

Adaptive synchronization of fractional-order discrete-time uncertain neural networks with leakage and discrete delays

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Keywords and phrases: Discrete-time neural networks, Fractional-order, Uncertain, Synchronization, Adaptive controller, Leakage delay, Discrete delay.

This manuscript is an extended and significantly improved version of a full paper entitled "Synchronization of delayed fractional-order discrete-time uncertain genetic regulatory networks" presented at ICRAMI 2025 (Tunisia, November 2025). Compared to the conference version, this paper extends the adaptive control methodology to a different class of systems (fractional-order neural networks with leakage and discrete delays), provides new complete synchronization conditions and includes additional numerical simulations.

The authors would like to thank the reviewers and editor for their constructive comments and valuable suggestions that improved the quality of our paper.

Abstract This paper investigates the complete synchronization (CS) problem for a class of fractional-order discrete-time neural networks subject to leakage, discrete delays, and parametric uncertainties. Fractional-order discrete-time models are capable of capturing long-term memory effects and hereditary dynamics, making them suitable for complex digital network implementations. To address the challenging problem of synchronization under combined uncertainties, an adaptive control strategy is proposed. Several novel sufficient conditions for achieving complete synchronization are derived using fractional calculus theory and Halanay-type inequalities, providing a rigorous theoretical foundation. The effectiveness and robustness of the proposed approach are demonstrated through numerical simulations, which illustrate its capability to synchronize networks with complex dynamics, unknown parameters, and time delays.

1 Introduction

Artificial neural networks (ANNs) have been widely recognized as effective tools for modeling complex nonlinear information-processing systems due to their parallel computation ability, adaptability, and robustness [1]. Inspired by biological neural mechanisms, ANNs have found numerous applications in areas such as secure communication [2], pattern recognition [3], optimization [4] and intelligent control [5]. Neural network models can be described using either continuous-time or discrete-time dynamics, depending on the nature of the system and its implementation. Continuous-time formulations are commonly adopted to represent analog neural circuits and physical processes evolving in real time, whereas discrete-time neural networks are more suitable for digital platforms and computer-based systems, where state variables are updated at discrete sampling instants. As a result, the analysis of dynamical behaviors and control problems for both continuous-time [6, 7, 8] and discrete-time [9, 10, 11] neural networks has attracted considerable research interest.

In recent years, fractional-order neural networks (FONNs) have received increasing attention due to their ability to model long-term memory effects and nonlocal dynamics that are not captured by classical integer-order models. Unlike integer-order neural networks, which rely on local derivatives and whose future behavior depends only on the current state, fractional-order models incorporate the influence of past states through fractional calculus, offering greater flexibility in capturing hereditary properties and complex dynamic behaviors [12]. In the continuous-time framework, fractional-order neural networks have been extensively studied for stability [13, 14], dissipativity [15, 16], and synchronization [17], demonstrating their enhanced capa-

bility in controlling and coordinating neural dynamics. Similarly, in the discrete-time setting, fractional-order difference operators allow the modeling of systems with memory while remaining compatible with digital implementations [18]. Several works have analyzed the stability and convergence of discrete-time FONNs [19, 20, 21], as well as their synchronization behaviors under various activation functions and delayed structures [22, 23, 24]. These results highlight the advantages of fractional-order models in providing a more accurate and versatile representation of neural network dynamics.

In practical neural network models, various factors such as leakage terms, discrete delays, and parametric uncertainties can significantly influence the system dynamics and complicate the analysis of stability and performance. Leakage terms, which represent the decay or dissipation of neuron states, are commonly included to more accurately reflect the intrinsic dynamics of the network [25]. Meanwhile, discrete-time delays naturally arise from finite signal transmission, processing times, or sampling effects in digital implementations, and they may destabilize the system or degrade performance if not properly addressed [26]. Additionally, uncertainties in system parameters, resulting from modeling errors, external disturbances, or environmental variations, pose further challenges for reliable network operation [27]. Consequently, numerous studies have focused on the stability and control of neural networks incorporating these effects, developing criteria that ensure robust behavior under delayed, leaky, and uncertain conditions. These investigations provide a theoretical foundation that is essential for designing effective synchronization strategies for complex neural network systems.

Synchronization is one of the most important phenomena in neural networks, as it reflects the coordinated behavior between the drive and response systems. Achieving complete synchronization ensures that the states of the response network asymptotically follow those of the drive network, which is essential in many applications such as secure communication and signal processing [28, 29]. In the context of fractional-order discrete-time neural networks, several studies have investigated different control strategies to achieve complete synchronization [22, 23, 30]. These works have provided valuable insights into how memory-dependent dynamics, network topology, and activation functions influence synchronization. However, most existing results have limitations that restrict their practical applicability. First, many studies ignore the combined effects of leakage terms, discrete delays, and parametric uncertainties, even though these factors are unavoidable in real-world networks. Second, some approaches assume full knowledge of system parameters, which is often unrealistic in practical scenarios where uncertainties and modeling errors exist. Third, a number of methods focus only on quasi-synchronization rather than complete synchronization, limiting the performance of the network in applications requiring exact state matching.

Recent literature demonstrates rapid methodological evolution in discrete-time fractional-order neural network synchronization. In [22], *Chen et al.* investigate complete synchronization of delayed discrete-time fractional-order competitive neural networks (DDFCNNs) using Caputo fractional difference inequalities and a linear feedback control strategy, providing sufficient conditions for synchronization via discrete-time Halanay-type inequalities. In [31], *Li et al.* investigate synchronization of discrete-time fractional-order quaternion-valued uncertain neural networks (DFQUNNs). Using discrete-time fractional calculus and quaternion properties, they establish Caputo difference inequalities and derive quasi- and complete synchronization criteria under nonlinear and adaptive quaternion-valued controllers, validated through numerical simulations. In [32], *Jebril et al.* investigated finite-time synchronization in a discrete reaction-diffusion FitzHugh–Nagumo model with unknown parameters, using a Lyapunov approach. Similarly, in [33], *Bendib et al.* established a finite-time stability criterion for fractional-order reaction-diffusion systems, providing a novel Lyapunov functional.

