

Adaptive synchronization of fractional-order neural networks with delays

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Abstract. This paper investigates the quasi-projective synchronization problem for a class of fractional-order cellular neural networks (FOCNNs). The proposed model incorporates complex dynamical features, including Caputo fractional derivatives, leakage delays, and time-varying delays. To address the challenges posed by discontinuous activation functions, which are often overlooked in existing literature, we employ the framework of nonsmooth analysis, utilizing Filippov solutions and differential inclusions. The primary technical contribution of this work is the design of a novel adaptive control scheme featuring update laws specifically formulated in the fractional domain. This approach allows for the effective management of uncertain parameters within the synchronization process. By constructing a specialized fractional Lyapunov–Krasovskii functional, we derive sufficient, analytically verifiable criteria to guarantee that the synchronization error converges to a predefined bound. This theoretical framework extends prior results on continuous fractional-order systems to a more generalized non-smooth context. Finally, numerical simulations are presented to validate the effectiveness of the adaptive controller and to illustrate how the fractional order influences the transient behavior and convergence of the network.

1 Introduction

As a development beyond traditional cellular neural networks (CNNs), FOCNNs have been introduced to better handle uncertainties and imprecision in neural network modeling through differential equations, as highlighted in [1]. FOCNNs are preferred over classical CNNs due to their enhanced adaptability in a range of image-processing applications.

In neural network implementations, time delays are of critical importance, as the finite switching speeds of amplifiers can induce instability in dynamic systems. Despite their importance, the qualitative effects of time delays on FOCNNs have not been extensively studied, which this work aims to address [3, 2].

Fractional-order systems have attracted considerable attention due to their unique features of infinite memory and hereditary properties. These systems find applications across fields, including electronics, engineering, and mechanics. Considerable research has been conducted on integer-order and fractional-order neural networks (FONNs), employing methods such as the Laplace transform and linear stability analysis. Nevertheless, these methods face limitations when applied to nonlinear FONNs ([4, 5]). The non-integer-order Lyapunov method was employed to analyze global stability in the sense of Mittag-Leffler to tackle these challenges [6]. For instance, stability conditions were initially derived using one-norm and infinity-norm Lyapunov functions, but these results were limited to individual matrix elements rather than entire matrices. To overcome this limitation, a two-norm Lyapunov function approach was proposed, providing improved results through linear matrix inequality (LMI) conditions [7].

Synchronization refers to the consistent behavior of chaotic systems over time, a phenomenon that has fascinated researchers due to its resemblance to natural clustering behaviors. Differ-

ent forms of synchronization have been identified in previous studies ([8], [9], [10]). Among these, quasi-synchronization is notable for its rather wide applicability. A particular form, quasi-projective synchronization, is characterized by a projection factor that can improve communication speed, making it a prominent subject of academic research with numerous significant results published ([11, 12]). Furthermore, while projective synchronization is a well-known concept, achieving quasi-projective synchronization which allows for faster communication through scalable projection factors in the presence of multiple delays and nonsmooth activations remains an open challenge.

Various controller schemes have been developed to achieve synchronization, including feedback, sliding mode, and adaptive controllers. Adaptive control schemes are often considered the most effective due to their ability to manage systems with uncertain parameters and disturbances. This study adopts an adaptive control approach. For further reference on synchronization issues, recent works are compiled in [13]. These studies demonstrate that FOCNNs can be designed and analyzed similarly to integer-order networks using Lyapunov stability theory. This paper bridges these gaps by proposing a robust theoretical framework for the quasi-projective synchronization of FOCNNs. By combining Caputo fractional calculus with Filippov's theory of differential inclusions and a novel adaptive control scheme, we derive new sufficient conditions to ensure the synchronization error converges to a predefined bounded set, even under parameter uncertainty.

While recent advancements have addressed various synchronization paradigms in fractional-order systems, research concerning the quasi-projective synchronization of delayed FOCNNs remains significantly underdeveloped. This investigation is specifically motivated by the critical need to account for the simultaneous impact of leakage delays, time-varying delays, and discontinuous activation functions complexities that are frequently overlooked in existing literature. By focusing on a generalized class of delayed FOCNNs, this work seeks to extend both the theoretical boundaries and the practical applicability of these models. To address the structural limitations of previous frameworks and provide a more rigorous dynamical analysis, the primary contributions of this study are outlined as follows

