

Global dissipativity of delayed stochastic quaternion-valued fuzzy neural networks

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Abstract. This study focuses on investigating the Global Dissipativity (GD) of stochastic fuzzy neural networks expressed in quaternion form and subject to time-varying delays. The approach begins by converting the initial quaternion-based model into an equivalent real-valued representation, using Hamilton's multiplication rules to split the system into four real components. Then, by constructing an appropriate Lyapunov functional and applying tools such as Jensen's inequality, Itô's formula, and several analytical techniques, we establish novel dissipativity conditions formulated as Linear Matrix Inequalities (LMIs). These conditions ensure the mean GD of the proposed model. In addition, the corresponding Globally Attractive (GA) in mean set is characterized. For the purpose of validating the results, a numerical example is introduced.

1 Introduction

Stochastic Neural Networks (SNNs) form an important class of dynamical systems in which randomness enters the dynamics through stochastic processes affecting one or several components of the model, leading to inherently random state trajectories. By explicitly incorporating uncertainties and external perturbations, SNNs provide an effective modeling framework for a wide range of practical problems, including nonlinear vibration analysis, engineering optimization, population dynamics, and control applications. These practical considerations have motivated growing research interest in developing rigorous theoretical tools for investigating essential system properties such as stability, robustness, and dissipativity. Nevertheless, despite this increasing attention, available theoretical results for stochastic neural networks remain comparatively limited. As an example, the authors of [1] analyzed the global dissipativity of a class of stochastic neural networks described by the following model:

$$\begin{aligned} dq_h(\rho) = & \left(-d_h q_h(\rho) + \sum_{z=1}^n a_{hz} \phi_z(q_z(\rho)) + \sum_{z=1}^n b_{hz} \phi_z(q_z(\rho - \ell)) + u_h \right) d\rho \\ & + \sum_{z=1}^n \sigma_{hz}(\rho, q_z(\rho), q_z(\rho - \ell)) d\omega_z(\rho), \end{aligned} \quad (1.1)$$

where $\omega_z(\rho)$ denotes a standard Brownian motion. Along similar lines, probabilistic stability and moment exponential stability for stochastic differential equations driven by Lévy noise were investigated in [2]. Furthermore, the existence and p th-moment asymptotic stability of mild solutions to nonlinear impulsive stochastic differential equations were studied in [3], while [4] addressed moment exponential stability for Markovian switching stochastic systems subject to impulsive effects. Additional related developments can be found in [5, 6, 7].

Quaternion algebra, originally introduced by William Rowan Hamilton in 1843, provides a natural extension of complex numbers and has proven useful in multidimensional signal processing

and neural network modeling. A quaternion-valued function $q(\rho)$ can be expressed as

$$q(\rho) = q^R(\rho) + iq^I(\rho) + jq^J(\rho) + kq^K(\rho),$$

where $q^R(\rho)$ denotes the real part and $q^I(\rho) + jq^J(\rho) + kq^K(\rho)$ represent the imaginary components. The imaginary units i, j , and k satisfy Hamilton's multiplication rules

$$\begin{cases} i^2 = j^2 = k^2 = ijk = -1, \\ ij = -ji = k, \\ jk = -kj = i \\ ki = -ik = j. \end{cases}$$

Let $q^*(\rho) = q^R(\rho) - iq^I(\rho) - jq^J(\rho) - kq^K(\rho)$ be the conjugate transpose of q , the modulus of q , denoted by $|q|$, is defined as:

$$|q| = \sqrt{qq^*} = \sqrt{(q^R)^2 + (q^I)^2 + (q^J)^2 + (q^K)^2}.$$

For a quaternion-valued vector $q(\rho) = (q_1(\rho), q_2(\rho), \dots, q_n(\rho))^T \in Q^n$, where each $q_h(\rho)$ is quaternion-valued, the induced norm is defined as

$$\|q\| = \sqrt{\sum_{h=1}^n |q_h|^2} = \sqrt{\sum_{h=1}^n (q_h^R)^2 + \sum_{h=1}^n (q_h^I)^2 + \sum_{h=1}^n (q_h^J)^2 + \sum_{h=1}^n (q_h^K)^2}.$$

In recent years, Quaternion-Valued Neural Networks (QVNNs) have attracted considerable attention. For instance, the existence and stability of Stepanov-like pseudo almost periodic solutions for QVNNs were studied in [8]. Stability issues for QVNNs with mixed delays were addressed in [9], while the global asymptotic stability of equilibrium points in delayed quaternion-valued fuzzy BAM neural networks was investigated in [10]. Further related results can be found in [11, 12, 13, 14].

On the other hand, fuzzy neural networks (FNNs) provide an effective hybrid modeling framework by integrating the learning capability of neural networks with the uncertainty-handling mechanism of fuzzy logic. This combination allows FNNs to cope efficiently with imprecise information and incomplete knowledge, making them particularly suitable for complex environments where exact mathematical models are difficult to obtain. The inclusion of fuzzy inference mechanisms enhances both the interpretability and robustness of the resulting models, which explains their widespread use in control engineering, decision-making, and pattern recognition applications [15, 16, 17]. Since the seminal work of Yang and Yang on fuzzy cellular neural networks in 1996 [18, 19], FNNs have been extensively studied and successfully applied to image processing and time-delay systems. Consequently, they have become a major research topic in the analysis of dynamical behaviors, stability properties, and robust control of uncertain systems [20, 21].

Dissipativity is a fundamental concept in dynamical system theory, introduced by Jan C. Willems in the early 1970s [22, 23] as a generalization of classical Lyapunov stability. Unlike traditional stability notions that emphasize convergence toward equilibrium points, dissipativity offers a broader analytical framework capable of addressing systems without equilibria or with non-convergent trajectories. This characteristic makes dissipativity theory particularly suitable for the analysis of complex neural network models [24, 25]. Within this framework, one can characterize globally attractive invariant sets and derive meaningful bounds on system trajectories, even in the presence of oscillatory or chaotic dynamics. As a result, dissipativity has become an essential tool for understanding long-term behavior and guaranteeing robust performance in nonlinear and uncertain systems.

Unlike previous studies focusing separately on stochastic, fuzzy, or quaternion-valued neural networks, this work establishes a unified framework that integrates all these features simultaneously. This unification allows us to capture more realistic dynamics arising in complex neural systems where uncertainty, fuzziness, and multidimensional interactions coexist.

Motivated by the above considerations, this paper investigates a class of stochastic neural networks with fuzzy effects formulated in the quaternion algebra framework, described by the following model:

$$\left\{ \begin{aligned} dq_h(\rho) &= \left(-\delta_h q_h(\rho) + \sum_{z=1}^n w_{hz}^1 \phi_z(q_z(\rho)) + \sum_{z=1}^n w_{hz}^2 \phi_z(q_z(\rho - \ell(\rho))) \right. \\ &\quad + \sum_{z=1}^n w_{hz}^3 \theta_z(\rho) + \bigwedge_{z=1}^n w_{hz}^4 \theta_z(\rho) + \bigwedge_{z=1}^n w_{hz}^5 \phi_z(q_z(\rho - \ell(\rho))) \\ &\quad + \bigvee_{z=1}^n w_{hz}^6 \phi_z(q_z(\rho - \ell(\rho))) + \bigvee_{z=1}^n w_{hz}^7 \theta_z(\rho) + u_h(\rho) \Big) d\rho \\ &\quad + \sum_{z=1}^n \sigma_{hz} \left(\rho, q_z(\rho), q_z(\rho - \ell(\rho)) \right) d\omega_z(\rho), \end{aligned} \right. \quad (1.2)$$