Motivated by the above discussion, this paper investigates the problem of adaptive complete synchronization for a class of fractional-order discrete-time neural networks with leakage, discrete delays, and parametric uncertainties. This manuscript is an extended and significantly improved version of a full paper [34] entitled "Synchronization of delayed fractional-order discrete-time uncertain genetic regulatory networks" presented at ICRAMI 2025 (Tunisia, November 2025). Compared to the conference version, this paper extends the adaptive control methodology to a different class of systems (fractional-order neural networks with leakage and discrete delays), provides new complete synchronization conditions and includes additional numerical simulations.

The main contributions of this study can be summarized as follows:

- (i) An adaptive control strategy is proposed to achieve complete synchronization between drive and response networks, explicitly considering leakage terms, discrete delays, and unknown parameters.
- (ii) Sufficient conditions for synchronization are derived using a Lyapunov-based method combined with fractional-order discrete calculus and Halanay-type inequality.
- (iii) A numerical example is provided to illustrate the effectiveness, robustness, and feasibility of the theoretical results.

The remainder of the paper is organized as follows. Section 2 introduces the basic concepts and fundamental lemmas of discrete fractional calculus. The mathematical formulation of the FODTUDNNS and the corresponding synchronization analysis are developed in Section 3. A numerical example illustrating the effectiveness of the proposed results is presented in Section 4.

2 Notation and Preliminaries

Notation : Let $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{R}^n$ denote the set of nonnegative integers, the set of real numbers and the set of n -dimensional real vector respectively. $\varrho(\xi) = \xi - h$, $(h\mathbb{N})_\psi = \{\psi, \psi + h, \psi + 2h, \dots\}$ and $(h\mathbb{N})_\psi^\delta = \{\psi, \psi + h, \psi + 2h, \dots, \delta\}$ where $\delta - \psi = kh$, k is non negative integer and $h > 0$

is the step size. For $u = \{u_1, \dots, u_n\} \in \mathbb{R}^n$, $\|u\| = \left(\sum_{i=1}^n |u_i|^2 \right)^{\frac{1}{2}}$ is the 2-norm of u . $\Gamma(\cdot)$ is the Gamma function and $\lceil \cdot \rceil$ signify the ceiling function.

In this section, several fundamental concepts and preliminary results related to discrete fractional calculus are presented.

Definition 2.1. [35] For $h > 0$, the backward h -difference operator is defined as

$$\nabla_h f(\xi) = \frac{f(\xi) - f(\xi - h)}{h}.$$

Definition 2.2. [36] Let $h > 0$, the γ -th order h -rising function is defined as

$$\xi_h^\gamma = h^\gamma \frac{\Gamma(\frac{\xi}{h} + \gamma)}{\Gamma(\frac{\xi}{h})}, \quad 0_h^\gamma = 0,$$

where $\gamma \in \mathbb{R} \setminus \{\dots, -2, -1\}$ and $\xi \in (h\mathbb{N})_0$.

Definition 2.3. [37] For $\gamma \in \mathbb{R} \setminus \{\dots, -2, -1\}$, the γ -th order fractional difference monomial is defined as

$$\mathcal{H}_\gamma^h(\xi, \psi) = \frac{(\xi - \psi)_h^\gamma}{\Gamma(\gamma + 1)}, \quad \xi \in (h\mathbb{N})_\psi.$$

Definition 2.4. [38, 22] For $h > 0$ and $g : (h\mathbb{N})_\psi \rightarrow \mathbb{R}$, the nabla γ -th order h -sum of $g(\xi)$ is defined as

$$\begin{aligned} {}_\psi \nabla_h^{-\gamma} g(\xi) &= \int_\psi^\xi \mathcal{H}_{\gamma-1}^h(\xi, \varrho(i)) g(i) \nabla_h i \\ &= \frac{h}{\Gamma(\gamma)} \sum_{i=\frac{\psi}{h}+1}^{\frac{\xi}{h}} (\xi - \varrho(ih))_h^{\gamma-1} g(ih), \end{aligned}$$

with $\xi \in (h\mathbb{N})_{\psi+h}$.

Definition 2.5. [39] For $g : (h\mathbb{N})_\psi \rightarrow \mathbb{R}$ and $n - 1 < \gamma < n$, $n \in \mathbb{N}^+$, the γ -th order Caputo h -difference is defined as

$${}_\psi^C \nabla_h^\gamma g(\xi) = {}_\psi \nabla_h^{-(n-\gamma)} \nabla_h^n g(\xi), \quad \xi \in (h\mathbb{N})_{\psi+nh}.$$

Remark 2.6. When $0 < \gamma < 1$, we have

$$\begin{aligned} {}^C_{\psi} \nabla_h^{\gamma} g(\xi) &= {}_{\psi} \nabla_h^{-(1-\gamma)} \nabla_h g(\xi) \\ &= \int_{\psi}^{\xi} \mathcal{H}_{-\gamma}^h(\xi, \varrho(i)) \nabla_h g(i) \nabla_h i \\ &= \frac{h}{\Gamma(1-\gamma)} \sum_{i=\frac{\psi}{h}+1}^{\frac{\xi}{h}} (\xi - \varrho(ih))_h^{\overline{-\gamma}} \nabla_h g(ih), \quad \xi \in (h\mathbb{N})_{\psi+h}. \end{aligned}$$

Lemma 2.7. [39] Let $\psi, \varkappa_j \in (h\mathbb{N})_0$ with $j = 1, \dots, M$, $\varkappa = \max\{\varkappa_1, \dots, \varkappa_M\}$, $v_1 : (h\mathbb{N})_{\psi-\varkappa} \rightarrow \mathbb{R}^+$ is bounded on $(h\mathbb{N})_{\psi-\varkappa}^{\psi}$ and $v_2(\xi)$ is nonnegative function such that $v_2(\psi) = 0$. Suppose that $v_1(\xi)$ and $v_2(\xi)$ can satisfy

$${}^C_{\psi} \nabla_h^{\gamma} (v_1(\xi) + v_2(\xi)) \leq -\lambda v_1(\xi) + \sum_{j=1}^M a_j \sup_{k \in (h\mathbb{N})_{-\varkappa_j}^0} v_1(\xi + k), \quad (2.1)$$

where λ, a_j are positive numbers and $\gamma \in (0, 1)$.

If $\lambda > \sum_{j=1}^M a_j$ and $0 < \lambda < h^{-\gamma}$ then $\lim_{\xi \rightarrow \infty} v_1(\xi) = 0$.