- (1) We propose a comprehensive FOCNN model that simultaneously incorporates leakage delays and time-varying delays under a fractional-order operator, offering a more general framework than studies focusing on a single delay type.
- (2) By employing Filippov solutions and nonsmooth analysis, this work successfully relaxes the standard Lipschitz continuity requirements, allowing the model to handle neural networks with discontinuous or switching activation functions.
- (3) We design a novel adaptive control law where the update rules are formulated directly in the fractional domain, distinguishing it from traditional integer-order adaptive schemes and providing better robustness against parameter variations .
- (4) Utilizing a specialized fractional Lyapunov-Krasovskii functional, we derive algebraic synchronization conditions that are easier to verify than complex linear matrix inequalities (LMIs) often found in literature.
- (5) The proposed framework provides a unified perspective that encompasses integer-order neural networks as special cases ($\alpha = 1$), thereby extending the applicability of our findings across different classes of dynamical systems.

The paper is structured as follows: Section 2 presents the model and fundamental concepts. In Section 3, we introduce effective criteria for controlling FOCNNs. To validate the theoretical findings, numerical simulations are presented, in Section 4. Section 5 then offers a concise discussion.

2 Preliminaries

2.1 Notations

- All fractional-order derivatives in this study are defined using the Caputo fractional differential operator ${}^C D_t^\alpha$.
- The symbol $\mathbb{R}$ stands for the domain of real numbers.

- $\mathbb{R}^n$ represents the n -dimensional real vector space with Euclidean norm.
- For $x \in \mathbb{R}^n$, we have $\|x\|_1 = \sum_{i=1}^n |x_i|$.
- Let $\text{sign}(\cdot)$ denote the signum function.

2.2 Fundamentals of fractional calculus

The calculus of fractional order generalizes integration and differentiation to operators of non-integer order. To outline the model, we first review definitions relevant to fractional-order calculus and subsequently introduce several useful lemmas.

Definition 2.1. [15] The integral of fractional order α for an integrable function $f(t) : [0, +\infty) \rightarrow \mathbb{R}$ is expressed by the following expression:

$${}_t^C I_t^\alpha f(t) = {}_t^C D_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{1}{(t-\tau)^{1-\alpha}} f(\tau) d\tau$$

where $\alpha > 0$ and $\Gamma(\cdot)$ denote the Gamma function, which is defined by

$$\Gamma(z) = \int_0^{+\infty} e^{-t} t^{z-1} dt$$

with $\text{Re}(z) > 0$ is the real part of z .

Definition 2.2. [15] The Caputo derivative of fractional-order α for a function $f(t) \in \mathcal{C}^1([0, +\infty), \mathbb{R})$ is defined as

$${}_t^C D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(\tau)}{(t-\tau)^\alpha} d\tau$$

where $t \geq 0$ and $0 < \alpha < 1$. The following are some properties of the Caputo fractional-order derivative (refer to [15]):

Lemma 2.3. [11] Let $V : [0, \infty) \rightarrow \mathbb{R}$ be a differentiable function such that

$${}_t^C D_t^\alpha V(t) \leq -\sigma_1 V(t) + \sigma_2, \quad t \geq 0,$$

where $\alpha \in (0, 1)$, $\sigma_1 > 0$ and $\sigma_2 \geq 0$. Then, we have

$$V(t) \leq V(0) - \frac{\sigma_2}{\sigma_1} E_\alpha(-\sigma_1 t^\alpha) + \frac{\sigma_2}{\sigma_1}, \quad t \geq 0.$$

Lemma 2.4. [16] Let $z(t) \in \mathcal{C}^1(\mathbb{R}^+, \mathbb{R})$. For any $0 < \alpha \leq 1$, the inequality below holds true:

$${}_t^C D_t^\alpha |z(t)| \leq \text{sign}(z(t)) {}_t^C D_t^\alpha z(t).$$

3 System description and problem formulation

3.1 System Description

Now, let us consider the FOCNNs presented below:

$${}_t^C D_t^\alpha z_i(t) = -d_i z_i(t - \delta) + \sum_{e=1}^n a_{ie} f_e(z_e(t)) + \sum_{e=1}^n b_{ie} f_e(z_e(t - \tau(t))) + I_i(t), \quad (3.1)$$

where $i \in \{1, 2, \dots, n\}$, ${}_t^C D_t^\alpha z_i(t)$ represents the Caputo fractional derivative of order α for $z_i(t)$, with $z_i(t) \in \mathbb{R}$ denoting the state variable. In this context, $d_i > 0$ signifies self-inhibition, and $a_{ie}, b_{ie} \in \mathbb{R}$ are the connection weights. $I_i(\cdot)$ represents the input together with the bias of the i -th neuron. We denote the activation function by $f_e(\cdot)$, $\tau(t)$ denotes a time-varying delay constrained by $0 \leq \tau(t) \leq \tilde{\tau}$ where τ is positive constant and $\delta > 0$ is the leakage delay. Assume that system (3.1) has an initial condition $z_i(s) = \phi_i(s)$, for $s \in [-\tilde{\tau}, 0]$, $\phi_i(s) \in \mathbb{C}([-\tilde{\tau}, 0], \mathbb{R})$.