where

- $q(\cdot) = [q_1(\cdot), \dots, q_n(\cdot)]^T \in \mathbb{Q}^n$, represent the state vector associated with the i th unit at instant s ;
- $\delta = \text{diag}\{\delta_1, \dots, \delta_n\} \in \mathbb{R}^{n \times n}$ be a diagonal matrix with strictly positive entries;
- $W^1 = (w_{hz}^1)_{n \times n} \in \mathbb{Q}^{n \times n}$, $W^2 = (w_{hz}^2)_{n \times n} \in \mathbb{Q}^{n \times n}$ represent the connection weight matrices from neuron i to neuron j .
- $W^3 = (w_{hz}^3)_{n \times n} \in \mathbb{Q}^{n \times n}$ is feed-forward template;
- $W^4 = (w_{hz}^4)_{n \times n} \in \mathbb{Q}^{n \times n}$, $W^5 = (w_{hz}^5)_{n \times n} \in \mathbb{Q}^{n \times n}$, $W^6 = (w_{hz}^6)_{n \times n} \in \mathbb{Q}^{n \times n}$, $W^7 = (w_{hz}^7)_{n \times n} \in \mathbb{Q}^{n \times n}$ represent the entries of the fuzzy feedback MIN and MAX templates;
- $\phi(\cdot) = [\phi_1(\cdot), \phi_2(\cdot), \dots, \phi_n(\cdot)]^T \in \mathbb{Q}^n$ is the activation functions;
- $\theta(\cdot) = [\theta_1(\cdot), \dots, \theta_n(\cdot)]^T \in \mathbb{Q}^n$ is the bias of the neuron with $|\theta(\rho)| \leq \check{\theta}$, ($\check{\theta}$ is a positive real constant);
- $u(\cdot) = [u_1(\cdot), u_2(\cdot), \dots, u_n(\cdot)]^T \in \mathbb{Q}^n$ is the external input vector with $|u(\rho)| \leq \check{u}$, ($\check{u}$ is a positive real constant);
- $\bigwedge$ is linked to the fuzzy conjunction (AND). $\bigvee$ represents the fuzzy disjunction (OR);
- $\ell(\cdot)$ is a time-varying delay;
- Let $\omega(\cdot)$ denote a one-dimensional Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$. Specifically, we work within a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_\rho\}_{\rho \geq 0}, \mathcal{P})$, where the filtration $\{\mathcal{F}_\rho\}_{\rho \geq 0}$ satisfies the usual conditions, namely right-continuity and the inclusion of all $\mathcal{P}$ -null sets in $\mathcal{F}_0$. Expectation with respect to the underlying probability measure $\mathcal{P}$ is represented by the operator $\mathbb{E}(\cdot)$.

The initial values of system (1.2) are

$$q(v) = \aleph(v), \quad v \in [-\bar{\ell}, 0]$$

where $\bar{\ell} = \sup_{v \in \mathbb{R}}(\ell(v))$ and $\aleph(\cdot) \in L^2_{\mathcal{F}_0}([-\bar{\ell}, 0], \mathbb{Q}^n)$ which is the family of all $\mathcal{F}_0$ -measurable $\mathcal{C}([-\bar{\ell}, 0], \mathbb{Q}^n)$ -valued random variables $\aleph = \{\aleph(v) : v \in [-\bar{\ell}, 0]\}$ satisfying

$$\sup_{0 \leq v \leq \bar{\ell}} \mathbb{E}(|\aleph(v)|^2) < \infty.$$

The main objective of this work is to investigate the global dissipativity in the mean sense for the SNNs described in (1.2). The main contributions of this paper can be summarized as follows:

- A novel class of stochastic fuzzy quaternion-valued neural networks with time-varying delays is proposed, which simultaneously incorporates quaternion dynamics, fuzzy logic, and stochastic perturbations in a unified framework.
- By exploiting Hamilton's multiplication rules, the quaternion-valued model is decomposed into four equivalent real-valued subsystems, thereby overcoming the difficulties arising from the non-commutativity of quaternion multiplication.
- By employing Lyapunov functionals together with the linear matrix inequality (LMI) technique, new sufficient conditions are derived to guarantee the global dissipativity in mean.
- Compared with existing results in the literature, the obtained LMI conditions are more general and can be reduced to several known results when stochastic or fuzzy components are removed.
- The proposed approach extends previous works by explicitly addressing the combined effects of stochastic disturbances and fuzzy rules, which introduce additional analytical challenges in establishing global dissipativity conditions for the considered system.
- A numerical example is provided to demonstrate the validity and effectiveness of the proposed theoretical results.

The remainder of this paper is organized as follows. In Section 2, several assumptions, definitions, and preliminary lemmas related to the dynamical model (1.2) are introduced. Section 3 is devoted to the derivation of new sufficient conditions ensuring global dissipativity. A numerical example is presented in Section 4 to illustrate the applicability of the obtained results. Finally, Section 5 concludes the paper and outlines future research directions.

2 Preliminaries

Throughout this manuscript, the below notation is used. The notations $\mathbb{Q}^{m \times n}$, $\mathbb{C}^{m \times n}$, and $\mathbb{R}^{m \times n}$ refer to the spaces of $m \times n$ matrices with quaternion, complex, and real entries, respectively. The identity matrix is denoted by I . For any matrix $\pi \in \mathbb{R}^{n \times n}$, the symbols $\pi > 0$ and $\pi \geq 0$ indicate that π is positive definite and positive semidefinite, respectively. The notations π^{-1} and π^T represent the inverse and transpose of π . The largest and smallest eigenvalues of π are denoted by $\lambda_{\max}(\pi)$ and $\lambda_{\min}(\pi)$, respectively. The symbol “ $\star$ ” stands for the symmetric term in block symmetric matrices.

We impose the following assumptions for the subsequent analysis:

Assumption 2.1. The retard $\ell(\rho)$ is differentiable function such that:

$$0 \leq \ell(\rho) \leq \bar{\ell}, \quad \dot{\ell}(\rho) \leq \tilde{\ell} < 1.$$

Assumption 2.2. The activation functions $\phi_z(q_z(\rho))$ break down into a real and an imaginary part as follows:

$$\phi_z(q(\rho)) = \phi_z^R(q(\rho)) + i\phi_z^I(q(\rho)) + j\phi_z^J(q(\rho)) + k\phi_z^K(q(\rho)), \quad z = 1, \dots, n,$$

with

$$\phi_z(\cdot) : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R},$$

satisfying:

$$\check{M} \leq \phi_z(s_1(\rho)) - \phi_z(s_2(\rho)) \leq \hat{M}$$

where

$$\check{M} = \check{l}_z^R(s_1^R(\rho) - s_2^R(\rho)) + \check{l}_z^I(s_1^I(\rho) - s_2^I(\rho)) + \check{l}_z^J(s_1^J(\rho) - s_2^J(\rho)) + \check{l}_z^K(s_1^K(\rho) - s_2^K(\rho)),$$

$$\hat{M} = \hat{l}_z^R(s_1^R(\rho) - s_2^R(\rho)) + \hat{l}_z^I(s_1^I(\rho) - s_2^I(\rho)) + \hat{l}_z^J(s_1^J(\rho) - s_2^J(\rho)) + \hat{l}_z^K(s_1^K(\rho) - s_2^K(\rho)),$$

$\check{l}_z^R, \check{l}_z^I, \check{l}_z^J, \check{l}_z^K, \hat{l}_z^R, \hat{l}_z^I, \hat{l}_z^J$ and $\hat{l}_z^K$ are all constants.

We denote

$$\hat{L}^R = \text{diag}\{\hat{l}_1^R, \dots, \hat{l}_n^R\}, \hat{L}^I = \text{diag}\{\hat{l}_1^I, \dots, \hat{l}_n^I\}, \hat{L}^J = \text{diag}\{\hat{l}_1^J, \dots, \hat{l}_n^J\}, \\ \hat{L}^K = \text{diag}\{\hat{l}_1^K, \dots, \hat{l}_n^K\},$$

$$\check{L}^R = \text{diag}\{\check{l}_1^R, \dots, \check{l}_n^R\}, \check{L}^I = \text{diag}\{\check{l}_1^I, \dots, \check{l}_n^I\}, \check{L}^J = \text{diag}\{\check{l}_1^J, \dots, \check{l}_n^J\}, \\ \check{L}^K = \text{diag}\{\check{l}_1^K, \dots, \check{l}_n^K\},$$

$$L_1 = \text{diag}\{\hat{l}_1^R \check{l}_1^R, \dots, \hat{l}_n^R \check{l}_n^R\}, L_2 = \text{diag}\{\frac{\hat{l}_1^R + \check{l}_1^R}{2}, \dots, \frac{\hat{l}_n^R + \check{l}_n^R}{2}\}, L_3 = \text{diag}\{\hat{l}_1^I \check{l}_1^I, \dots, \hat{l}_n^I \check{l}_n^I\}, \\ L_4 = \text{diag}\{\frac{\hat{l}_1^I + \check{l}_1^I}{2}, \dots, \frac{\hat{l}_n^I + \check{l}_n^I}{2}\}, L_5 = \text{diag}\{\hat{l}_1^J \check{l}_1^J, \dots, \hat{l}_n^J \check{l}_n^J\}, L_6 = \text{diag}\{\frac{\hat{l}_1^J + \check{l}_1^J}{2}, \dots, \frac{\hat{l}_n^J + \check{l}_n^J}{2}\}, \\ L_7 = \text{diag}\{\hat{l}_1^K \check{l}_1^K, \dots, \hat{l}_n^K \check{l}_n^K\}, L_8 = \text{diag}\{\frac{\hat{l}_1^K + \check{l}_1^K}{2}, \dots, \frac{\hat{l}_n^K + \check{l}_n^K}{2}\}.$$

