_1 \leq \varpi \|\nu_i(\dagger_\ell)\|_1 \tag{3.12}$$

Consequently, the triggering instant $\dagger_{\ell+1}$ is defined by:

$$\dagger_{\ell+1} = \inf\{t : t > \dagger_\ell, \|\varepsilon_i(\dagger)\|_1 \geq \varpi \|\nu_i(\dagger_\ell)\|_1\} \tag{3.13}$$

with $\varpi > 0$ is the triggering constant.

Remark 3.2. In [24], synchronization of fractional-order neural networks is achieved using a continuous control strategy, which requires continuous communication. In contrast, the proposed event-triggered control updates the controller only when necessary. This significantly reduces communication cost while maintaining satisfactory synchronization performance under uncertainties and time-varying delays. Therefore, the proposed approach is more suitable for practical systems with limited communication and computational resources.

Theorem 3.3. Assume that Assumptions 2.11 and 2.12 holds, (2.4) and (2.5) achieve QS under the event triggered control (3.11) if this parameters satisfy

$$\begin{aligned}\rho_1 &= \min_{1 \leq i \leq \aleph} \left\{ c_i + \frac{\chi_i}{1 - \varpi} - \sum_{j=1}^{\aleph} (a''_{ij} + \Delta a_{ij}) M_i \right\}; \\ \rho_2 &= \sum_{i=1}^{\aleph} \sum_{j=1}^{\aleph} (b''_{jn} + \Delta b_{jn}) M_i; \\ \Xi &= 2 \sum_{i=1}^{\aleph} \sum_{j=1}^{\aleph} \left[(a''_{ij} - a'_{ij}) + (b''_{ij} - b'_{ij}) \right] F_j.\end{aligned}$$

Furthermore, the error bounded can be estimated as $\frac{\Xi}{2(\rho_1 - \rho_2)}$.

A function is introduced via

$$\Lambda(\dagger) = 2\|\nu_i(\dagger)\|_1 = \sum_{i=1}^{\aleph} \left[\overline{\text{sign}(\nu_i(\dagger))}(\nu_i(\dagger)) + (\overline{\nu_i(\dagger)}) \text{sign}(\nu_i(\dagger)) \right].$$

Then,

$$\begin{aligned}D^\epsilon \Lambda(\dagger) &\leq \sum_{i=1}^{\aleph} \overline{\text{sign}(\nu_i(\dagger))} D^\epsilon[\nu_i(\dagger)] + D^\epsilon(\overline{\nu_i(\dagger)}) \text{sign}(\nu_i(\dagger)) \\ &\leq \sum_{i=1}^{\aleph} \overline{\text{sign}(\nu_i(\dagger))} \left\{ -c_i \nu_i(\dagger) + \sum_{j=1}^{\aleph} [a_{ij}^{**}(\dagger) + \Delta a_{ij}(\dagger)] (f_j(z_j(\dagger)) - f_j(y_j(\dagger))) \right. \\ &\quad \left. + \sum_{j=1}^{\aleph} [b_{ij}^{**}(\dagger) + \Delta b_{ij}(\dagger)] (f_j(z_j(\dagger - \tau(\dagger))) - f_j(y_j(\dagger - \tau(\dagger)))) \right. \\ &\quad \left. + \sum_{j=1}^{\aleph} [a_{ij}^{**}(\dagger) - a_{ij}^*(\dagger)] f_j(y_j(\dagger)) + \sum_{j=1}^{\aleph} [b_{ij}^{**}(\dagger) - b_{ij}^*(\dagger)] f_j(y_j(\dagger - \tau(\dagger))) + u_i(\dagger) \right\} \\ &\quad + \sum_{i=1}^{\aleph} \left\{ -c_i \nu_i(\dagger) + \sum_{j=1}^{\aleph} [a_{ij}^{**}(\dagger) + \Delta a_{ij}(\dagger)] (f_j(z_j(\dagger)) - f_j(y_j(\dagger))) \right. \\ &\quad \left. + \sum_{j=1}^{\aleph} [b_{ij}^{**}(\dagger) + \Delta b_{ij}(\dagger)] (f_j(z_j(\dagger - \tau(\dagger))) - f_j(y_j(\dagger - \tau(\dagger)))) \right. \\ &\quad \left. + \sum_{j=1}^{\aleph} [a_{ij}^{**}(\dagger) - a_{ij}^*(\dagger)] f_j(y_j(\dagger)) + \sum_{j=1}^{\aleph} [b_{ij}^{**}(\dagger) - b_{ij}^*(\dagger)] f_j(y_j(\dagger - \tau(\dagger))) + u_i(\dagger) \right\} \text{sign}(\nu_i(\dagger)).\end{aligned}\tag{3.14}$$