By treating system (3.1) as the driving system, the corresponding response system is formulated as follows:

$$\begin{aligned} {}^C_{t_0}D_t^\alpha w_i(t) &= -d_i w_i(t - \delta) + \sum_{e=1}^n a_{ie} f_e(w_e(t)) + \sum_{e=1}^n b_{ie} f_e(w_e(t - \tau(t))) + I_i(t) \\ &+ u_i(t), \end{aligned} \quad (3.2)$$

where the expression ${}^C_{t_0}D_t^\alpha w_i(t)$ represents the fractional Caputo derivative of order α applied to $w_i(t)$. The variable $w_i(t) \in \mathbb{R}$ denotes the state variable of the system (3.2), while $u_i(t) \in \mathbb{R}$ represents the controller. We assume that the system (3.2) has an initial condition given by $w_i(s) = \Phi_i(s)$. To establish the stability properties of the proposed nonlinear FONNs, it is essential to utilize the following assumptions that confirm the boundedness of the solutions.

- A_1 : [9] The discontinuous activation function $f_e(\cdot) : \mathbb{R} \rightarrow \mathbb{R}$ is bounded, satisfying $|f_e(\cdot)| \leq \mathcal{F}_e$. It is continuous except for a finite number of jump discontinuities γ_m within any bounded interval. Furthermore, for each discontinuity γ_m , both the left limit $f_e(\gamma_m^-)$ and the right limit $f_e(\gamma_m^+)$ exist.
- A_2 : In the context of FOCNNs, the discontinuous activation function satisfies the linear growth conditions. Specifically, for $\alpha_e(t) \in \overline{\text{co}}[f_e(z_e(t))]$ and $\alpha_e^*(t) \in \overline{\text{co}}[f_e(w_e(t))]$, there are positive constants χ_e and χ_e^* such that

$$\sup |\alpha_e(t) - \alpha_e^*(t)| \leq \chi_e |z_e(t) - w_e(t)| + \chi_e^*$$

where

$$\overline{\text{co}}[f_e(\cdot)] = [\min\{f_e^-(\cdot), f_e^+(\cdot)\}, \max\{f_e^-(\cdot), f_e^+(\cdot)\}].$$

- A_3 : For arbitrary function $I_i(\cdot)$, there exist real number $\mathcal{I}_i \in \mathbb{R}$ that satisfy:

$$|I_i(\nu)| \leq \mathcal{I}_i, \quad \forall \nu \in \mathbb{R}.$$

By employing set-valued maps and differential inclusions, systems (3.1) and (3.2) can be rewritten as follows:

$${}^C_{t_0}D_t^\alpha z_i(t) \in -d_i z_i(t - \delta) + \sum_{e=1}^n a_{ie} \overline{\text{co}}[f_e(z_e(t))] + \sum_{e=1}^n b_{ie} \overline{\text{co}}[f_e(z_e(t - \tau(t)))] + I_i(t),$$

and

$$\begin{aligned} {}^C_{t_0}D_t^\alpha w_i(t) &\in -d_i w_i(t - \delta) + \sum_{e=1}^n a_{ie} \overline{\text{co}}[f_e(w_e(t))] + \sum_{e=1}^n b_{ie} \overline{\text{co}}[f_e(w_e(t - \tau(t)))] + I_i(t) \\ &+ u_i(t), \quad t \geq 0. \end{aligned}$$

In the case of discontinuous activation functions, the systems are analyzed using differential inclusions and Filippov solutions. Based on the drive system (3.1) and the response system (3.2), we utilize measurable selections $\alpha_e(t) \in \overline{\text{co}}[f_e(z_e(t))]$, $\alpha_e^*(t) \in \overline{\text{co}}[f_e(w_e(t))]$, $\beta_e(t) \in \overline{\text{co}}[f_e(z_e(t - \tau(t)))]$ and $\beta_e^*(t) \in \overline{\text{co}}[f_e(w_e(t - \tau(t)))]$, to account for discontinuities:

$${}^C_{t_0}D_t^\alpha z_i(t) = -d_i z_i(t - \delta) + \sum_{e=1}^n a_{ie} \alpha_e(t) + \sum_{e=1}^n b_{ie} \beta_e(t) + I_i(t), \quad (3.3)$$

and

$${}^C_{t_0}D_t^\alpha w_i(t) = -d_i w_i(t - \delta) + \sum_{e=1}^n a_{ie} \alpha_e^*(t) + \sum_{e=1}^n b_{ie} \beta_e^*(t) + I_i(t) + u_i(t). \quad (3.4)$$