Assumption 2.3. Let $\sigma(\rho, q(\rho), q(\rho - \ell(\rho))) : \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$ is locally Lipschitz continuous and there exist matrices $H_j \geq 0$, $j = 1, 2$, such that

$$\text{trace}\{\sigma^T[\rho, q(\rho), q(\rho - \ell(\rho))]\sigma[\rho, q(\rho), q(\rho - \ell(\rho))]\} \leq q^T(\rho)H_1q(\rho) + q^T(\rho - \ell(\rho))H_2q(\rho - \ell(\rho)).$$

To transform the quaternion-valued system (1.2) into an equivalent real-valued representation, the following assumption is required:

Assumption 2.4. Let $q_h(\rho) = q_h^R(\rho) + iq_h^I(\rho) + jq_h^J(\rho) + kq_h^K(\rho)$. Then, the activation functions $\phi_z(q_z)$ can be expressed as

$$\phi_z(z) = \phi_z^R(q_z^R, q_z^I, q_z^J, q_z^K) + i\phi_z^I(q_z^R, q_z^I, q_z^J, q_z^K) + j\phi_z^J(q_z^R, q_z^I, q_z^J, q_z^K) \\ + k\phi_z^K(q_z^R, q_z^I, q_z^J, q_z^K), z = 1, \dots, n,$$

where $\phi_z^\Lambda : \mathbb{R}^4 \rightarrow \mathbb{R}$, $z = 1, \dots, n$, and $\Lambda = R, I, J, K$.

Under assumption (2.4), system (1.2) can be rewritten as four real-valued subsystems:

$$\left\{ \begin{aligned} dq_h^R(\rho) &= \left(-\delta_h q_h^R(\rho) + \sum_{z=1}^n [(w_{hz}^1)^R \phi_z^R(q_z(\rho)) - (w_{hz}^1)^I \phi_z^I(q_z(\rho)) - (w_{hz}^1)^J \phi_z^J(q_z(\rho)) \right. \\ &\quad - (w_{hz}^1)^K \phi_z^K(q_z(\rho))] + \sum_{z=1}^n [(w_{hz}^2)^R \phi_z^R(q_z(\rho - \ell(\rho))) - (w_{hz}^2)^I \phi_z^I(q_z(\rho - \ell(\rho))) \\ &\quad - (w_{hz}^2)^J \phi_z^J(q_z(\rho - \ell(\rho))) - (w_{hz}^2)^K \phi_z^K(q_z(\rho - \ell(\rho)))] \\ &\quad + \sum_{z=1}^n [(w_{hz}^3)^R \theta_z^R(\rho) - (w_{hz}^3)^I \theta_z^I(\rho) - (w_{hz}^3)^J \theta_z^J(\rho) - (w_{hz}^3)^K \theta_z^K(\rho)] \\ &\quad + \bigwedge_{z=1}^n [(w_{hz}^4)^R \theta_z^R(\rho) - (w_{hz}^4)^I \theta_z^I(\rho) - (w_{hz}^4)^J \theta_z^J(\rho) - (w_{hz}^4)^K \theta_z^K(\rho)] \\ &\quad + \bigwedge_{z=1}^n [(w_{hz}^5)^R \phi_z^R(q_z(\rho - \ell(\rho))) - (w_{hz}^5)^I \phi_z^I(q_z(\rho - \ell(\rho))) - (w_{hz}^5)^J \phi_z^J(q_z(\rho - \ell(\rho))) \\ &\quad - (w_{hz}^5)^K \phi_z^K(q_z(\rho - \ell(\rho)))] + \bigvee_{z=1}^n [(w_{hz}^6)^R \phi_z^R(q_z(\rho - \ell(\rho))) - (w_{hz}^6)^I \phi_z^I(q_z(\rho - \ell(\rho))) \\ &\quad - (w_{hz}^6)^J \phi_z^J(q_z(\rho - \ell(\rho))) - (w_{hz}^6)^K \phi_z^K(q_z(\rho - \ell(\rho)))] + \bigvee_{z=1}^n [(w_{hz}^7)^R \theta_z^R(\rho) \\ &\quad - (w_{hz}^7)^I \theta_z^I(\rho) - (w_{hz}^7)^J \theta_z^J(\rho) - (w_{hz}^7)^K \theta_z^K(\rho)] + u_h^R(\rho) \bigg) d\rho \\ &\quad + \sum_{z=1}^n \sigma_{hz}^R(\rho, q_z^R(\rho), q_z^R(\rho - \ell(\rho))) d\omega_z(\rho), \\ q^R(v) &= \aleph^R(v), v \in [-\bar{\ell}, 0], \end{aligned} \right. \quad (2.1)$$

$$\left\{ \begin{aligned} dq_h^I(\rho) &= \left(-\delta_h q_h^I(\rho) + \sum_{z=1}^n [(w_{hz}^1)^R \phi^I(q_z(\rho)) + (w_{hz}^1)^I \phi^R(q_z(\rho)) - (w_{hz}^1)^J \phi^K(q_z(\rho)) \right. \\ &\quad - (w_{hz}^1)^K \phi^J(q_z(\rho))] + \sum_{z=1}^n [(w_{hz}^2)^R \phi^I(q_z(\rho - \ell(\rho))) + (w_{hz}^2)^I \phi_z^R(q_z(\rho - \ell(\rho))) \\ &\quad - (w_{hz}^2)^J \phi_z^K(q_z(\rho - \ell(\rho))) - (w_{hz}^2)^K \phi_z^J(q_z(\rho - \ell(\rho)))] + \sum_{z=1}^n [(w_{hz}^3)^R \theta_z^I(\rho) + (w_{hz}^3)^I \theta_z^R(\rho) \\ &\quad - (w_{hz}^3)^J \theta_z^K(\rho) - (w_{hz}^3)^K \theta_z^J(\rho)] + \bigwedge_{z=1}^n [(w_{hz}^4)^R \theta^I(\rho) + (w_{hz}^4)^I \theta^R(\rho) - (w_{hz}^4)^J \theta_z^K(\rho) \\ &\quad - (w_{hz}^4)^K \theta_z^J(\rho)] + \bigwedge_{z=1}^n [(w_{hz}^5)^R \phi_z^I(q_z(\rho - \ell(\rho))) + (w_{hz}^5)^I \phi_z^R(q_z(\rho - \ell(\rho))) \\ &\quad - (w_{hz}^5)^J \phi_z^K(q_z(\rho - \ell(\rho))) - (w_{hz}^5)^K \phi_z^J(q_z(\rho - \ell(\rho)))] + \bigvee_{z=1}^n [(w_{hz}^6)^R \phi_z^I(q_z(\rho - \ell(\rho))) \\ &\quad + (w_{hz}^6)^I \phi^R(q_z(\rho - \ell(\rho))) - (w_{hz}^6)^J \phi_z^K(q_z(\rho - \ell(\rho))) - (w_{hz}^6)^K \phi_z^J(q_z(\rho - \ell(\rho)))] \\ &\quad + \bigvee_{z=1}^n [(w_{hz}^7)^R \theta^I(\rho) + (w_{hz}^7)^I \theta^R(\rho) - (w_{hz}^7)^J \theta^K(\rho) - (w_{hz}^7)^K \theta_z^J(\rho)] + u_h^I(\rho) \Big) d\rho \\ &\quad + \sum_{z=1}^n \sigma_{hz}^I(\rho, q_z^I(\rho), q_z^I(\rho - \ell(\rho))) d\omega_z(\rho), \\ q^I(v) &= \aleph^I(v), \quad v \in [-\bar{\ell}, 0], \end{aligned} \right. \tag{2.2}$$