Lemma 2.8. [36] Let $f : (h\mathbb{N})_{\psi} \rightarrow \mathbb{R}$ and $0 < \gamma < 1$, then

$${}^C_{\psi} \nabla_h^{\gamma} f^2(\xi) \leq 2f(\xi) {}^C_{\psi} \nabla_h^{\gamma} f(\xi), \quad \xi \in (h\mathbb{N})_{\psi+h}.$$

3 Main results

In this section, we consider a class of fractional-order discrete-time uncertain networks with leakage and discrete delays, and derive sufficient conditions to ensure complete synchronization. of the proposed model under adaptive controller.

Consider the following FODTUDNNS

$$\left\{ \begin{aligned} {}^C_{\psi} \nabla_h^{\gamma} x_i(\xi) &= -(a_i + \Delta a_i(\xi)) x_i(\xi - \pi_1) + \sum_{j=1}^n (b_{ij} + \Delta b_{ij}(\xi)) f_j(x_j(\xi)) \\ &\quad + \sum_{j=1}^n (c_{ij} + \Delta c_{ij}(\xi)) g_j(x_j(\xi - \pi_2)) + q_i(\xi), \quad \xi \in (h\mathbb{N})_{\psi+h}, \\ x_i(s) &= \varphi_i(s), \quad s \in (h\mathbb{N})_{\psi-\pi}^{\psi}, \end{aligned} \right. \quad (3.1)$$

where $0 < \gamma < 1$, $i = \{1, \dots, n\}$, $j = \{1, \dots, n\}$, $x_i(\xi)$ is the state of i -th neuron at time ξ , $a_i > 0$ represents self feedback connection weight, b_{ij} and c_{ij} are respectively, the connection weight and the discrete delay connection weight, π_1 and π_2 mean separately leakage delay and discrete delay, $\pi = \max\{\pi_1, \pi_2\}$, $f_j(\cdot)$ and $g_j(\cdot)$ are the activation function, and $\Delta a_i(\xi)$, $\Delta b_{ij}(\xi)$ and $\Delta c_{ij}(\xi)$ are parametric uncertainties and $q_i(\xi)$ represent the external input.

Consider (3.1) as the drive system, the controlled response system is described as

$$\left\{ \begin{aligned} {}^C_{\psi} \nabla_h^{\gamma} y_i(\xi) &= -(a_i + \Delta a_i(\xi)) y_i(\xi - \pi_1) + \sum_{j=1}^n (b_{ij} + \Delta b_{ij}(\xi)) f_j(y_j(\xi)) \\ &\quad + \sum_{j=1}^n (c_{ij} + \Delta c_{ij}(\xi)) g_j(y_j(\xi - \pi_2)) + q_i(\xi) + u_i(\xi), \quad \xi \in (h\mathbb{N})_{\psi+h}, \\ y_i(s) &= \bar{\varphi}_i(s), \quad s \in (h\mathbb{N})_{\psi-\pi}^{\psi}, \end{aligned} \right. \quad (3.2)$$

where $u_i(\xi)$ is the controller to be designed later in order to achieve complete synchronization and the remaining symbols are identical to those in system (3.1).

The error between the drive system (3.1) and the response system (3.2) is given as

$$e_i(\xi) = y_i(\xi) - x_i(\xi).$$

Subtracting (3.1) from (3.2), the error system is expressed as

$$\begin{aligned} {}^C\nabla_h^\gamma e_i(\xi) = & -(a_i + \Delta a_i(\xi))e_i(\xi - \pi_1) + \sum_{j=1}^n (b_{ij} + \Delta b_{ij}(\xi))(f_j(y_j(\xi)) - f_j(x_j(\xi))) \\ & + \sum_{j=1}^n (c_{ij} + \Delta c_{ij}(\xi))(g_j(y_j(\xi - \pi_2)) - g_j(x_j(\xi - \pi_2))) + u_i(\xi), \quad \xi \in (h\mathbb{N})_{\psi+h}. \end{aligned} \quad (3.3)$$

To deal with the synchronization problem between (3.1) and (3.2), the following definition and assumptions are introduced.

Definition 3.1. [39] The CS between (3.1) and (3.2) is achieved if $\lim_{\xi \rightarrow \infty} \|E(\xi)\| = 0$, where $E(\xi) = (e_1(\xi), \dots, e_n(\xi))$.

Assumption 3.2. For arbitrary $v_1, v_2 \in \mathbb{R}$ and $j \in \{1, 2, \dots, n\}$, there are positives constants $\mathcal{F}_j$ and $\mathcal{G}_j$ satisfying

$$|f_j(v_1) - f_j(v_2)| \leq \mathcal{F}_j |v_1 - v_2| \text{ and } |g_j(v_1) - g_j(v_2)| \leq \mathcal{G}_j |v_1 - v_2|.$$

Assumption 3.3. For all $i, j \in \{1, 2, \dots, n\}$, there are positives constants $\tilde{a}_i$, $\tilde{b}_{ij}$ and $\tilde{c}_{ij}$ satisfying

$$|\Delta a_i(\xi)| \leq \tilde{a}_i, |\Delta b_{ij}(\xi)| \leq \tilde{b}_{ij}, \text{ and } |\Delta c_{ij}(\xi)| \leq \tilde{c}_{ij}.$$

Remark 3.4. The above assumptions are standard in the literature on dynamic neural networks.

- Assumption 3.2 is satisfied by most activation functions commonly used in practice, such as the sigmoid function, the hyperbolic tangent function, and bounded threshold-type functions.
- Assumption 3.3 represents a conventional way of modeling parameter uncertainties: they are assumed to be bounded but otherwise arbitrary, which accounts for variations caused by measurement errors, environmental fluctuations, or modeling inaccuracies, while preserving the generality of the proposed framework.

Therefore, these assumptions are not restrictive and ensure that the considered model remains broad enough to cover a wide range of practical applications.

To achieve complete synchronization, we design the following adaptive linear controller

$$\begin{cases} u_i(\xi) &= -\mathfrak{R}_i(\xi)e_i(\xi), \\ {}^C\nabla_h^\gamma \mathfrak{R}_i(\xi) &= \rho_i e_i^2(\xi), \end{cases} \quad (3.4)$$

where $0 < \gamma < 1$, $i = \{1, \dots, n\}$ and ρ_i are positives constants.