Remark 3.1. In previous studies on FONNs [14, 17, 2], f and g were required to meet standard Lipschitz continuity assumptions; however, the present work relaxes these requirements. Furthermore, unlike the outcomes for non-fractional-order neural networks with non-continuous activations, our approach does not impose requirements on the boundedness [18] and monotonicity [19] of the activation functions. Consequently, our assumptions have greater generality. The existence of non-continuous activations f_e (for $e = 1, 2, \dots, n$) introduces discontinuity in (3.1) and (3.2), leading to the absence of a classical solution. Consequently, we investigate solutions distinct from the classical solution (see [20]).

To obtain the main findings, the following lemmas and a definition are provided.

Definition 3.2. Systems (3.1) and (3.2) achieve quasi-projective synchronization with an error bound $\epsilon > 0$ if the error $r(t) = w(t) - rz(t)$ exponentially converges into a compact set defined as:

$$\mathbb{D} := \psi(s) = \left\{ (r_1(s), \dots, r_n(s))^T \in \mathbb{C}([- \tilde{\tau}, 0], \mathbb{R}^n) \mid \|r(t)\|_1 \leq \epsilon \right\} \quad \text{as } t \rightarrow +\infty.$$

3.2 Principal results

In this part, our objective is to establish several conditions for quasi-projective synchronization of FOCNNs described by equations (3.1) and (3.2), by designing a novel controller. We assume:

$$u_i(t) = -\tilde{\gamma}_i(t)r_i(t) - \tilde{\delta}_i(t)\text{sign}(r_i(t))|r_i(t - \tau(t))|. \quad (3.5)$$

Moreover, the adaptive control is formulated as:

$${}^C D_t^\alpha \tilde{\gamma}_i(t) = \varepsilon_i |r_i(t)|, \quad {}^C D_t^\alpha \tilde{\delta}_i(t) = \eta_i |r_i(t - \tau(t))|,$$

where ε_i, η_i are arbitrarily chosen positive constants.

Remark 3.3. The choice of an adaptive control scheme for the synchronization of fractional-order cellular neural networks (FOCNNs) is justified by its superior ability to manage complex dynamical systems characterized by uncertain parameters and external disturbances. While other methods such as feedback or sliding mode control exist, adaptive control is particularly effective in this context because it allows the controller gains to adjust dynamically to the system's evolving state. Furthermore, the specific controller designed in this work is novel because its update laws are formulated directly in the fractional domain, distinguishing it from traditional integer-order schemes. This approach is essential for maintaining robustness in the presence of leakage delays, time-varying delays, and discontinuous activation functions, ensuring that the synchronization error converges to a predefined compact set even when the exact system parameters are unknown.

By substituting these inequalities into the Lyapunov derivative and grouping the terms for

$|r_i(t)|$ and $|r_i(t - \tau(t))|$, we obtain:

$$\begin{aligned}
{}_t^C D_t^\alpha V(t) &\leq \sum_{i=1}^n \left[-d_i |r_i(t - \delta)| + \sum_{e=1}^n |a_{ie}| |\alpha_e^*(t) - \alpha_e^{**}(t)| + \sum_{e=1}^n |a_{ie}| |\alpha_e^{**}(t) - r \alpha_e(t)| \right. \\
&\quad + \sum_{e=1}^n |b_{ie}| |\beta_e^*(t) - \beta_e^{**}(t)| + \sum_{e=1}^n |b_{ie}| |\beta_e^{**}(t) - r \beta_e(t)| + |1 - r| |I_i(t)| \\
&\quad \left. - \varepsilon_i |\delta_i(t)| - \eta_i |r_i(t - \tau(t))| \right], \\
&\leq \sum_{i=1}^n \left[-d_i |r_i(t - \delta)| + \sum_{e=1}^n |a_{ie}| [\chi_e |r_e(t)| + \chi_e^*] + (1 + r) \mathcal{F}_i \sum_{e=1}^n |a_{ie}| \right. \\
&\quad + \sum_{e=1}^n |b_{ie}| [\chi_e |r_e(t - \tau(t))| + \chi_e^*] + (1 + r) \mathcal{F}_i \sum_{e=1}^n |b_{ie}| + |1 - r| |\mathcal{I}_i - \varepsilon_i |r_i(t)| \\
&\quad \left. - \eta_i |r_i(t - \tau(t))| \right] \\
&\leq \sum_{i=1}^n \left[\sum_{e=1}^n |a_{ei}| \chi_i - \varepsilon_i - d_i \right] |r_i(t)| + \sum_{i=1}^n \left[\sum_{e=1}^n |b_{ei}| \chi_i - \eta_i \right] |r_i(t - \tau(t))| \\
&\quad + \sum_{i=1}^n \left[\sum_{e=1}^n |a_{ei}| \chi_i^* + \sum_{e=1}^n |b_{ei}| \chi_i^* + (1 + r) \mathcal{F}_i \sum_{e=1}^n (|a_{ei}| + |b_{ei}|) + |1 - r| |\mathcal{I}_i| \right] \\
&\leq -\min_{1 \leq i \leq n} \left\{ d_i + \varepsilon_i - \sum_{e=1}^n |a_{ei}| \chi_i \right\} \sum_{i=1}^n |r_i(t)| + \max_{1 \leq i \leq n} \left\{ \sum_{e=1}^n |b_{ei}| \chi_i - \eta_i \right\} \\
&\quad \times \sum_{i=1}^n |r_i(t - \tau(t))| + \sum_{i=1}^n \left[\sum_{e=1}^n |a_{ei}| \chi_i^* + \sum_{e=1}^n |b_{ei}| \chi_i^* + (1 + r) \mathcal{F}_i \sum_{e=1}^n (|a_{ei}| \right. \\
&\quad \left. + |b_{ei}|) + |1 - r| |\mathcal{I}_i| \right], \\
&\leq -\tilde{\rho}_1 \sum_{i=1}^n |r_i(t)| + \tilde{\rho}_2 \sum_{i=1}^n |r_i(t - \tau(t))| + \mathcal{N}_2, \\
&\leq -\tilde{\rho}_1 \sum_{i=1}^n V(t) + \tilde{\rho}_2 \sum_{i=1}^n V(t - \tau(t)) + \mathcal{N}_2, \tag{3.6}
\end{aligned}$$

To bridge the gap between this expression and the application of Lemma 2.3, we introduce the following Razumikhin-type condition:

Hypothesis: We assume that there exists a constant $\mu > 1$ such that for any $\theta \in [0, \tilde{\tau}]$, the following condition holds:

$$V(t - \theta) \leq \mu V(t) \tag{3.7}$$

Applying this hypothesis to the time-varying delay $\tau(t)$, inequality (3.6) is simplified as:

$$\begin{aligned}
{}_0^C D_t^\alpha V(t) &\leq -\tilde{\rho}_1 V(t) + \tilde{\rho}_2 \mu V(t) + \mathcal{N}_2, \\
&\leq -(\tilde{\rho}_1 - \mu \tilde{\rho}_2) V(t) + \mathcal{N}_2, \\
&\leq -\tilde{\rho}_3 V(t) + \mathcal{N}_2, \tag{3.8}
\end{aligned}$$

where $\tilde{\rho}_3 = \tilde{\rho}_1 - \mu \tilde{\rho}_2 > 0$ as defined in (3.8). This formulation now strictly satisfies the conditions of Lemma 2.3, leading to:

$$V(t) \leq \left(V(0) - \frac{\mathcal{N}_2}{\tilde{\rho}_3} \right) E_\alpha(-\tilde{\rho}_3 t^\alpha) + \frac{\mathcal{N}_2}{\tilde{\rho}_3} \tag{3.9}$$

where

$$\begin{aligned}
\bullet \quad \tilde{\rho}_1 &= \min_{1 \leq i \leq n} \{d_i + \epsilon_i - \sum_{e=1}^n |a_{ei}| \chi_i\}, \quad \tilde{\rho}_2 = \max_{1 \leq i \leq n} \left\{ \sum_{e=1}^n |b_{ei}| \chi_i - \eta_i \right\} \\
\bullet \quad N_2 &= \sum_{i=1}^n \left[\sum_{e=1}^n |a_{ei}| \chi_i^* + \sum_{e=1}^n |b_{ei}| \chi_i^* + (1+r) \mathcal{F}_i \sum_{e=1}^n (|a_{ei}| + |b_{ei}|) + |1-r| \mathcal{I}_i \right].
\end{aligned}$$

This confirms that the synchronization error converges to the compact set D as $t \rightarrow \infty$, thus completing the proof of quasi-projective synchronization.