$$\left\{ \begin{aligned} dq_h^J(\rho) &= \left(-\delta_h q_h^J(\rho) + \sum_{z=1}^n [(w_{hz}^1)^R \phi_z^J(q_z(\rho)) - (w_{hz}^1)^I \phi_z^K(q_z(\rho)) - (w_{hz}^1)^J \phi_z^R(q_z(\rho)) \right. \\ &\quad + (w_{hz}^1)^K \phi_z^I(q_z(\rho))] + \sum_{z=1}^n [(w_{hz}^2)^R \phi_z^J(q_z(\rho - \ell(\rho))) - (w_{hz}^2)^I \phi_z^K(q_z(\rho - \ell(\rho))) \\ &\quad - (w_{hz}^2)^J \phi_z^R(q_z(\rho - \ell(\rho))) + (w_{hz}^2)^K \phi_z^I(q_z(\rho - \ell(\rho)))] + \sum_{z=1}^n [(w_{hz}^3)^R \theta_z^J(\rho) - (w_{hz}^3)^I \theta_z^K(\rho) \\ &\quad - (w_{hz}^3)^J \theta_z^R(\rho) + (w_{hz}^3)^K \theta_z^I(\rho)] + \bigwedge_{z=1}^n [(w_{hz}^4)^R \theta_z^J(\rho) - (w_{hz}^4)^I \theta_z^K(\rho) \\ &\quad - (w_{hz}^4)^J \theta_z^R(\rho) + (w_{hz}^4)^K \theta_z^I(\rho)] + \bigwedge_{z=1}^n [(w_{hz}^5)^R \phi_z^J(q_z(\rho - \ell(\rho))) - (w_{hz}^5)^I \phi_z^K(q_z(\rho - \ell(\rho))) \\ &\quad - (w_{hz}^5)^J \phi_z^R(q_z(\rho - \ell(\rho))) + (w_{hz}^5)^K \phi_z^I(q_z(\rho - \ell(\rho)))] + \bigvee_{z=1}^n [(w_{hz}^6)^R \phi_z^J(q_z(\rho - \ell(\rho))) \\ &\quad - (w_{hz}^6)^I \phi_z^K(q_z(\rho - \ell(\rho))) - (w_{hz}^6)^J \phi_z^R(q_z(\rho - \ell(\rho))) + (w_{hz}^6)^K \phi_z^I(q_z(\rho - \ell(\rho)))] \\ &\quad + \bigvee_{z=1}^n [(w_{hz}^7)^R \theta_z^J(\rho) - (w_{hz}^7)^I \theta_z^K(\rho) - (w_{hz}^7)^J \theta_z^R(\rho) + (w_{hz}^7)^K \theta_z^I(\rho)] + u_h^J(\rho) \Big) d\rho \\ &\quad + \sum_{z=1}^n \sigma_{hz}^J(\rho, q_z^J(\rho), q_z^J(\rho - \ell(\rho))) d\omega_z(\rho), \\ q^J(v) &= \aleph^J(v), \quad v \in [-\bar{\ell}, 0], \end{aligned} \right. \tag{2.3}$$

$$\left\{ \begin{aligned} dq_{hp}^K(\rho) &= \left(-\delta_h q_h^K(\rho) + \sum_{z=1}^n [(w_{hz}^1)^R \phi_z^K(q_z(\rho)) + (w_{hz}^1)^I \phi_z^J(q_z(\rho)) - (w_{hz}^1)^J \phi_z^I(q_z(\rho)) \right. \\ &\quad + (w_{hz}^1)^K \phi_z^R(q_z(\rho))] + \sum_{z=1}^n [(w_{hz}^2)^R \phi_z^K(q_z(\rho - \ell(\rho))) + (w_{hz}^2)^I \phi_z^J(q_z(\rho - \ell(\rho))) \\ &\quad - (w_{hz}^2)^J \phi_z^I(q_z(\rho - \ell(\rho))) + (w_{hz}^2)^K \phi_z^R(q_z(\rho - \ell(\rho)))] + \sum_{z=1}^n [(w_{hz}^3)^R \theta_z^K(\rho) + (w_{hz}^3)^I \theta_z^J(\rho) \\ &\quad - (w_{hz}^3)^J \theta_z^I(\rho) + (w_{hz}^3)^K \theta_z^R(\rho)] + \bigwedge_{z=1}^n [(w_{hz}^4)^R \theta_z^K(\rho) + (w_{hz}^4)^I \theta_z^J(\rho) - (w_{hz}^4)^J \theta_z^I(\rho) \\ &\quad + (w_{hz}^4)^K \theta_z^R(\rho)] + \bigwedge_{z=1}^n [(w_{hz}^5)^R \phi_z^K(q_z(\rho - \ell(\rho))) + (w_{hz}^5)^I \phi_z^J(q_z(\rho - \ell(\rho))) \\ &\quad - (w_{hz}^5)^J \phi_z^I(q_z(\rho - \ell(\rho))) + (w_{hz}^5)^K \phi_z^R(q_z(\rho - \ell(\rho)))] + \bigvee_{z=1}^n [(w_{hz}^6)^R \phi_z^K(q_z(\rho - \ell(\rho))) \\ &\quad + (w_{hz}^6)^I \phi_z^J(q_z(\rho - \ell(\rho))) - (w_{hz}^6)^J \phi_z^I(q_z(\rho - \ell(\rho))) + (w_{hz}^6)^K \phi_z^R(q_z(\rho - \ell(\rho)))] \\ &\quad + \bigvee_{z=1}^n [(w_{hz}^7)^R \theta_z^K(\rho) + (w_{hz}^7)^I \theta_z^J(\rho) - (w_{hz}^7)^J \theta_z^I(\rho) + (w_{hz}^7)^K \theta_z^R(\rho)] + u_h^K(\rho) \Big) d\rho \\ &\quad + \sum_{z=1}^n \sigma_{hz}^K(\rho, q_z^K(\rho), q_z^K(\rho - \ell(\rho))) d\omega_z(\rho), \\ q^K(v) &= \mathfrak{K}^K(v), \quad v \in [-\bar{\ell}, 0]. \end{aligned} \right. \tag{2.4}$$

where $\phi_z^\Lambda(q_z(\rho_z - \ell(\rho))) = \phi_z^\Lambda(q_z^R(\rho_z - \ell(\rho)), q_z^I(\rho_z - \ell(\rho)), q_z^J(\rho_z - \ell(\rho)), q_z^K(\rho_z - \ell(\rho)))$,

and

$$w_{hz}^1 = (w_{hz}^1)^R + i(w_{hz}^1)^I + j(w_{hz}^1)^J + k(w_{hz}^1)^K, w_{hz}^2 = (w_{hz}^2)^R + i(w_{hz}^2)^I + j(w_{hz}^2)^J + k(w_{hz}^2)^K,$$

$$w_{hz}^3 = (w_{hz}^3)^R + i(w_{hz}^3)^I + j(w_{hz}^3)^J + k(w_{hz}^3)^K, w_{hz}^4 = (w_{hz}^4)^R + i(w_{hz}^4)^I + j(w_{hz}^4)^J + k(w_{hz}^4)^K,$$

$$w_{hz}^5 = (w_{hz}^5)^R + i(w_{hz}^5)^I + j(w_{hz}^5)^J + k(w_{hz}^5)^K, w_{hz}^6 = (w_{hz}^6)^R + i(w_{hz}^6)^I + j(w_{hz}^6)^J + k(w_{hz}^6)^K,$$

$$w_{hz}^7 = (w_{hz}^7)^R + i(w_{hz}^7)^I + j(w_{hz}^7)^J + k(w_{hz}^7)^K, \theta_z = \theta_z^R + i\theta_z^I + j\theta_z^J + k\theta_z^K,$$

$$\bar{\sigma}_{hz}(\rho, q_z(\rho), q_z(\rho - \ell(\rho))) = \sigma_{hz}^R(\rho, q_z^R(\rho), q_z^R(\rho - \ell(\rho))) + i\sigma_{hz}^I(\rho, q_z^I(\rho), q_z^I(\rho - \ell(\rho)))$$

$$+ j\sigma_{hz}^J(\rho, q_z^J(\rho), q_z^J(\rho - \ell(\rho))) + k\sigma_{hz}^K(\rho, q_z^K(\rho), q_z^K(\rho - \ell(\rho))).$$