Theorem 3.5. Under Assumption 3.2, Assumption 3.3 and controller (3.4), systems (3.1) and (3.2) achieve complete synchronization if $0 < \chi < h^{-\gamma}$ and $\sum_{k=1}^2 \theta_k < \chi$, where

$$\begin{aligned} \chi = & \min_{1 \leq i \leq n} \left\{ 2\mathfrak{R}_i(\psi) - a_i - \tilde{a}_i - \sum_{j=1}^n \left[(|b_{ij}| + \tilde{b}_{ij})\mathcal{F}_j + (|b_{ji}| + \tilde{b}_{ji})\mathcal{F}_i + (|c_{ij}| + \tilde{c}_{ij})\mathcal{G}_j \right] \right\}, \\ \theta_1 = & \max_{1 \leq i \leq n} \{a_i + \tilde{a}_i\} \text{ and } \theta_2 = \max_{1 \leq i \leq n} \left\{ \sum_{j=1}^n (|c_{ji}| + \tilde{c}_{ji})\mathcal{G}_i \right\}. \end{aligned}$$

Proof. Select the following Lyapunov function

$$V(\xi) = V_1(\xi) + V_2(\xi)$$

where

$$\begin{cases} V_1(\xi) &= \frac{1}{2} \sum_{i=1}^n e_i^2(\xi) \\ V_2(\xi) &= \frac{1}{2} \sum_{i=1}^n \frac{1}{\rho_i} (\mathfrak{R}_i(\xi) - \mathfrak{R}_i(\psi))^2. \end{cases}$$

Using Lemma 2.8, the γ -order Caputo fractional difference of $V(\xi)$ along system (3.3) is computed as follows

$$\begin{aligned} {}^C_{\psi} \nabla_h^\gamma V(\xi) &\leq \sum_{i=1}^n e_i(\xi) {}^C_{\psi} \nabla_h^\gamma e_i(\xi) + \sum_{i=1}^n \frac{1}{\rho_i} (\mathfrak{R}_i(\xi) - \mathfrak{R}_i(\psi)) {}^C_{\psi} \nabla_h^\gamma (\mathfrak{R}_i(\xi) - \mathfrak{R}_i(\psi)) \\ &= \sum_{i=1}^n e_i(\xi) \left\{ -(a_i + \Delta a_i(\xi)) e_i(\xi - \pi_1) + \sum_{j=1}^n (b_{ij} + \Delta b_{ij}(\xi)) (f_j(y_j(\xi)) - f_j(x_j(\xi))) \right. \\ &\quad \left. + \sum_{j=1}^n (c_{ij} + \Delta c_{ij}(\xi)) (g_j(y_j(\xi - \pi_2)) - g_j(x_j(\xi - \pi_2))) + u_i(\xi) \right\} \\ &\quad + \sum_{i=1}^n (\mathfrak{R}_i(\xi) - \mathfrak{R}_i(\psi)) e_i^2(\xi) \\ &= - \sum_{i=1}^n (a_i + \Delta a_i(\xi)) e_i(\xi) e_i(\xi - \pi_1) - \sum_{i=1}^n \mathfrak{R}_i(\psi) e_i^2(\xi) \\ &\quad + \sum_{i=1}^n \sum_{j=1}^n (b_{ij} + \Delta b_{ij}(\xi)) e_i(\xi) (f_j(y_j(\xi)) - f_j(x_j(\xi))) \\ &\quad + \sum_{i=1}^n \sum_{j=1}^n (c_{ij} + \Delta c_{ij}(\xi)) e_i(\xi) (g_j(y_j(\xi - \pi_2)) - g_j(x_j(\xi - \pi_2))). \end{aligned} \tag{3.5}$$

Employing Assumption 3.3 and using the fact that $2|\mu_1||\mu_2| \leq \mu_1^2 + \mu_2^2$, we obtain

$$\begin{aligned} &- \sum_{i=1}^n (a_i + \Delta a_i(\xi)) e_i(\xi) e_i(\xi - \pi_1) \\ &\leq \sum_{i=1}^n |a_i + \Delta a_i(\xi)| |e_i(\xi)| |e_i(\xi - \pi_1)| \\ &\leq \sum_{i=1}^n (a_i + \tilde{a}_i) |e_i(\xi)| |e_i(\xi - \pi_1)| \\ &\leq \frac{1}{2} \sum_{i=1}^n (a_i + \tilde{a}_i) (e_i^2(\xi) + e_i^2(\xi - \pi_1)) \\ &= \frac{1}{2} \sum_{i=1}^n (a_i + \tilde{a}_i) e_i^2(\xi) + \frac{1}{2} \sum_{i=1}^n (a_i + \tilde{a}_i) e_i^2(\xi - \pi_1). \end{aligned} \tag{3.6}$$

Using Assumption 3.2, Assumption 3.3 and using the fact that $2|\mu_1||\mu_2| \leq \mu_1^2 + \mu_2^2$, we can get

$$\begin{aligned}
& \sum_{i=1}^n \sum_{j=1}^n (b_{ij} + \Delta b_{ij}(\xi)) e_i(\xi) (f_j(y_j(\xi)) - f_j(x_j(\xi))) \\
& \leq \sum_{i=1}^n \sum_{j=1}^n |b_{ij} + \Delta b_{ij}(\xi)| |e_i(\xi)| |f_j(y_j(\xi)) - f_j(x_j(\xi))| \\
& \leq \sum_{i=1}^n \sum_{j=1}^n (|b_{ij}| + \tilde{b}_{ij}) \mathcal{F}_j |e_i(\xi)| |e_j(\xi)| \\
& \leq \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n (|b_{ij}| + \tilde{b}_{ij}) \mathcal{F}_j (e_i^2(\xi) + e_j^2(\xi)) \\
& = \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n (|b_{ij}| + \tilde{b}_{ij}) \mathcal{F}_j e_i^2(\xi) + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n (|b_{ji}| + \tilde{b}_{ji}) \mathcal{F}_i e_i^2(\xi),
\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}
& \sum_{i=1}^n \sum_{j=1}^n (c_{ij} + \Delta c_{ij}(\xi)) e_i(\xi) (g_j(y_j(\xi - \pi_2)) - g_j(x_j(\xi - \pi_2))) \\
& \leq \sum_{i=1}^n \sum_{j=1}^n |c_{ij} + \Delta c_{ij}(\xi)| |e_i(\xi)| |g_j(y_j(\xi - \pi_2)) - g_j(x_j(\xi - \pi_2))| \\
& \leq \sum_{i=1}^n \sum_{j=1}^n (|c_{ij}| + \tilde{c}_{ij}) \mathcal{G}_j |e_i(\xi)| |e_j(\xi - \pi_2)| \\
& \leq \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n (|c_{ij}| + \tilde{c}_{ij}) \mathcal{G}_j (e_i^2(\xi) + e_j^2(\xi - \pi_2)) \\
& = \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n (|c_{ij}| + \tilde{c}_{ij}) \mathcal{G}_j e_i^2(\xi) + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n (|c_{ji}| + \tilde{c}_{ji}) \mathcal{G}_i e_i^2(\xi - \pi_2).
\end{aligned} \tag{3.8}$$