4 Simulation example

To demonstrate the applicability of Theorem 1, this section presents numerical simulation results. We consider the following system:

$${}_0^C D_t^\alpha z_i(t) = -d_i z_i(t) + \sum_{e=1}^2 a_{ie} f_e(z_e(t)) + \sum_{e=1}^2 b_{ie} f_e(z_e(t - \tau(t))) + I_i(t), \quad (4.1)$$

$$\begin{aligned}
{}_0^C D_t^\alpha w_i(t) &= -d_i w_i(t) + \sum_{e=1}^2 a_{ie} f_e(w_e(t)) + \sum_{e=1}^2 b_{ie} f_e(w_e(t - \tau(t))) + I_i(t) \\
&+ u_i(t), \quad i = 1, 2,
\end{aligned} \quad (4.2)$$

where

$$A = \begin{pmatrix} -0.5 & -0.3 \\ 0.4 & 0.6 \end{pmatrix}, \quad B = \begin{pmatrix} 0.2 & 0 \\ 0 & 0.1 \end{pmatrix}$$

$$\alpha = 0.98, \quad d_1 = 1, \quad d_2 = 1.2$$

$$f(\cdot) = 0.5 \times \text{sign}(\cdot), \quad I_1(t) = 1, \quad I_2(t) = -1.$$

Moreover, $\tau(t) = \frac{e^t}{e^t + 1}$ and $\delta = 1$. We set projective coefficients $r_1 = r_2 = 0.06$ and devise a comfortable adaptive controller as follows:

$$\begin{aligned}
u_1(t) &= -\tilde{\gamma}_1(t) r_1(t) - \tilde{\delta}_1(t) \text{sign}(r_1(t)) |r_1(t - \tau(t))|, \\
u_2(t) &= -\tilde{\gamma}_2(t) r_2(t) - \tilde{\delta}_2(t) \text{sign}(r_2(t)) |r_2(t - \tau(t))|
\end{aligned}$$

Moreover, the adaptive update law is specified as:

$$\begin{aligned}
{}_0^C D_t^{0.98} \tilde{\gamma}_1(t) &= \varepsilon_1 |r_1(t)| = 1 |r_1(t)|, \\
{}_0^C D_t^{0.98} \tilde{\gamma}_2(t) &= \varepsilon_2 |r_2(t)| = 2 |r_2(t)|, \\
{}_0^C D_t^{0.98} \tilde{\delta}_1(t) &= \eta_1 |r_1(t - \tau(t))| = 0.35 |r_1(t - \tau(t))|, \\
{}_0^C D_t^{0.98} \tilde{\delta}_2(t) &= \eta_2 |r_2(t - \tau(t))| = 0.35 |r_2(t - \tau(t))|.
\end{aligned}$$

Now, the following calculations verify that the chosen constants satisfy the theoretical requirements established in Section 2.2.

- **Assumption A_1** : Given $f_j(p) = 0.5 \times \text{sign}(p)$, we have: $|f_j(p)| \leq |0.5 \times \text{sign}(p)|$. Since, $|0.5 \times \text{sign}(p)| \leq 0.5$, it follows that: $|f_j(p)| \leq 0.5$. Thus, setting $\mathcal{F}_1 = \mathcal{F}_2 = 0.5$ satisfies the assumption.
- **Assumption A_2** : For the set-valued map, $\text{co}[f_j(\cdot)]$, the condition is:
 $\sup \|\alpha_j(t) - \alpha_j^*(t)\| \leq \chi_j |z_j(t) - w_j(t)| + \chi_j^*$. The jump discontinuity $0.5 \times \text{sign}(p)$ has a maximum leap of 1. By choosing: $\chi_j = 2$ and $\chi_j^* = 1$, we ensure that: $1 \cdot |z_j - w_j| + 1 \leq 2|z_j - w_j| + 1$. Therefore, Assumption A_2 is satisfied.
- **Assumption A_3** : The external bias is given as $I_1(t) = 1, I_2(t) = -1$. By selecting $\mathcal{I}_1 = \mathcal{I}_2 = 1$, we clearly have: $|1| \leq 1$ and $|-1| \leq 1$. Thus, Assumption A_3 is verified.