To handle the control input appearing in (3.11), we now employ the event-triggered control law and derive an upper bound for the corresponding terms.

$$u_i(\dagger) = \chi_i(\dagger) \nu_i(\dagger_\ell) = \chi_i(\dagger) [\varepsilon_i(\dagger) + \nu_i(\dagger)]$$

or,

$$\|\nu_i(\dagger_\ell)\|_1 \leq \|\varepsilon_i(\dagger)\|_1 + \|\nu_i(\dagger)\|_1 \leq \varpi \|\nu_i(\dagger_\ell)\|_1 + \|\nu_i(\dagger)\|_1$$

then,

$$\|\nu_i(\dagger_\ell)\|_1 \leq \frac{1}{1 - \varpi} \|\nu_i(\dagger)\|_1,$$

and

$$\sum_{i=1}^N \overline{\text{sign}(\nu_i(\dagger))} u_i(\dagger) + \sum_{i=1}^N \overline{u_i(\dagger)} \text{sign}(\nu_i(\dagger)) \leq 2 \sum_{i=1}^N \chi_i \|\nu_i(\dagger_\ell)\|_1 \leq 2 \sum_{i=1}^N \frac{\chi_i}{1-\varpi} \|\nu_i(\dagger)\|_1 \quad (3.15)$$

By (3.3)-(3.7) and (3.15) into (3.14), we obtain

$$\begin{aligned} D^\epsilon \Lambda(\dagger) &\leq 2 \sum_{i=1}^N \left(-c_i - \frac{\chi_i}{1-\varpi} + \sum_{j=1}^N (a''_{jn} + \Delta a_{jn}) M_i \right) \|\nu_i(\dagger)\|_1 \\ &\quad + 2 \sum_{i=1}^N \sum_{j=1}^N (b''_{jn} + \Delta b_{jn}) M_i \|\nu_i(\dagger - \tau(\dagger))\|_1 \\ &\quad + 2 \sum_{i=1}^N \sum_{j=1}^N \left[(a''_{ij} - a'_{ij}) + (b''_{ij} - b'_{ij}) \right] F_j \\ &\leq -\rho_1 \Lambda(\dagger) + \rho_2 \Lambda(\dagger - \tau(\dagger)) + \Xi \\ &\leq -\rho_1 \Lambda(\dagger) + \rho_2 \sup_{-\tau \leq \sigma \leq 0} \Lambda(\dagger + \sigma) + \Xi. \end{aligned} \quad (3.16)$$

From Lemma 2.8, it find that

$$\lim_{\dagger \rightarrow +\infty} \Lambda(\dagger) = \frac{\Xi}{\rho_1 - \rho_2},$$

which implies that

$$\lim_{\dagger \rightarrow +\infty} \|\nu_i(\dagger)\|_1 = \frac{\Xi}{\rho_1 - \rho_2}.$$

Hence $\|\nu_i(\dagger)\|_1 \leq \frac{\Xi}{\rho_1 - \rho_2}$, $\dagger \rightarrow +\infty$ by Definition 2.13, we can conclude that (2.4) and (2.5) achieve QS under the event triggered control (3.11).

Remark 3.4. If the triggering parameter is set to $\varpi = 0$, the event-triggered condition (3.12) becomes inactive, and the control scheme simplifies to a conventional continuous feedback law. Consequently, the standard periodic control can be viewed as a limiting case of the proposed strategy, which highlights the inherent flexibility and superior resource efficiency of our event-triggered design.

Remark 3.5. Note that the non-commutative nature of quaternion multiplication does not affect the Lyapunov analysis in this work. Indeed, the Lyapunov functional is constructed based on real-valued norms of quaternion vectors, which map quaternion quantities into the real field. Therefore, all terms involved in the stability analysis become scalar-valued, and the fractional-order derivative is applied to real scalar functions. As a result, the derivation of the Lyapunov inequalities is carried out in the real domain, while the quaternion structure of the system is fully preserved.

Theorem 3.6. Under the designed control strategy (3.11) and presented event triggering condition (3.13), the $Z_\ell = \dagger_{\ell+1} - \dagger_\ell$ zeno behavior can be precluded.