Putting (3.6)-(3.8) into (3.5), we can get

$$\begin{aligned}
C_\psi \nabla_h^\gamma V(\xi) & \leq \frac{1}{2} \sum_{i=1}^n (a_i + \tilde{a}_i) e_i^2(\xi) + \frac{1}{2} \sum_{i=1}^n (a_i + \tilde{a}_i) e_i^2(\xi - \pi_1) - \sum_{i=1}^n \Re_i(\psi) e_i^2(\xi) \\
& + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n (|b_{ij}| + \tilde{b}_{ij}) \mathcal{F}_j e_i^2(\xi) + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n (|b_{ji}| + \tilde{b}_{ji}) \mathcal{F}_i e_i^2(\xi) \\
& + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n (|c_{ij}| + \tilde{c}_{ij}) \mathcal{G}_j e_i^2(\xi) + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n (|c_{ji}| + \tilde{c}_{ji}) \mathcal{G}_i e_i^2(\xi - \pi_2) \\
& = -\frac{1}{2} \sum_{i=1}^n \{2\Re_i(\psi) - a_i - \tilde{a}_i - \sum_{j=1}^n [(|b_{ij}| + \tilde{b}_{ij}) \mathcal{F}_j + (|b_{ji}| + \tilde{b}_{ji}) \mathcal{F}_i \\
& + (|c_{ij}| + \tilde{c}_{ij}) \mathcal{G}_j]\} e_i^2(\xi) + \frac{1}{2} \sum_{i=1}^n \{a_i + \tilde{a}_i\} e_i^2(\xi - \pi_1) \\
& + \frac{1}{2} \sum_{i=1}^n \left\{ \sum_{j=1}^n (|c_{ji}| + \tilde{c}_{ji}) \mathcal{G}_i \right\} e_i^2(\xi - \pi_2)
\end{aligned}$$

$$\begin{aligned}
&\leq -\chi V_1(\xi) + \theta_1 V_1(\xi - \pi_1) + \theta_2 V_1(\xi - \pi_2) \\
&\leq -\chi V_1(\xi) + \theta_1 \sup_{s \in (h\mathbb{N})^0_{-\pi_1}} V_1(\xi + s) + \theta_2 \sup_{s \in (h\mathbb{N})^0_{-\pi_2}} V_1(\xi + s) \\
&= -\chi V_1(\xi) + \sum_{k=1}^2 \theta_k \sup_{s \in (h\mathbb{N})^0_{-\pi_k}} V_1(\xi + s).
\end{aligned} \tag{3.9}$$

According to the definition of $V_2(\xi)$, $V_2(\psi) = 0$. Under Lemma 2.7, we get $\lim_{\xi \rightarrow \infty} V_1(\xi) = 0$. Finally, from Definition 3.1, we conclude that (3.1) and (3.2) reach complete synchronization. $\square$

Remark 3.6. The main advantage of the adaptive controller (3.4) is that it dynamically adjusts the control gains $\mathfrak{R}_i(\xi)$ according to the fractional-order update law ${}^C\nabla_h^\gamma \mathfrak{R}_i(\xi) = \rho_i e_i^2(\xi)$, without requiring prior knowledge of the system parameters or the exact bounds of uncertainties. Unlike constant-gain linear feedback, the adaptive gains evolve automatically based on the synchronization error, which eliminates manual tuning and provides greater robustness against time-varying parametric uncertainties. This self-adaptation is the key reason for employing this type of controller in the presence of unknown parameters.

Remark 3.7. In [22], *Chen et al.* investigated the complete synchronization of delayed discrete-time fractional-order competitive neural networks using linear feedback control. In [23], *Zhang et al.* addressed the quasi-projective and complete synchronization of discrete-time fractional-order delayed neural networks. Moreover, in [31], *Li et al.* studied synchronization of discrete-time fractional-order quaternion-valued uncertain neural networks. The present paper simultaneously accounts for leakage delays, discrete delays, and parametric uncertainties within a unified adaptive control framework. Furthermore, our approach guarantees complete synchronization under only bounded uncertainties, without needing exact parameter knowledge, and the adaptive law automatically adjusts the control gains. These features constitute the main novelty and advantages of our work over existing results.

Remark 3.8. This study addresses three main challenges. (1) The discrete-time fractional-order dynamics prevent the direct use of integer-order methods; we overcome this by constructing Lyapunov functions adapted to fractional differences (Lemma 2.8). (2) The leakage and discrete delays introduce memory effects that hinder synchronization; we handle them via a discrete-time Halanay-type inequality (Lemma 2.7). (3) Unknown parametric uncertainties are dealt with by an adaptive controller whose Lyapunov analysis uses only the known bounds of the uncertainties. Theorem 3.5 synthesizes these elements to achieve complete synchronization.

Let control gain $\mathfrak{R}_i(\xi)$ become constants $\mathfrak{R}_i > 0$, then the controller (3.4) can degenerate into the following linear controller

$$u_i(\xi) = -\mathfrak{R}_i e_i(\xi). \tag{3.10}$$

Corollary 3.9. Under Assumption 3.2, Assumption 3.3 and controller (3.10), systems (3.1) and (3.2) achieve complete synchronization if $0 < \chi < h^{-\gamma}$ and $\sum_{k=1}^2 \theta_k < \chi$, where $\chi = \min_{1 \leq i \leq n} \left\{ 2\mathfrak{R}_i - a_i - \tilde{a}_i - \sum_{j=1}^n \left[(|b_{ij}| + \tilde{b}_{ij})\mathcal{F}_j + (|b_{ji}| + \tilde{b}_{ji})\mathcal{F}_i + (|c_{ij}| + \tilde{c}_{ij})\mathcal{G}_j \right] \right\}$, $\theta_1 = \max_{1 \leq i \leq n} \{a_i + \tilde{a}_i\}$ and $\theta_2 = \max_{1 \leq i \leq n} \left\{ \sum_{j=1}^n (|c_{ji}| + \tilde{c}_{ji})\mathcal{G}_i \right\}$.