According to (2.1)–(2.4), one can obtain that

$$\begin{aligned} d\bar{q}(\rho) &= \left(-\bar{\delta}_h \bar{q}_h(\rho) + \sum_{z=1}^n \bar{W}_{hz}^1 g_z(\bar{q}_z(\rho)) + \sum_{z=1}^n \bar{W}_{hz}^2 g_z(\bar{q}_z(\rho - \ell(\rho))) + \sum_{z=1}^n \bar{W}_{hz}^3 \bar{\theta}_z(\rho) \right. \\ &\quad + \bigwedge_{z=1}^n \bar{W}_{hz}^4 \bar{\theta}_z(\rho) + \bigwedge_{z=1}^n \bar{W}_{hz}^5 g_z(\bar{q}_z(\rho - \ell(\rho))) + \bigvee_{z=1}^n \bar{W}_{hz}^6 g_z(\bar{q}_z(\rho - \ell(\rho))) \\ &\quad \left. + \bigvee_{z=1}^n \bar{W}_{hz}^7 \bar{\theta}_z(\rho) + \bar{u}_h(\rho) \right) d\rho + \sum_{z=1}^n \check{\sigma}_{hz}(\rho) d\omega_z(\rho), \end{aligned} \tag{2.5}$$

where

$$\begin{aligned}
\bar{\delta}_h &= \text{diag}\{\delta_h, \delta_h, \delta_h, \delta_h\}, \quad \bar{q}_h(\rho) = ((q_h^R(\rho))^T, (q_h^I(\rho))^T, (q_h^J(\rho))^T, (q_h^K(\rho))^T)^T, \\
g_z(\bar{q}_z(\rho)) &= ((\phi_z^R(q_z(\rho)))^T, (\phi_z^I(q_z(\rho)))^T, (\phi_z^J(q_z(\rho)))^T, (\phi_z^K(q_z(\rho)))^T)^T, \\
g_z(\bar{q}_z(\rho - \ell(\rho))) &= ((\phi_z^R(q_z(\rho - \ell(\rho))))^T, (\phi_z^I(q_z(\rho - \ell(\rho))))^T, (\phi_z^J(q_z(\rho - \ell(\rho))))^T, \\
&\quad (\phi_z^K(q_z(\rho - \ell(\rho))))^T)^T, \\
\bar{\theta}_z(\rho) &= ((\theta_z^R(\rho))^T, (\theta_z^I(\rho))^T, (\theta_z^J(\rho))^T, (\theta_z^K(\rho))^T)^T, \quad (\text{with } |\bar{\theta}_z(\rho)| \leq \check{\theta}_z), \\
\bar{u}_h(\rho) &= ((u_h^R(\rho))^T, (u_h^I(\rho))^T, (u_h^J(\rho))^T, (u_h^K(\rho))^T)^T, \quad (\text{with } |\bar{u}_h(\rho)| \leq \check{u}_h), \\
\check{\sigma}_{hz}(\rho) &= \left(\sigma_{hz}^R(\rho, q_z^R(\rho), q_z^R(\rho - \ell(\rho))), \sigma_{hz}^I(\rho, q_z^I(\rho), q_z^I(\rho - \ell(\rho))), \right. \\
&\quad \left. \sigma_{hz}^J(\rho, q_z^J(\rho), q_z^J(\rho - \ell(\rho))), \sigma_{hz}^K(\rho, q_z^K(\rho), q_z^K(\rho - \ell(\rho))) \right)^T,
\end{aligned}$$

and

$$\begin{aligned}
\bar{W}_{hz}^1 &= \begin{pmatrix} (w_{hz}^1)^R & -(w_{hz}^1)^I & -(w_{hz}^1)^J & -(w_{hz}^1)^K \\ (w_{hz}^1)^I & (w_{hz}^1)^R & -(w_{hz}^1)^K & -(w_{hz}^1)^J \\ (w_{hz}^1)^J & (w_{hz}^1)^K & (w_{hz}^1)^R & -(w_{hz}^1)^I \\ (w_{hz}^1)^K & -(w_{hz}^1)^J & (w_{hz}^1)^I & (w_{hz}^1)^R \end{pmatrix}, \\
\bar{W}_{hz}^2 &= \begin{pmatrix} (w_{hz}^2)^R & -(w_{hz}^2)^I & -(w_{hz}^2)^J & -(w_{hz}^2)^K \\ (w_{hz}^2)^I & (w_{hz}^2)^R & -(w_{hz}^2)^K & -(w_{hz}^2)^J \\ (w_{hz}^2)^J & (w_{hz}^2)^K & (w_{hz}^2)^R & -(w_{hz}^2)^I \\ (w_{hz}^2)^K & -(w_{hz}^2)^J & (w_{hz}^2)^I & (w_{hz}^2)^R \end{pmatrix}, \\
\bar{W}_{hz}^3 &= \begin{pmatrix} (w_{hz}^3)^R & -(w_{hz}^3)^I & -(w_{hz}^3)^J & -(w_{hz}^3)^K \\ (w_{hz}^3)^I & (w_{hz}^3)^R & -(w_{hz}^3)^K & -(w_{hz}^3)^J \\ (w_{hz}^3)^J & (w_{hz}^3)^K & (w_{hz}^3)^R & -(w_{hz}^3)^I \\ (w_{hz}^3)^K & -(w_{hz}^3)^J & (w_{hz}^3)^I & (w_{hz}^3)^R \end{pmatrix}, \\
\bar{W}_{hz}^4 &= \begin{pmatrix} (w_{hz}^4)^R & -(w_{hz}^4)^I & -(w_{hz}^4)^J & -(w_{hz}^4)^K \\ (w_{hz}^4)^I & (w_{hz}^4)^R & -(w_{hz}^4)^K & -(w_{hz}^4)^J \\ (w_{hz}^4)^J & (w_{hz}^4)^K & (w_{hz}^4)^R & -(w_{hz}^4)^I \\ (w_{hz}^4)^K & -(w_{hz}^4)^J & (w_{hz}^4)^I & (w_{hz}^4)^R \end{pmatrix}, \\
\bar{W}_{hz}^5 &= \begin{pmatrix} (w_{hz}^5)^R & -(w_{hz}^5)^I & -(w_{hz}^5)^J & -(w_{hz}^5)^K \\ (w_{hz}^5)^I & (w_{hz}^5)^R & -(w_{hz}^5)^K & -(w_{hz}^5)^J \\ (w_{hz}^5)^J & (w_{hz}^5)^K & (w_{hz}^5)^R & -(w_{hz}^5)^I \\ (w_{hz}^5)^K & -(w_{hz}^5)^J & (w_{hz}^5)^I & (w_{hz}^5)^R \end{pmatrix}, \\
\bar{W}_{hz}^6 &= \begin{pmatrix} (w_{hz}^6)^R & -(w_{hz}^6)^I & -(w_{hz}^6)^J & -(w_{hz}^6)^K \\ (w_{hz}^6)^I & (w_{hz}^6)^R & -(w_{hz}^6)^K & -(w_{hz}^6)^J \\ (w_{hz}^6)^J & (w_{hz}^6)^K & (w_{hz}^6)^R & -(w_{hz}^6)^I \\ (w_{hz}^6)^K & -(w_{hz}^6)^J & (w_{hz}^6)^I & (w_{hz}^6)^R \end{pmatrix},
\end{aligned}$$

$$\bar{W}_{hz}^7 = \begin{pmatrix} (w_{hz}^7)^R & -(w_{hz}^7)^I & -(w_{hz}^7)^J & -(w_{hz}^7)^K \\ (w_{hz}^7)^I & (w_{hz}^7)^R & -(w_{hz}^7)^K & -(w_{hz}^7)^J \\ (w_{hz}^7)^J & (w_{hz}^7)^K & (w_{hz}^7)^R & -(w_{hz}^7)^I \\ (w_{hz}^7)^K & -(w_{hz}^7)^J & (w_{hz}^7)^I & (w_{hz}^7)^R \end{pmatrix},$$

$$\check{\sigma}(\rho) = \begin{pmatrix} \sigma^R(\rho, q_z^R(\rho), q_z^R(\rho - \ell(\rho))) \\ \sigma^I(\rho, q_z^I(\rho), q_z^I(\rho - \ell(\rho))) \\ \sigma^J(\rho, q_z^J(\rho), q_z^J(\rho - \ell(\rho))) \\ \sigma^K(\rho, q_z^K(\rho), q_z^K(\rho - \ell(\rho))) \end{pmatrix}.$$