For $t \in [\dagger_\ell, \dagger_{\ell+1})$, one has $D^\epsilon \varepsilon_i(\dagger) = -D^\epsilon \nu_i(\dagger)$ or,

$$\begin{aligned} D^\epsilon \|\varepsilon_i(\dagger)\|_1 &\leq \frac{1}{2} \left[\overline{\text{sign}(\varepsilon_i(\dagger))} (D^\epsilon \varepsilon_i(\dagger)) + (\overline{D^\epsilon \varepsilon_i(\dagger)}) \text{sign}(\varepsilon_i(\dagger)) \right] \\ &\leq -D^\epsilon \|\nu_i(\dagger)\|_1 \end{aligned} \quad (3.17)$$

From Theorem 3.3, the synchronization error satisfies

$$\|\nu_i(\dagger)\|_1 \leq \frac{\Xi}{\rho_1 - \rho_2},$$

which implies that the state trajectories are bounded. Moreover, under Assumptions (A1)–(A2), the nonlinear functions are continuous and Lipschitz, which ensures that $D^\epsilon \|\nu_i(\dagger)\|_1$ is also bounded. Hence, there exists a constant $\Theta > 0$ such that

$$D^\epsilon \|\nu_i(\dagger)\|_1 \leq \Theta.$$

By integrating both sides of inequality (3.17) from $\dagger_\ell$ to $\dagger$, we obtain:

$$\begin{aligned} \|\varepsilon_i(\dagger)\| &\leq \frac{1}{\Gamma(\epsilon)} \int_{\dagger_\ell}^{\dagger} (\dagger - s)^{\epsilon-1} D^\epsilon \|\nu_i(\dagger)\| ds \\ &\leq \frac{\Theta}{\Gamma(\epsilon)} \int_{\dagger_\ell}^{\dagger} (\dagger - s)^{\epsilon-1} ds \\ &\leq \frac{\Theta(\dagger - \dagger_\ell)^\epsilon}{\Gamma(\epsilon + 1)}. \end{aligned}$$

By the triggered condition (3.13), when $\dagger = \dagger_{\ell+1}$, we can get $\|\varepsilon_i(t_{\ell+1})\| \geq \varpi \|\nu_i(\dagger_\ell)\|$. Thus

$$\frac{\Theta(\dagger_{\ell+1} - \dagger_\ell)^\epsilon}{\Gamma(\epsilon + 1)} \geq \varpi \|\nu_i(\dagger_\ell)\|_1 \geq \varpi \sigma$$

ones derives

$$Z_\ell = \dagger_{\ell+1} - \dagger_\ell \geq \left\lceil \frac{\varpi \sigma \Gamma(\epsilon + 1)}{\Theta} \right\rceil \frac{1}{\epsilon}.$$

Therefore, the Zeno behavior can be precluded.

Remark 3.7. In [24], Tan *et al.* studied the global Mittag–Leffler synchronization of fractional-order nonautonomous neural networks with mixed delays using a linear feedback controller. In [17], Yang *et al.* investigated QS and CS of fractional-order fuzzy BAM neural networks via nonlinear control. Moreover, in [22], Yang *et al.* addressed complete and quasi-synchronization of fractional-order memristive neural networks with time-varying delays. Compared with these works, the present study considers uncertain fractional-order quaternion-valued memristive neural networks with time-varying delays and non-identical connection weights between the drive and response systems. In addition, two distinct control strategies are developed: an adaptive nonlinear controller to achieve CS under system uncertainties, and an event-triggered control scheme to realize QS.

4 Numerical simulations

To illustrate the applicability of the proposed theoretical results, this section presents two numerical examples.

Example 4.1. Considering system (2.4) and (2.5) with the following parameter values:

$$\begin{aligned} a_{11}^*(y_1(\dagger)) &= \begin{cases} 1.8 + 1.29i + 1.12j + 1.9k, & |y_1(\dagger)| \leq 1 \\ 1.6 + 1.2i + 1.17j + k, & |y_1(\dagger)| > 1 \end{cases} \\ a_{12}^*(y_2(\dagger)) &= \begin{cases} -0.1 + 0.09i - 0.07j - 0.11k, & |y_2(\dagger)| \leq 1 \\ -0.12 + 0.1i - 0.1j - 0.12k, & |y_2(\dagger)| > 1 \end{cases} \end{aligned}$$