Proof. We construct the following Lyapunov function

$$V(\xi) = \frac{1}{2} \sum_{i=1}^n e_i^2(\xi)$$

and the proof of reasoning is similar to that of Theorem 3.5. $\square$

Next, we consider the case where parameter uncertainties are neglected, then systems (3.2) and (3.1) degenerate into

$$\begin{aligned} {}^C\nabla_h^\gamma x_i(\xi) &= -a_i x_i(\xi - \pi_1) + \sum_{j=1}^n b_{ij} f_j(x_j(\xi)) + \sum_{j=1}^n c_{ij} g_j(x_j(\xi - \pi_2)) \\ &\quad + q_i(\xi), \quad \xi \in (h\mathbb{N})_{\psi+h}, \end{aligned} \quad (3.11)$$

and

$$\begin{aligned} {}^C\nabla_h^\gamma y_i(\xi) &= -a_i y_i(\xi - \pi_1) + \sum_{j=1}^n b_{ij} f_j(y_j(\xi)) + \sum_{j=1}^n c_{ij} g_j(y_j(\xi - \pi_2)) \\ &\quad + q_i(\xi) + u_i(\xi), \quad \xi \in (h\mathbb{N})_{\psi+h}. \end{aligned} \quad (3.12)$$

Corollary 3.10. *Under Assumption 3.2 and controller (3.4), systems (3.11) and (3.12) achieve complete synchronization if $0 < \chi < h^{-\gamma}$ and $\sum_{k=1}^2 \theta_k < \chi$, where $\chi = \min_{1 \leq i \leq n} \left\{ 2\Re_i(\psi) - a_i - \sum_{j=1}^n \left(|b_{ij}| \mathcal{F}_j + |b_{ji}| \mathcal{F}_i + |c_{ij}| \mathcal{G}_j \right) \right\}$, $\theta_1 = \max_{1 \leq i \leq n} \{a_i\}$ and $\theta_2 = \max_{1 \leq i \leq n} \left\{ \sum_{j=1}^n |c_{ji}| \mathcal{G}_i \right\}$.*

Corollary 3.11. *Under Assumption 3.2 and controller (3.10), systems (3.11) and (3.12) can achieve complete synchronization.*

4 Numerical simulations

To verify the theoretical findings, a numerical example is introduced in this section.

Consider the following FODTUDNNS

$$\begin{aligned} {}^C\nabla_h^\gamma x_i(\xi) &= -(a_i + \Delta a_i(\xi))x_i(\xi - \pi_1) + \sum_{j=1}^n (b_{ij} + \Delta b_{ij}(\xi))f_j(x_j(\xi)) \\ &\quad + \sum_{j=1}^n (c_{ij} + \Delta c_{ij}(\xi))g_j(x_j(\xi - \pi_2)) + q_i(\xi), \end{aligned} \quad (4.1)$$

where $\psi = 0$, $\gamma = 0.96$, $h = 0.1$, $n = 2$, $\pi_1 = 0.1$, $\pi_2 = 0.3$, $a_1 = 0.3$, $a_2 = 0.3$, $b_{11} = 0.54$, $b_{12} = 0.16$, $b_{21} = -0.36$, $b_{22} = 0.32$, $c_{11} = 0.6$, $c_{12} = -0.35$, $c_{21} = 0.4$, $c_{22} = -1.21$, $f_j(x_j(\xi)) = \tanh(x_j(\xi))$, $g_j(x_j(\xi - \pi_2)) = \frac{1}{1 + e^{x_j(\xi - \pi_2)}}$, $q_1(\xi) = q_2(\xi) = \sin(2\xi)$,

$\Delta A(\xi) = -\sin(\xi) \begin{pmatrix} 0.1 & 0 \\ 0 & 0.14 \end{pmatrix}$, $\Delta B(\xi) = -\cos(\xi) \begin{pmatrix} 0.14 & -0.13 \\ -0.17 & 0.1 \end{pmatrix}$ and $\Delta C(\xi) = -\sin(\xi) \begin{pmatrix} -0.29 & 0.1 \\ -0.3 & 0.1 \end{pmatrix}$. It can be checked that Assumption 3.2 holds with $\mathcal{F}_j = 1$ and

$\mathcal{G}_j = 0.25$, and Assumption 3.3 holds with $\tilde{a}_1 = 0.1$, $\tilde{a}_2 = 0.14$, $\tilde{b}_{11} = 0.14$, $\tilde{b}_{12} = 0.13$, $\tilde{b}_{21} = 0.17$, $\tilde{b}_{22} = 0.1$, $\tilde{c}_{11} = 0.29$, $\tilde{c}_{12} = 0.1$, $\tilde{c}_{21} = 0.3$ and $\tilde{c}_{22} = 0.1$.

Figure 1 displays the dynamics of $x_1(\xi)$ and $x_2(\xi)$ with initial conditions $x_i(s) = (-2.9, -1.3)^T$, $s \in (h\mathbb{N})_{-0.3}^0$, and Figure 2 illustrates their chaotic trajectories with the above conditions. Note that the chaotic behavior observed in Figure 2 is intrinsic to the neural network dynamics and is not merely a forced response to the sinusoidal inputs $q_1(\xi)$ and $q_2(\xi)$; indeed, even with $q_1(\xi) = q_2(\xi) = 0$ the system remains chaotic.