Based on these direct computations, $\chi_1 = \chi_2 = 2$, $\chi_1^* = \chi_2^* = 1$ (Satisfies A_2), $\mathcal{F}_1 = \mathcal{F}_2 = 0.5$ (Satisfies A_1), $\mathcal{I}_1 = \mathcal{I}_2 = 1$ (Satisfies A_3).

Moreover, we have:

$$\begin{aligned}
 \mathcal{N}_2 &= \sum_{i=1}^2 \left[\sum_{e=1}^2 2 \times |a_{ei}| + \sum_{e=1}^2 2 \times |b_{ei}| + (1 + 0.06) \times 0.5 \sum_{e=1}^2 (|a_{ei}| + |b_{ei}|) + |1 - 0.06| \times 1 \right], \\
 &= [(0.5 + 0.4) \times 1 + (0.2 \times 1) + 1.06 \times 0.5 \times (0.5 + 0.4 + 0.2 + 0) + 0.94] + [(0.3 + 0.6) \times 1 \\
 &\quad + (0 + 0.1) \times 1 + 1.06 \times 0.5 \times (0.3 + 0.6 + 0 + 0.1) + 0.94], \\
 &= 2.623 + 2.47, \\
 &= 5.093 > 0, \\
 \tilde{\rho}_1 &= \min_{i=1,2} \left\{ d_i + \varepsilon_i - \sum_{e=1}^2 |a_{ei}| \chi_i \right\}, \\
 &= \min_{i=1,2} \left\{ d_1 + \varepsilon_1 - \sum_{e=1}^2 |a_{e1}| \chi_1, d_2 + \varepsilon_2 - \sum_{e=1}^2 |a_{e2}| \chi_2 \right\}, \\
 &= \min_{i=1,2} \left\{ 1 + 1 - 1.8, 1.2 + 1 - 1.8 \right\}, \\
 &= \min_{i=1,2} \left\{ 0.2, 0.4 \right\}, \\
 &= 0.2 > 0, \\
 \tilde{\rho}_2 &= \max_{i=1,2} \left\{ \sum_{e=1}^2 2|b_{ei}| - \eta_1, \sum_{e=1}^2 2|b_{e2}| - \eta_2 \right\}, \\
 &= \max_{i=1,2} \left\{ (2 \times 0.2) - 0.35, (2 \times 0.1) - 0.35 \right\}, \\
 &= \max_{i=1,2} \{0.05, -0.15\} \\
 &= 0.05 > 0, \\
 \tilde{\rho}_3 &= \tilde{\rho}_1 - \mu \tilde{\rho}_2, \\
 &= 0.2 - (2 \times 0.05) \quad \text{with} \quad \mu = 2 > 1 \\
 &= 0.1 > 0.
 \end{aligned}$$

Meeting all the requirements of Theorem 1 confirms that the controller (3.5) successfully achieves quasi-projective synchronization between systems (4.1) and (4.2). The simulations demonstrate that synchronization occurs within a settling time of 2 units.

For the simulations, the initial conditions are set as follows: $z_1(t) = 0.5$, $z_2(t) = -0.5$, $w_1(t) = 1$, $w_2(t) = -1$, $t \in [-1, 0]$. Figures 1 and 2 illustrate the trajectories of (4.1) and (4.2). At the start of the simulation ($t = 0$), there is a significant discrepancy between the drive and response trajectories due to the different initial conditions ($z_1(0) = 0.5$, $z_2(0) = -0.5$, $w_1(0) = 1$, $w_2(0) = -1$). Once the adaptive controller is activated, the trajectories of the slave system (w_i) rapidly adjust to follow the scaled trajectories of the master system (z_i). This visual alignment confirms that the systems have entered a synchronized state despite the presence of leakage delays and discontinuous activations.

Additionally, Figure 3 displays the synchronization error states $r_\eta(t)$ ($\eta = 1, 2$) over time. The errors start at non-zero values (reflecting the initial mismatch) but converge rapidly toward the origin. The simulation shows that the synchronization error achieves a steady state within a settling time of approximately 2 units. This convergence to a near-zero or bounded value (compact set) provides a direct numerical proof of the effectiveness of the designed adaptive update laws.

Remark 4.1. A central challenge in our model is the use of non-continuous activation functions,

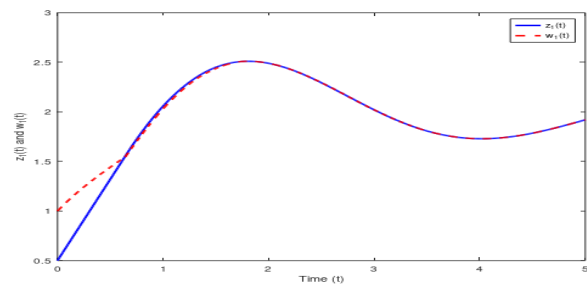


Figure 1. The evolution of $z_1(t)$ and $w_1(t)$ over time.