$$\begin{aligned}
a_{21}^*(y_1(\dagger)) &= \begin{cases} -2.26 - 2.37i - 2.29j - 1.7k, |y_1(\dagger)| \leq 1 \\ -2.1 - 2.2i - 2.3j - 1.3k, |y_1(\dagger)| > 1 \end{cases} \\
a_{22}^*(y_2(\dagger)) &= \begin{cases} 3 + 3.4i + 2.97j + 3k, |y_2(\dagger)| \leq 1 \\ 3 + 3.3i + 3j + 2k, |y_2(\dagger)| > 1 \end{cases} \\
b_{11}^*(y_1(\dagger)) &= \begin{cases} -1.5 - 2.16i - 1.43j - 1.5k, |y_1(\dagger)| \leq 1 \\ -1.3 - 0.19i - 1.3j - 1.1k, |y_1(\dagger)| > 1 \end{cases} \\
b_{12}^*(y_2(\dagger)) &= \begin{cases} -0.1 - 0.12i - 0.1j - 0.12k, |y_2(\dagger)| \leq 1 \\ 0.1 + 0.1i - 0.1j + 0.1k, |y_2(\dagger)| > 1 \end{cases} \\
b_{21}^*(y_1(\dagger)) &= \begin{cases} -0.31 - 0.12i - 0.12j + 0.13k, |y_1(\dagger)| \leq 1 \\ 0.29 + 0.13i + 0.1j + 0.1k, |y_1(\dagger)| > 1 \end{cases} \\
b_{22}^*(y_2(\dagger)) &= \begin{cases} -1.7 - 2.2i - 2j - 1.9k, |y_2(\dagger)| \leq 1 \\ -1.39 - 2.34i - 1.89j - 1.92k, |y_2(\dagger)| > 1 \end{cases} \\
a_{11}^{**}(z_1(\dagger)) &= \begin{cases} 1.13 + 1.3i + 1.26j + 1.35k, |y_1(\dagger)| \leq 1 \\ 0.87 + 1.1i + 1.2j + 1.3k, |y_1(\dagger)| > 1 \end{cases} \\
a_{12}^{**}(z_2(\dagger)) &= \begin{cases} -0.23 - 0.2i - 0.26j - 0.23k, |y_2(\dagger)| \leq 1 \\ -0.31 + 0.3i - 0.27j - 0.24k, |y_2(\dagger)| > 1 \end{cases} \\
a_{21}^{**}(z_1(\dagger)) &= \begin{cases} -1.2 - 1.24i - 1.24j - 1.2k, |y_1(\dagger)| \leq 1 \\ -1.11 - 1.29i - 1.1j - 1.22k, |y_1(\dagger)| > 1 \end{cases} \\
a_{22}^{**}(z_2(\dagger)) &= \begin{cases} 3.1 + 2.8i + 2.3j + 2.9k, |y_2(\dagger)| \leq 1 \\ 2.8 + 2.4i + 2j + 2.8k, |y_2(\dagger)| > 1 \end{cases} \\
b_{11}^{**}(z_1(\dagger)) &= \begin{cases} -1.5 - 1.2i - 1.17j - 1.23k, |y_1(\dagger)| \leq 1 \\ -1.48 - 0.9i - 1.15j - 0.49k, |y_1(\dagger)| > 1 \end{cases} \\
b_{12}^{**}(z_2(\dagger)) &= \begin{cases} -0.21 - 0.2i - 0.3j - 0.21k, |y_2(\dagger)| \leq 1 \\ 0.23 + 0.23i + 0.2j + 0.2k, |y_2(\dagger)| > 1 \end{cases} \\
b_{21}^{**}(z_1(\dagger)) &= \begin{cases} -0.2 - 0.23i - 0.2j - 0.27k, |y_1(\dagger)| \leq 1 \\ 0.7 - 0.21i + 0.2j + 0.11k, |y_1(\dagger)| > 1 \end{cases} \\
b_{22}^{**}(z_2(\dagger)) &= \begin{cases} 0.7 - 2.23i - 2.4j - 2.26k, |y_2(\dagger)| \leq 1 \\ -1.7 - 2i - 2.2j - 2.22k, |y_2(\dagger)| > 1 \end{cases}
\end{aligned}$$

Furthermore, the parameters are selected as $\epsilon = 0.98$ is the fractional order, $c_1 = c_2 = 5$, $\tau(\dagger) = \frac{e^\dagger}{1 + e^\dagger}$, $\Delta a_{ij}(\dagger) = 0.1 \sin(\dagger)$ and $\Delta b_{ij}(\dagger) = 0.1 \cos(\dagger)$. In this example, $y_i(\dagger) = y_i^R(\dagger) + iy_i^I(\dagger) + jy_i^J(\dagger) + ky_i^K(\dagger)$ the activation function for each neuron is defined as $f(y_i(\dagger)) = g(y_i(\dagger)) = \tanh(y_i^R(\dagger) + iy_i^I(\dagger) + jy_i^J(\dagger) + ky_i^K(\dagger))$ with initial condition $y_1(0) = -1.2 - 1.3i - 1.4j + \frac{3}{2}k$, $z_1(0) = 3 - 2.5i - 3j - 2k$, $y_2(0) = 1.4 + 1.25i + 4j + 4k$ and $z_2(0) = -3 - 4.5i - 2.5j - 4k$. The dynamic evolutions of systems (2.4) and (2.5) are depicted in Figure 1. Without the