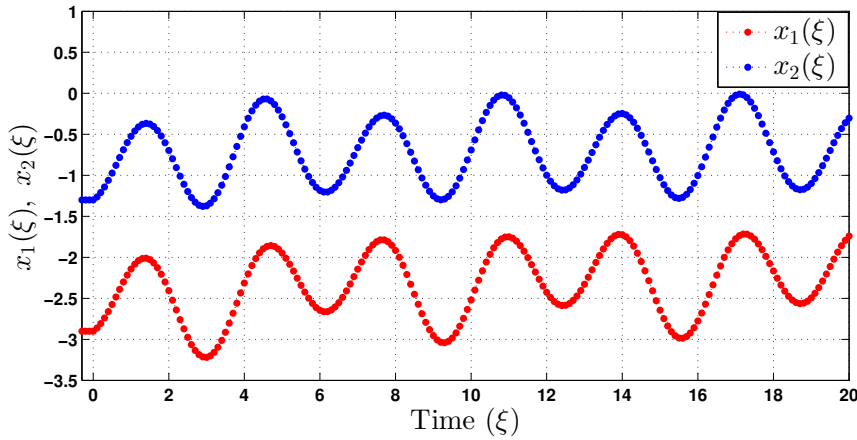


Figure 1. Dynamic behavior of $x_1(\xi)$ and $x_2(\xi)$.

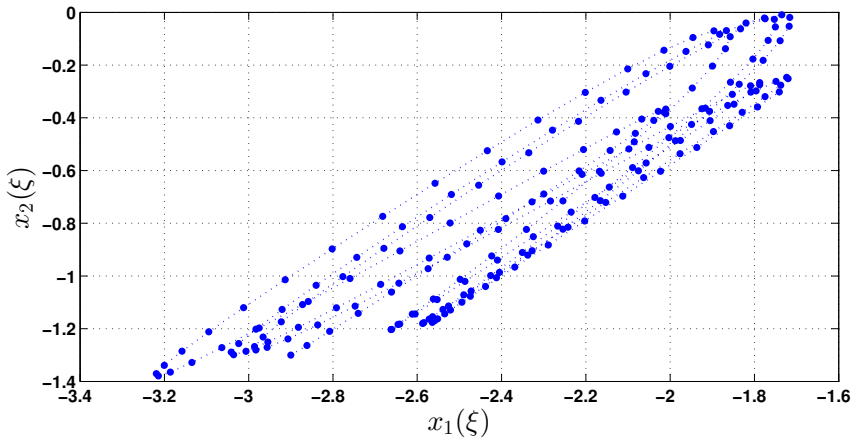


Figure 2. Chaotic trajectory of system (4.1).

The response system is given as

$$\begin{aligned} {}^C_{\psi} \nabla_h^{\gamma} y_i(\xi) = & -(a_i + \Delta a_i(\xi)) y_i(\xi - \pi_1) + \sum_{j=1}^n (b_{ij} + \Delta b_{ij}(\xi)) f_j(y_j(\xi)) \\ & + \sum_{j=1}^n (c_{ij} + \Delta c_{ij}(\xi)) g_j(y_j(\xi - \pi_2)) + q_i(\xi) + u_i(\xi), \end{aligned} \quad (4.2)$$

where all coefficients are the same as those of (4.1) and the design of controller $u_i(\xi)$ is consistent with (3.4).

First, let $\mathfrak{R}_1(\xi) = \mathfrak{R}_2(\xi) = 0$, then system (4.2) is not controlled. Figure 3 displays the dynamics of $y_1(\xi)$ and $y_2(\xi)$ with initial conditions $y_i(s) = (1.6, -2.72)^T$, $s \in (h\mathbb{N})_{-0.3}^0$.

In Figure 4, we can see that the system (4.1) and (4.2) cannot realize complete synchronization.

Now, for the controller (3.4), select $\mathfrak{R}_1(0) = 4.1$, $\mathfrak{R}_2(0) = 4.3$, $\rho_1 = 1.23$ and $\rho_2 = 1.1$. Throughout calculations, we obtain $\chi = 5.29$, $\sum_{k=1}^2 \theta_k = 0.88$. It's clear that $0 < \chi < h^{\gamma} = 9.12$

and $\sum_{k=1}^2 \theta_k < \chi$ which satisfies all conditions of Theorem 3.5.

On the basis of Theorem 3.5, FODTUDNNS (4.1) and (4.2) can achieve complete synchronization under adaptive controller (3.4), as illustrated in Figure 5.

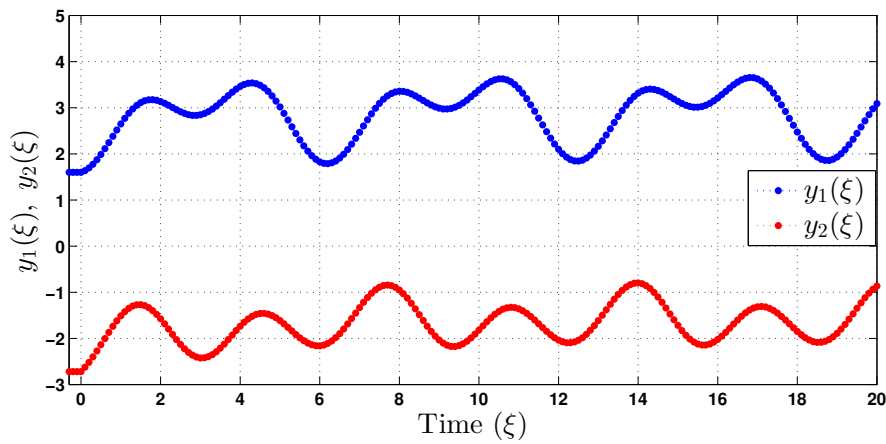


Figure 3. Dynamic behavior of $y_1(\xi)$ and $y_2(\xi)$.

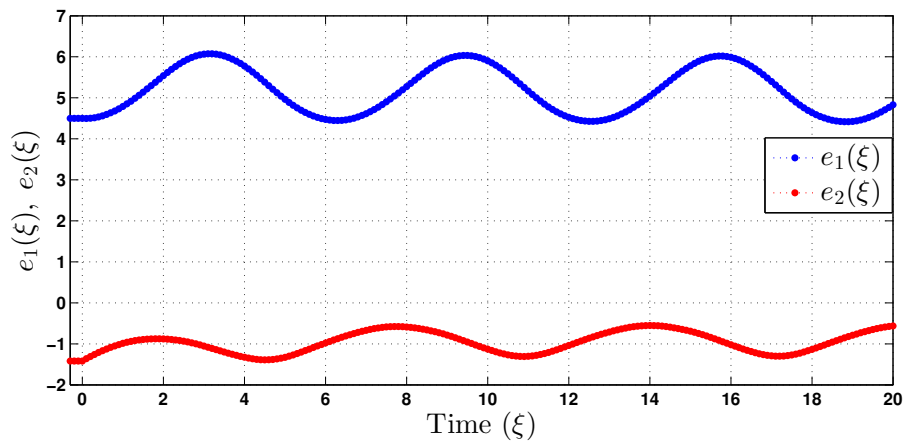


Figure 4. Trajectories of $e_1(\xi)$ and $e_2(\xi)$ without controller.