Remark 2.5. In system (2.5), the fuzzy operators AND ($\wedge$) and OR ($\vee$) are specified as the minimum and maximum of the weighted functions, respectively:

$$\bigwedge_{z=1}^n \bar{W}_{hz} g_z(q_z((\rho - \ell(\rho)))) = \min \{ \bar{W}_{hz} g_z(q_z((\rho - \ell(\rho)))) \},$$

$$\bigvee_{z=1}^n \bar{W}_{hz} g_z(q_z((\rho - \ell(\rho)))) = \max \{ \bar{W}_{hz} g_z(q_z((\rho - \ell(\rho)))) \}.$$

To proceed, we first state the following lemma:

Lemma 2.6. [27] Let $\zeta_z, \xi_z, F_{hz}, \nu_{hz} \in \mathbb{R}$, and let $\vartheta_z : \mathbb{R} \rightarrow \mathbb{R}$ are continuous for all $h, z = 1, 2, \dots, n$. The relations given below are fulfilled.

$$\left| \bigwedge_{z=1}^n F_{hz} \vartheta_z(\zeta_z) - \bigwedge_{z=1}^n F_{hz} \vartheta_z(\xi_z) \right| \leq \sum_{z=1}^n |F_{hz}| |\vartheta_z(\zeta_z) - \vartheta_z(\xi_z)|,$$

$$\left| \bigvee_{z=1}^n \nu_{hz} \vartheta_z(\zeta_z) - \bigvee_{z=1}^n \nu_{hz} \vartheta_z(\xi_z) \right| \leq \sum_{z=1}^n |\nu_{hz}| |\vartheta_z(\zeta_z) - \vartheta_z(\xi_z)|.$$

By applying Lemma 2.6, we obtain

$$\begin{aligned} d\bar{q}(\rho) \leq & \left(-\bar{\delta}_h \bar{q}_h(\rho) + \sum_{z=1}^n \bar{W}_{hz}^1 g_z(\bar{q}_z(\rho)) + \sum_{z=1}^n \bar{W}_{hz}^2 g_z(\bar{q}_z(\rho - \ell(\rho))) + \sum_{z=1}^n \bar{W}_{hz}^3 \bar{\theta}_z(\rho) \right. \\ & + \sum_{z=1}^n \bar{W}_{hz}^4 \bar{\theta}_z(\rho) + \sum_{z=1}^n \bar{W}_{hz}^5 g_z(\bar{q}_z(\rho - \ell(\rho))) + \sum_{z=1}^n \bar{W}_{hz}^6 g_z(\bar{q}_z(\rho - \ell(\rho))) \\ & \left. + \sum_{z=1}^n \bar{W}_{hz}^7 \bar{\theta}_z(\rho) + \bar{u}_h(\rho) \right) d\rho + \sum_{z=1}^n \check{\sigma}_{hz}(\rho) d\omega_z(\rho), \end{aligned} \quad (2.6)$$

Rewritten the neural networks (2.6) in the vector form as :

$$\begin{aligned} d\bar{q}(\rho) \leq & \left(-\bar{\delta} \bar{q}(\rho) + \bar{W}^1 g(\bar{q}(\rho)) + \bar{W}^2 g(\bar{q}(\rho - \ell(\rho))) + \bar{W}^3 \bar{\theta}(\rho) + \bar{W}^4 \bar{\theta}(\rho) + \bar{W}^5 g(\bar{q}(\rho - \ell(\rho))) \right. \\ & \left. + \bar{W}^6 g(\bar{q}(\rho - \ell(\rho))) + \bar{W}^7 \bar{\theta}(\rho) + \bar{u}(\rho) \right) d\rho + \check{\sigma}(\rho) d\omega(\rho), \end{aligned} \quad (2.7)$$

It can be observed that

$$\bar{L}^- \leq \frac{g(\bar{s}_1) - g(\bar{s}_2)}{\bar{s}_1 - \bar{s}_2} \leq \bar{L}^+$$

where $\bar{L}^- = \text{diag}\{\check{L}^R, \check{L}^I, \check{L}^J, \check{L}^K\}$, $\bar{L}^+ = \text{diag}\{\hat{L}^R, \hat{L}^I, \hat{L}^J, \hat{L}^K\}$, we denote

$$\tilde{G}_1 = \text{diag}\{L_1, L_3, L_5, L_7\}, \quad \tilde{G}_2 = \text{diag}\{L_2, L_4, L_6, L_8\}.$$

The system (2.5) is initialized with the following conditions:

$$\bar{q}(v) = \bar{\aleph}(v), \quad v \in [-\bar{\ell}, 0], \quad (2.8)$$

where $\bar{\aleph}(v) = [\aleph^R(v), \aleph^I(v), \aleph^J(v), \aleph^K(v)]^T$.

Definition 2.7. [27] Model (1.2) is GD in mean if there is a compact set $\Xi \subset \mathbb{Q}^n$ such that, for any initial function $\aleph_0 \in L^2_{\mathcal{F}_0}([-\bar{\ell}, 0]; \mathbb{Q}^n)$, there exists a time $\mathcal{T}(\aleph_0) > \rho_0$, satisfying $\rho \geq \rho_0 + \mathcal{T}(\aleph_0) \Rightarrow \mathbb{E}q(\rho_0, \rho, \aleph_0) \subseteq \Xi$ where $q(\rho_0, \rho, \aleph_0)$ represents time evolution of model (1.2) starting from the initial state $\aleph_0$ at time ρ_0 . In this situation, Ξ is referred to as a GA set in mean. Moreover, the set Ξ is referred to as a positively invariant in mean if, for any initial value $\aleph_0 \in L^2_{\mathcal{F}_0}([-\bar{\ell}, 0]; \mathbb{Q}^n)$, the expected trajectory remains inside $\mathbb{E}q(\rho_0, \rho, \aleph) \subseteq \Xi$ for $\rho \geq \rho_0$.

Lemma 2.8. [27]: For any $\omega, \varpi \in \mathbb{R}^{n \times n}$ if $\phi > 0$ is a definite matrix with appropriate dimensions, then

$$2\omega^T \varpi \leq \omega^T \phi^{-1} \omega + \varpi^T \phi \varpi.$$

Lemma 2.9. [28]: Let ω, ϖ and P be real matrices with $P^T P \leq I$ and ε a scalar. Then

$$\omega P \varpi + (\omega P \varpi)^T \leq \frac{1}{\varepsilon} \omega^T \omega + \varepsilon \varpi^T \varpi, \quad \forall \varepsilon > 0.$$

Lemma 2.10. [28]: Given three constant matrices ϕ_1, ϕ_3 and ϕ_2 where $\phi_1 = \phi_1^T, \phi_2 = \phi_2^T > 0$, then

$$\phi_1 + \phi_3^T \phi_2^{-1} \phi_3 < 0,$$

if and only if

$$\begin{pmatrix} \phi_1 & \phi_3^T \\ \phi_3 & -\phi_2 \end{pmatrix} < 0, \text{ or } \begin{pmatrix} -\phi_2 & \phi_3 \\ \phi_3^T & \phi_1 \end{pmatrix} < 0.$$

Lemma 2.11. [29] Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a given probability space, and assume that $\hbar$ is an integrable random variable defined on it. Consider a measurable convex function $\Xi : \mathbb{R} \rightarrow \mathbb{R}$ such that $\mathbb{E}(|\Xi(\hbar)|) < \infty$. Then, the following inequality holds:

$$\Xi(\mathbb{E}(\hbar)) < \mathbb{E}(|\Xi(\hbar)|).$$

Consider a general stochastic system described by

$$d\gamma(\rho) = \psi(\rho, \gamma(\rho)) d\rho + \sigma(\rho, \gamma(\rho)) dw(\rho),$$

where $w(\rho)$ is an n -dimensional Brownian motion defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and the functions $\psi(\cdot), \sigma(\cdot) : \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ are suitably smooth. For a twice differentiable scalar function $\Phi : \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, the associated stochastic differential operator $\mathcal{L}$ is defined as

$$\mathcal{L}\Phi(\rho, \gamma) = \Phi_\rho(\rho, \gamma) + \Phi_\gamma(\rho, \gamma)\psi(\rho, \gamma) + \frac{1}{2}\text{trace}[\sigma^T(\rho, \gamma)\Phi_{\gamma\gamma}(\rho, \gamma)\sigma(\rho, \gamma)].$$

where

$$\begin{aligned} \Phi_\rho(\rho, \gamma) &= \frac{\partial \Phi(\rho, \gamma)}{\partial \rho}, \quad \Phi_\gamma(\rho, \gamma) = \left(\frac{\partial \Phi(\rho, \gamma)}{\partial \gamma_1}, \dots, \frac{\partial \Phi(\rho, \gamma)}{\partial \gamma_n} \right), \\ \Phi_{\gamma\gamma}(\rho, \gamma) &= \left(\frac{\partial^2 \Phi(\rho, \gamma)}{\partial \gamma_h \partial \gamma_z} \right)_{n \times n}. \end{aligned}$$

Remark 2.12. The proposed approach extends previous works by explicitly addressing the combined effects of stochastic disturbances and fuzzy rules, which introduce additional analytical challenges in establishing global dissipativity conditions. In particular, the presence of stochastic terms requires the construction of an augmented Lyapunov–Krasovskii functional and results in additional LMI constraints. Moreover, when the stochastic terms vanish, the obtained results reduce to those reported in [27, 28], showing that the present results generalize existing ones.