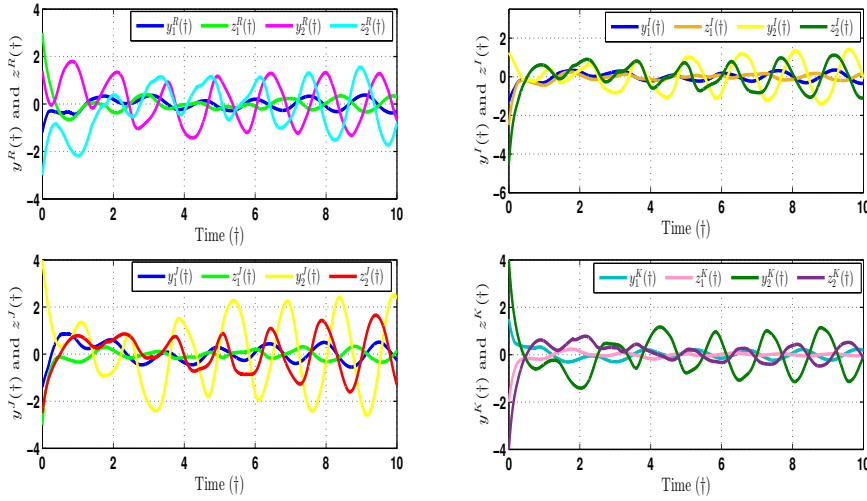


Figure 1. The trajectories of $y_i(\dagger)$ and $z_i(\dagger)$ for systems (2.4) and (2.5) without control.

controller, the error trajectories displayed in Figure 2 demonstrate that synchronization is not attainable.

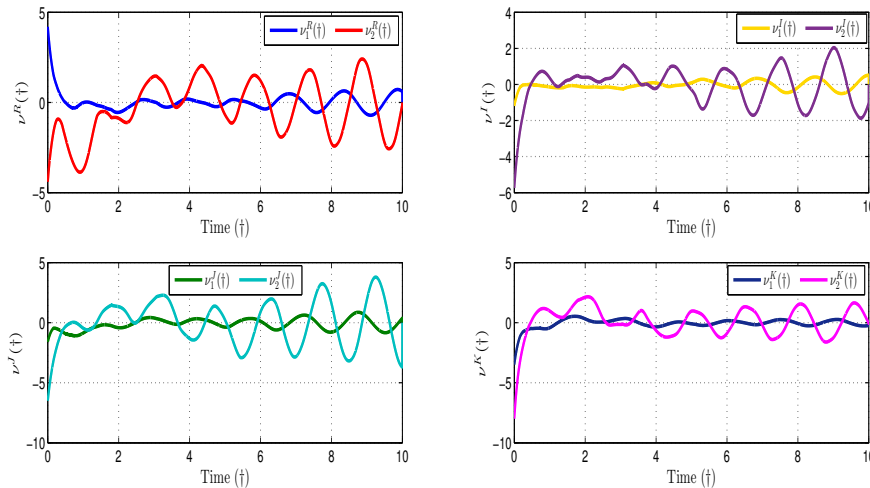


Figure 2. Error state trajectories without controller

The controller parameters are given by $\tilde{k}_1 = \tilde{k}_2 = 7$, $\lambda(\dagger) = e^{-0.1\dagger}$, $\alpha_1 = \alpha_2 = 0.5$, $k_z(0) = (1, 1)^T$. The Figure 3 portrays the error system (2.7).

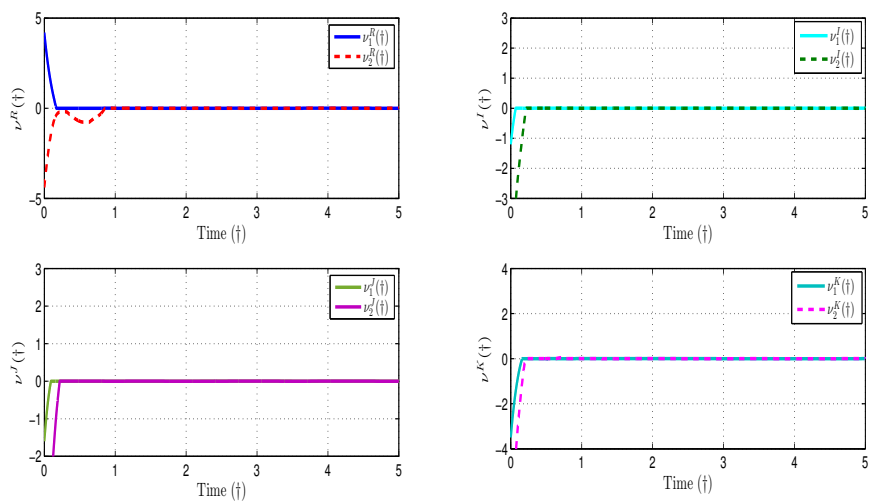


Figure 3. Error state trajectories under the control (3.1)

Furthermore, Figure 4 illustrates the corresponding control-gain curves.