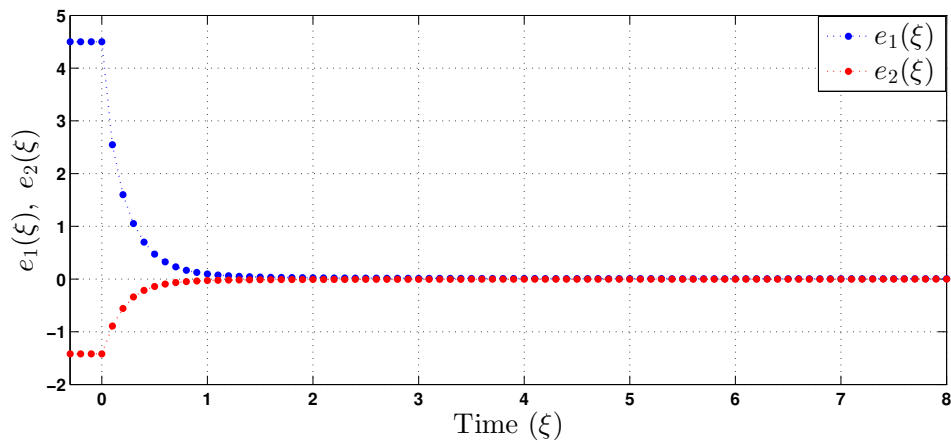


Figure 5. Trajectories of $e_1(\xi)$ and $e_2(\xi)$ under controller (3.4).

The evolution of adaptive gains $\mathfrak{R}_1(\xi)$ and $\mathfrak{R}_2(\xi)$ are illustrated in Figure 6.

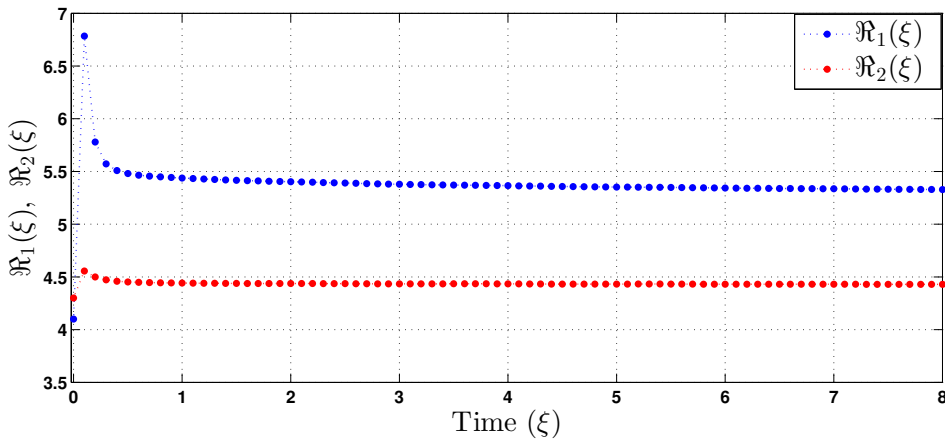


Figure 6. Trajectories of adaptive gains $\mathfrak{R}_1(\xi)$ and $\mathfrak{R}_2(\xi)$.

These numerical results confirm the theoretical findings of Theorem 3.5: under the adaptive controller (3.4), the synchronization errors converge to zero despite the leakage delay, discrete delay, and parametric uncertainties. The uncontrolled case (Figure 4) fails to synchronize, illustrating the necessity of the proposed control strategy.

Remark 4.1. The parameters in the numerical simulations are chosen to satisfy the hypotheses and sufficient conditions of Theorem 3.5. Specifically, the fractional order $\gamma = 0.96$ and step size $h = 0.1$ satisfy $0 < \gamma < 1$ and $h > 0$. The leakage delay $\pi_1 = 0.1$ and discrete delay $\pi_2 = 0.3$ are positive multiples of h meeting the definitions. The initial adaptive gains $\mathfrak{R}_1(0) = 4.1$ and $\mathfrak{R}_2(0) = 4.3$ are selected to fulfill the synchronization condition $\chi > \sum_{k=1}^2 \theta_k$ required by Theorem 3.5. All other parameters (parametric uncertainties, activation functions) satisfy Assumptions 3.2 and 3.3.

Remark 4.2. It is important to point out that the proposed adaptive controller has a theoretical limitation. The mathematical necessity forces the initial condition of the adaptive gain $\mathfrak{R}_i(\psi)$ to be the constant used in calculating χ (where $\mathfrak{R}_1(0) = 4.1$ and $\mathfrak{R}_2(0) = 4.3$ are selected to obtain $\chi = 5.29$). Consequently, the controller does not enjoy full adaptability in the sense that the gains cannot start from arbitrarily small or zero values; they must be preselected to satisfy the synchronization condition. This restriction stems directly from the use of the discrete-time Halanay-type inequality (Lemma 2.7) in the Lyapunov analysis.

5 Conclusions

In this paper, we have investigated the problem of complete synchronization for a class of fractional-order discrete-time neural networks with leakage, discrete delays, and parametric uncertainties. An adaptive control strategy has been proposed, capable of handling memory-dependent dynamics, time delays, and unknown parameters. Rigorous theoretical analysis has been conducted using Lyapunov-based techniques combined with discrete-time fractional-order difference inequalities, and sufficient conditions for synchronization have been established. Numerical simulations have been provided to illustrate the effectiveness and robustness of the proposed method, demonstrating its ability to achieve synchronization under various challenging scenarios. The results show that the approach can successfully coordinate networks even in the presence of leakage, delays, and uncertainties, highlighting its practical applicability for digital implementations and real-world systems.

Future research may extend these results to more general network topologies such as complex networks and coupled systems, to other control objectives like quasi-synchronization and finite-time synchronization, and to practical applications including secure communication and image encryption. Stochastic disturbances, higher-order fractional systems, and event-triggered control are also promising directions.

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