3 Main results

In what follows, we will present new conditions that allow us to demonstrate the GD in mean for SNNs (1.2).

Denote $\Pi = \bar{W}^2 + \bar{W}^5 + \bar{W}^6$ and $\kappa = \bar{W}^3 + \bar{W}^4 + \bar{W}^7$.

Theorem 3.1. *Assume that assumptions 2.1-2.3 are satisfied. Suppose there exist positive semi-definite matrices $H_1 \geq 0$, $H_2 \geq 0$ such that the diffusion term satisfies*

$$\text{trace}\{\sigma^T[\rho, q(\rho), q(\rho - \ell(\rho))]\sigma[\rho, q(\rho), q(\rho - \ell(\rho))]\} \leq q^T(\rho)H_1q(\rho) + q^T(\rho - \ell(\rho))H_2q(\rho - \ell(\rho)).$$

Moreover, assume the existence of positive diagonal matrices P , R_1 , R_2 , W , $\tilde{\chi}$ and positive scalar $\varrho > 0$ such that

$$\Sigma_1 = \begin{pmatrix} \psi_{11} & \psi_{12} & P\Pi & 0 \\ \star & \psi_{22} & 0 & 0 \\ \star & \star & \psi_{33} & 0 \\ \star & \star & \star & \psi_{44} \end{pmatrix} < 0, \quad P \leq \varrho I, \quad (3.1)$$

where

$$\begin{aligned} \psi_{11} &= -P\bar{\delta} - \bar{\delta}^T P + R_1 + \kappa^T P \kappa + P - \tilde{G}_1 \tilde{\chi} + \varrho H_1 + W, \quad \psi_{12} = P\bar{W}^1 + \tilde{G}_2 \tilde{\chi}, \quad \psi_{22} = R_2 - \tilde{\chi}, \\ \psi_{33} &= -(1 - \tilde{\ell})R_2, \quad \psi_{44} = -(1 - \tilde{\ell})R_1 + \varrho H_2. \end{aligned}$$

Then the SNNs (1.2) is GD in mean, and

$$\Xi_1 \triangleq \left\{ \bar{q} \in \mathbb{Q}^n \setminus \bar{q}^T(\rho)W\bar{q}(\rho) \leq \gamma \right\}$$

is GA set in mean of (1.2). Where $\gamma = \check{u}^T P \check{u} + \check{\theta}^T P \check{\theta}$.

Proof. The Lyapunov functional associated with the model (1.2) is given as follows:

$$\Phi(\rho) = \bar{q}^T(\rho)P\bar{q}(\rho) + \int_{\rho - \ell(\rho)}^{\rho} \bar{q}^T(v)R_1\bar{q}(v)dv + \int_{\rho - \ell(\rho)}^{\rho} g^T(\bar{q}(v))R_2g(\bar{q}(v))dv.$$

By Itô's formula, the stochastic derivative of $\Phi(\cdot)$ along the model (1.2) is

$$d\Phi(\rho) = \mathcal{L}(\Phi(\rho))d\rho + 2\bar{q}^T(\rho)P\bar{\sigma}(\rho)d\omega(\rho),$$

by using the inequality (2.7), we get

$$\begin{aligned}
\mathcal{L}\Phi(\rho) &\leq 2\bar{q}^T(\rho)P[-\bar{\delta}\bar{q}(\rho) + \bar{W}^1g(\bar{q}(\rho)) + \bar{W}^2g(\bar{q}(\rho - \ell(\rho))) + \bar{W}^3\bar{\theta}(\rho) + \bar{W}^4\bar{\theta}(\rho) \\
&\quad + \bar{W}^5g(\bar{q}(\rho - \ell(\rho))) + \bar{W}^6g(\bar{q}(\rho - \ell(\rho)))\bar{W}^7\bar{\theta}(\rho) + \bar{u}(\rho)] + \text{trace}(\check{\sigma}^T(\rho)Q\check{\sigma}(\rho)) \\
&\quad + \bar{q}^T(\rho)R_1\bar{q}(\rho) - (1 - \ell(\rho))\bar{q}^T(\rho - \ell(\rho))R_1\bar{q}(\rho - \ell(\rho)) + g^T(\bar{q}(\rho))R_2g(\bar{q}(\rho)) \\
&\quad - (1 - \ell(\rho))g^T(\bar{q}(\rho - \ell(\rho)))R_2g(\bar{q}(\rho - \ell(\rho))) \\
&\leq \bar{q}^T(\rho)[-P\bar{\delta} - \bar{\delta}^TP]\bar{q}(\rho) + 2\bar{q}^T(\rho)P\bar{W}^1g(\bar{q}(\rho)) + 2\bar{q}^T(\rho)P\bar{W}^2g(\bar{q}(\rho - \ell(\rho))) \\
&\quad + 2\bar{q}^T(\rho)P\bar{W}^3\bar{\theta}(\rho) + 2\bar{q}^T(\rho)P\bar{W}^4\bar{\theta}(\rho) + 2\bar{q}^T(\rho)P\bar{W}^5g(\bar{q}(\rho - \ell(\rho))) \\
&\quad + 2\bar{q}^T(\rho)P\bar{W}^6g(\bar{q}(\rho - \ell(\rho))) + 2\bar{q}^T(\rho)P\bar{W}^7\bar{\theta}(\rho) + 2\bar{q}^T(\rho)P\bar{u}(\rho)] + \text{trace}(\check{\sigma}^T(\rho)Q\check{\sigma}(\rho)) \\
&\quad + \bar{q}^T(\rho)R_1\bar{q}(\rho) - (1 - \tilde{\ell})\bar{q}^T(\rho - \ell(\rho))R_1\bar{q}(\rho - \ell(\rho)) + g^T(\bar{q}(\rho))R_2g(\bar{q}(\rho)) \\
&\quad - (1 - \tilde{\ell})g^T(\bar{q}(\rho - \ell(\rho)))R_2g(\bar{q}(\rho - \ell(\rho))) \\
&= \bar{q}^T(\rho)[-P\bar{\delta} - \bar{\delta}^TP]\bar{q}(\rho) + 2\bar{q}^T(\rho)P\bar{W}^1g(\bar{q}(\rho)) + 2\bar{q}^T(\rho)P\Pi g(\bar{q}(\rho - \ell(\rho))) \\
&\quad + 2\bar{q}^T(\rho)P\kappa\bar{\theta}(\rho) + 2\bar{q}^T(\rho)P\bar{u}(\rho) + \text{trace}(\check{\sigma}^T(\rho)Q\check{\sigma}(\rho)) + \bar{q}^T(\rho)R_1\bar{q}(\rho) \\
&\quad - (1 - \tilde{\ell})\bar{q}^T(\rho - \ell(\rho))R_1\bar{q}(\rho - \ell(\rho)) + g^T(\bar{q}(\rho))R_2g(\bar{q}(\rho)) \\
&\quad - (1 - \tilde{\ell})g^T(\bar{q}(\rho - \ell(\rho)))R_2g(\bar{q}(\rho - \ell(\rho))).
\end{aligned}$$