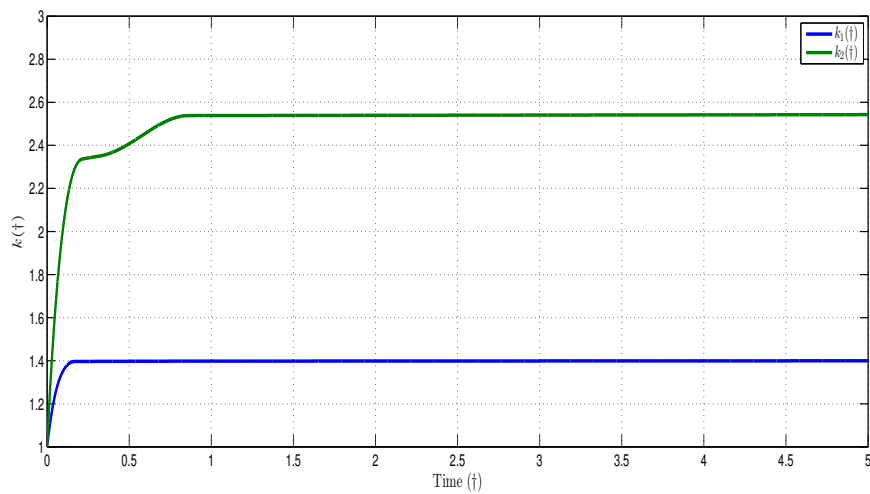


Figure 4. Temporal profiles of the control gains $k_i(\dagger)$, $i = 1, 2$.

Example 4.2. With the same the memristive feedback connection weights as Example 4.1 for systems (2.4) and (2.5). The parameters are selected as $\epsilon = 0.8$ is the fractional order, $c_1 = c_2 = 4.3$, $\tau(\dagger) = \frac{e^\dagger}{1 + e^\dagger}$, $\Delta a_{ij}(\dagger) = 0.5 \sin(\dagger)$ and $\Delta b_{ij}(\dagger) = 0.5 \cos(\dagger)$. In this example, $y_i(\dagger) = y_i^R(\dagger) + iy_i^I(\dagger) + jy_i^J(\dagger) + ky_i^K(\dagger)$ the activation function for each neuron is defined as $f(y_i(\dagger)) = g(y_i(\dagger)) = 1.5 \tanh(y_i^R(\dagger) + iy_i^I(\dagger) + jy_i^J(\dagger) + ky_i^K(\dagger))$ with initial condition $y_1(0) = -1.2 - 1.3i - 1.4j + \frac{3}{2}k$, $z_1(0) = 3 - 2.5i - 3j - 2k$, $y_2(0) = 1.4 + 1.25i + 4j + 4k$ and $z_2(0) = -3 - 4.5i - 2.5j - 4k$.

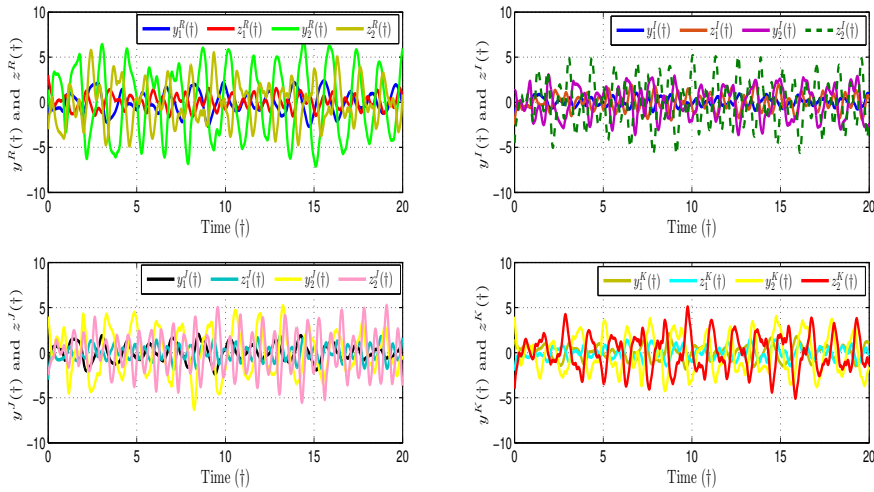


Figure 5. The trajectories of $y_i(t)$ and $z_i(t)$ for systems (2.4) and (2.5) without control.