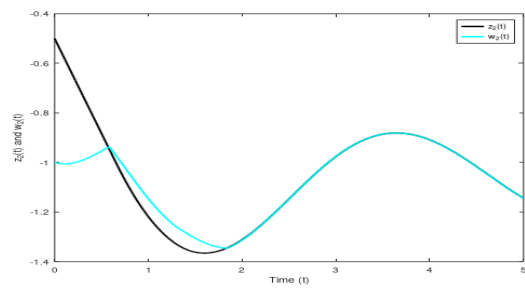


Figure 2. The evolution of $z_2(t)$ and $w_2(t)$ over time.

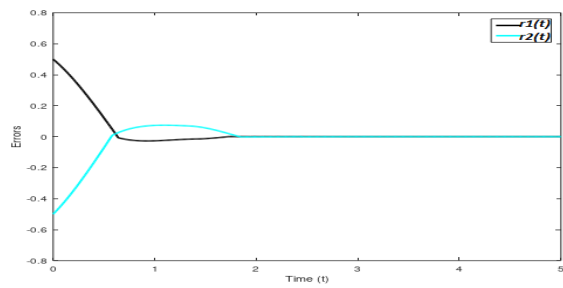


Figure 3. The states of error $r_\eta(t)$ ($\eta = 1, 2$).

which prevents the existence of classical solutions. To address this, we have strictly employed the Filippov solution framework. By utilizing nonsmooth analysis and differential inclusion theory, we have rigorously demonstrated the existence of globally defined solutions for delayed fractional-order systems. The novelty of the proposed adaptive controller lies in its unique update law, which is defined directly within the fractional domain. Unlike traditional integer-order controllers, this scheme is specifically designed to handle the infinite memory and hereditary properties inherent in fractional-order dynamics, making it more robust against unpredictable parameter variations. We have focused on quasi-projective synchronization due to its practical relevance in improving communication speed through the use of a projection factor. Our results extend existing findings on continuous delayed FONNs, offering a more general and less conservative framework for the dynamical analysis of neural networks. Finally, theoretical rigor is maintained by applying the fractional-order theorem to address the stability of systems with time-varying delays ($\tau(t)$). This ensures that our synchronization criteria are not only verifiable but also mathematically sound for non-constant delay scenarios.

Remark 4.2. This manuscript is an extended and significantly improved version of the paper entitled "Adaptive Synchronization of Neural Networks Characterized by Fractional-Order Dynamics, Leakage, and Variable Time Delays" [21] presented at ICRAMI 2025 (Tunisia, November 2025). The version presented at the conference established the foundational framework for the quasi-projective synchronization of fractional-order cellular neural networks (FOCNNs) in the real domain, incorporating both leakage delays and time-varying delays with discontinuous activation function.

In contrast, this expanded work provides a more comprehensive and rigorous treatment of the problem through the following improvements: 1) While the preliminary work introduced the basic control scheme, this version provides a much more detailed and robust dynamical analysis. It utilizes a specialized fractional Lyapunov–Krasovskii functional combined with a Razumikhin-type condition to strictly address the stability of systems with non-constant delays. 2) The derived synchronization conditions in this paper are more exhaustive and formulated as analytically verifiable algebraic criteria, making them easier to apply than the results in the conference version. 3) This manuscript includes in-depth numerical simulations and a more thorough discussion of the results. It specifically illustrates how the fractional order α and the adaptive update laws formulated directly in the fractional domain enhance the network's robustness and influence its transient behavior.

5 Conclusion

In this work, we have addressed the quasi-projective synchronization of FOCNNs by employing tools from nonsmooth analysis, differential inclusion theory, and stability results specific to fractional-order systems. We have derived sufficient conditions to guarantee synchronization within a predefined settling time. Although the analysis focuses mainly on the case $0 \leq \alpha \leq 1$, the proposed results remain valid for the case of integer order $\alpha = 1$. Based on the available knowledge, the predefined-time synchronization of FOCNNs particularly those with inertial terms and time-varying delays has been scarcely explored in prior work, underlining both the originality and significance of our contribution.

Parameter uncertainties, stochastic effects, external disturbances, and other factors are inherently unavoidable in nature; therefore, the dynamic analysis of FOCNNs considering one or more of these factors could be a focus of our future research.

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