Based on the event-triggered control mechanism described in equations (3.11), (3.12), and (3.13), the following control parameters are selected: $\varpi_1 = \varpi_2 = 0.7$ and $\chi_1 = \chi_2 = 5$. The assumptions of Theorem 3.3 are fully achieved with the error bound $\frac{\Xi}{\rho_1 - \rho_2} \approx 2.516$ then system (2.5) is quasi-synchronized with system (2.4), with a synchronization bound is 2.516.

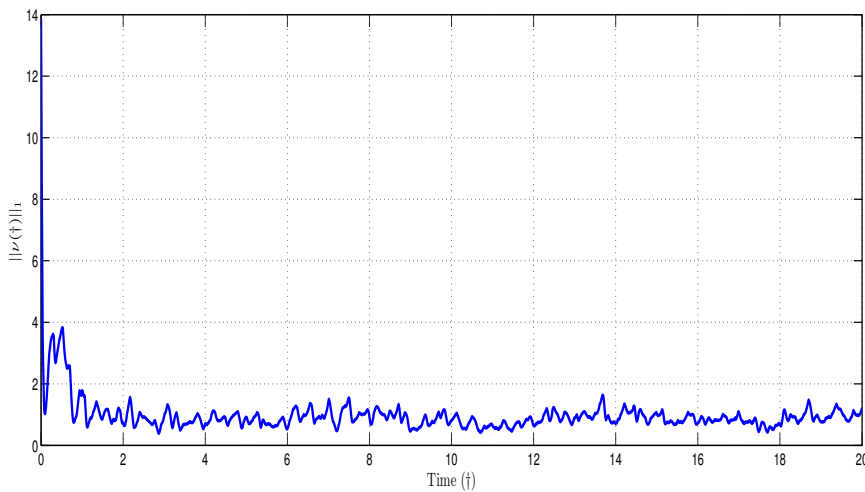


Figure 6. The curve of $\|\nu(t)\|_1$ under controller (3.11).

The simulation results are illustrated in Figs. 5–7. Specifically, Figs. 5 depict the temporal evolution of the states $y_i(t)$ and $z_i(t)$ for systems (2.4) and (2.5). The evaluation of the corresponding synchronization errors is presented in Fig6. Fig 7 displays the distribution of event-triggering intervals, demonstrating that the inter-event times remain strictly positive. This effectively confirms that the Zeno phenomenon does not occur, under the proposed control mechanism.

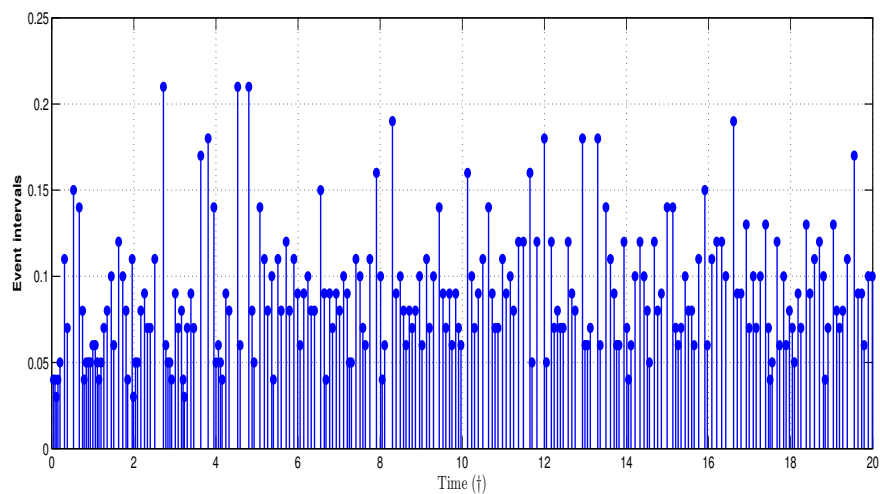


Figure 7. Inter event times.

Example 4.3. To further investigate the influence of the fractional order, time delay, and uncertainties, we consider the same setup as in Example 4.2, while only modifying these specific parameters. $\epsilon = 0.6$, $\tau(\dagger) = |\cos(\dagger)|$, $\Delta a_{ij}(\dagger) = \sin(\dagger)$ and $\Delta b_{ij}(\dagger) = \cos(\dagger)$. Numerical verification confirms that the assumptions of Theorem 3.3 are fulfilled with the parameters. Consequently, the theoretical error bound is determined to be 1.901, establishing that system (2.5) is quasi-synchronized with system (2.4) under the proposed control law.

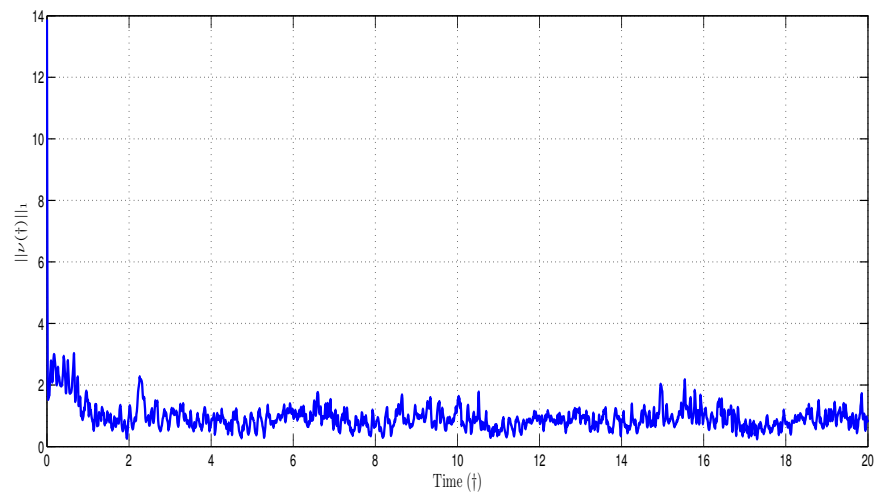


Figure 8. The curve of $||\nu(\dagger)||_1$ under controller (3.11).

The simulation outcomes are shown in Figures 8 and 9. Specifically, the corresponding synchronization errors are presented in Figure 8. Figure 9 displays the sequence of event-triggering instants; the time gaps between consecutive triggers are always positive, thereby confirming that the Zeno phenomenon does not occur under the proposed control mechanism.

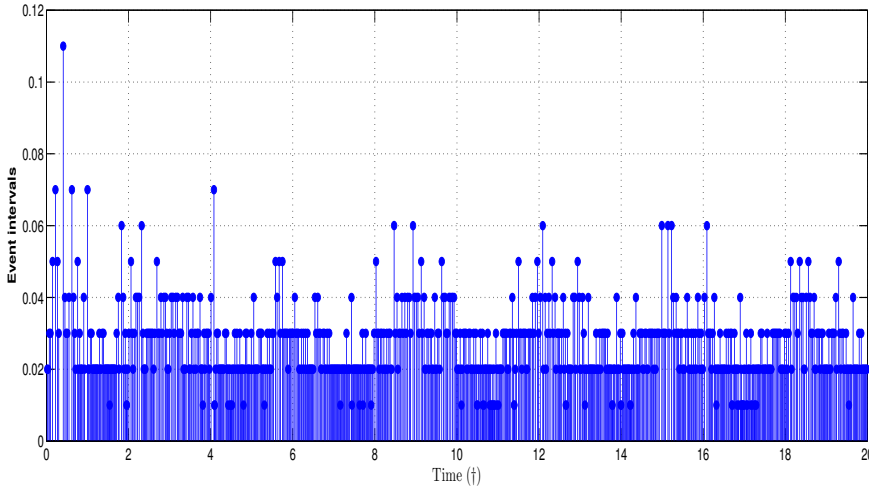


Figure 9. Inter event times.

Remark 4.4. By comparing the simulation results for different fractional orders (e.g., $\epsilon = 0.6$ and $\epsilon = 0.8$), as illustrated in Figures 6 and 8, one can clearly observe the influence of the fractional order on the quasi-synchronization behavior of the neural network with time delays. When $\epsilon = 0.6$, the convergence process is relatively slow, and the quasi-synchronization error requires a longer time to enter a small neighborhood around zero (approximately $t > 3$). In this case, the system exhibits a strong memory effect, and the initial disturbance significantly affects the transient dynamics, leading to noticeable fluctuations in the error trajectory. In contrast, when $\epsilon = 0.8$, the error converges much faster, typically within $t < 1$, to a small bounded neighborhood around zero. These results confirm that increasing the fractional order reduces memory effects and enhances the quasi-synchronization speed of the system.

5 Conclusion

In conclusion, this paper has addressed the synchronization challenges of uncertain FOQVMNNs by accounting for the presence of non-identical connection weights between the drive and response systems. Through the design of a nonlinear adaptive control law and an event-triggered mechanism, we have successfully achieved complete and quasi-synchronization while significantly reducing the communication burden. The direct quaternion-domain analysis proved effective in preserving the system's algebraic integrity and reducing computational overhead. Looking forward, these results open several promising avenues for future research. One such direction involves extending these control strategies to finite-time and fixed-time synchronization frameworks to enhance convergence speed.

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