In accordance with Lemma 2.8, we obtain

$$\begin{aligned}
2\bar{q}^T(\rho)P\kappa\bar{\theta}(\rho) &\leq \bar{q}^T(\rho)\kappa^TP\kappa\bar{q}(\rho) + |\bar{\theta}^T(\rho)|P|\bar{\theta}(\rho)| \leq \bar{q}^T(\rho)\kappa^TP\kappa\bar{q}(\rho) + \check{\theta}^TP\check{\theta}, \\
2\bar{q}^T(\rho)P|\bar{u}(\rho)| &\leq \bar{q}^T(\rho)P\bar{q}(\rho) + |\bar{u}^T(\rho)|P|\bar{u}(\rho)| \leq \bar{q}^T(\rho)P\bar{q}(\rho) + \check{u}^TP\check{u}.
\end{aligned}$$

Based on Assumption 2.1, the below inequality holds for any real matrix

$$\tilde{\chi} = \text{diag}\{\hat{\chi}_1, \hat{\chi}_2, \dots, \hat{\chi}_p\} > 0$$

$$[g_p(\bar{q}_p(\rho)) - \bar{L}^-\bar{q}_p(\rho)]\hat{\chi}_p[\bar{L}^+\bar{q}_p(\rho) - g_p(\bar{q}_p(\rho))] \geq 0,$$

which imply

$$\begin{pmatrix} \bar{q}^T(\rho) \\ g^T(\bar{q}(\rho)) \end{pmatrix} \begin{pmatrix} -\tilde{G}_1\tilde{\chi} & \tilde{G}_2\tilde{\chi} \\ \star & -\tilde{\chi} \end{pmatrix} \begin{pmatrix} \bar{q}(\rho) \\ g(\bar{q}(\rho)) \end{pmatrix} \geq 0, \quad (3.2)$$

so

$$\begin{aligned}
\mathcal{L}\Phi(\rho) &\leq \bar{q}^T(\rho)[-P\bar{\delta} - \bar{\delta}^T P + R_1 + \kappa^T P \kappa + P - \tilde{G}_1 \tilde{\chi} + \varrho H_1 + W]\bar{q}(\rho) \\
&\quad - \bar{q}^T(\rho)W\bar{q}(\rho) + 2\bar{q}^T(\rho)[P\bar{W}^1 + \tilde{G}_2 \tilde{\chi}]g(\bar{q}(\rho)) + 2\bar{q}^T(\rho)P\Pi g(\bar{q}(\rho - \ell(\rho))) \\
&\quad + \bar{q}^T(\rho - \ell(\rho))[-(1 - \tilde{\ell})R_1 + \varrho H_2]\bar{q}(\rho - \ell(\rho)) + g^T(\bar{q}(\rho))[R_2 - \tilde{\chi}]g(\bar{q}(\rho)) \\
&\quad - (1 - \tilde{\ell})g^T(\bar{q}(\rho - \ell(\rho)))R_2 g(\bar{q}(\rho - \ell(\rho))) + \check{u}^T P \check{u} + \check{\theta}^T P \check{\theta} \\
&\leq -\bar{q}^T(\rho)W\bar{q}(\rho) + \check{u}^T P \check{u} + \check{\theta}^T P \check{\theta} + \xi^T(\rho)\Sigma\xi(\rho) \\
&\leq -\bar{q}^T(\rho)W\bar{q}(\rho) + \check{u}^T P \check{u} + \check{\theta}^T P \check{\theta},
\end{aligned}$$

where, $\xi(\rho) = [\bar{q}(\rho), g(\bar{q}(\rho)), g(\bar{q}(\rho - \ell(\rho))), \bar{q}(\rho - \ell(\rho))]^T$. Utilizing the expectation operator and Jensen's inequality, we derive the following

$$\begin{aligned}
d\mathbb{E}\Phi(\rho, \bar{q}(\rho)) &= \mathbb{E}\mathcal{L}\Phi(\rho, \bar{q}(\rho))d\rho \\
&\leq \left(-\lambda_{\min}(W)\mathbb{E}|\bar{q}|^2 + \check{u}^T P \check{u} + \check{\theta}^T P \check{\theta} \right) d\rho \quad (3.3)
\end{aligned}$$

$$\leq \left(-\lambda_{\min}(W)\mathbb{E}|\bar{q}|^2 + \check{u}^T P \check{u} + \check{\theta}^T P \check{\theta} \right) d\rho < 0, \quad (3.4)$$

where, $\mathbb{E}\bar{q} \in \mathbb{Q}^n \setminus \Xi_1$. In virtue of Itô's formula, we have

$$\mathbb{E}\Phi(\rho, \bar{q}(\rho)) = \mathbb{E}\Phi(\rho_0, \bar{q}_0) + \int_{\rho_0}^{\rho} \mathbb{E}\mathcal{L}\Phi(v, \bar{q}(v))dv.$$

On the other hand,

$$\mathbb{E}\Phi(\rho, \bar{q}(\rho)) \geq \lambda_{\min}(P)\mathbb{E}|\bar{q}(\rho)|^2 \geq \lambda_{\min}(P)\mathbb{E}|\bar{q}_0|^2. \quad (3.5)$$

There exists a time $\mathcal{T}(\rho_0, \bar{\aleph})$ such that $\mathbb{E}\bar{q}(\rho) \in \Xi_1$ for $\rho > \rho_0 + \mathcal{T}(\rho_0, \bar{\aleph})$. By definition 2.7, the SNNs (1.2) is GD in mean and Ξ_1 is an GA set in mean of it. This completes the proof. $\square$

Remark 3.2. By the proof of Theorem 3.1, we can find that as soon as $\mathbb{E}q(\rho_0, \rho, \aleph)$ remains in Ξ_1 since it enters Ξ_1 . Consequently, Ξ_1 is one of positive invariant set in mean of the differential equation (1.2).

Theorem 3.3. Assume that assumptions 2.1-2.3 hold and there exist positive matrices $H_1 \geq 0$, $H_2 \geq 0$ such that

$$\text{trace}\{\sigma^T(\rho, q(\rho), q(\rho - \ell(\rho)))\sigma(\rho, q(\rho), q(\rho - \ell(\rho)))\} \leq q^T(\rho)H_1 q(\rho) + q^T(\rho - \ell(\rho))H_2 q(\rho - \ell(\rho)).$$

Suppose that there exist real positive diagonal matrices P , R_1 , R_2 and a scalars $\varrho > 0$, $\alpha > 0$ and $0 < \zeta < 1$ such that the following LMIs hold:

$$\Sigma_2 = \begin{pmatrix} \psi_{11} & P\bar{W}^1 & P\Pi & 0 \\ \star & \psi_{22} & 0 & 0 \\ \star & \star & \psi_{33} & 0 \\ \star & \star & \star & \psi_{44} \end{pmatrix} < 0, \quad P \leq \varrho I, \quad (3.6)$$

where

$$\begin{aligned}
\psi_{11} &= -P\bar{\delta} - \bar{\delta}^T P + \varrho H_1 + R_1 + \alpha(\bar{L}^+)^T \bar{L}^+, \quad \psi_{22} = R_2 - \alpha I, \\
\psi_{33} &= -(1 - \tilde{\ell})R_2, \quad \psi_{44} = -(1 - \tilde{\ell})R_1 + \varrho H_2.
\end{aligned}$$

Then the SDEs (1.2) is globally dissipative in mean, and the set:

$$\Xi_2 \triangleq \left\{ \bar{q} \in \mathbb{Q}^n \setminus |\bar{q}| \leq \frac{\sqrt{\varepsilon_2 |P\check{u}|^2 + \varepsilon_1 |P\check{\kappa}\check{\theta}|^2}}{\sqrt{\varepsilon_1 \varepsilon_2 \lambda_{\min}(\Theta)}} \right\}$$

is globally attractive in mean of (1.2), where $\Theta = (1 - \zeta)[P\bar{\delta} + \bar{\delta}^T P] - (\varepsilon_1 + \varepsilon_2)I$, $0 < \zeta < 1$ is a constant to be determined and $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ are selected for $\Theta > 0$. Here, ζ is defined as follows: letting $\psi_{11}(\beta) = -\beta[P\bar{\delta} + \bar{\delta}^T P] + \varrho H_1 + R_1 + \alpha(\bar{L}^+)^T \bar{L}^+$, and replacing ψ_{11} by $\psi_{11}(\beta)$ in Σ_2 , we obtain $\Sigma_2(\beta)$, then define $\zeta \triangleq \inf\{\beta | \Sigma_2(\beta) \leq 0, 0 < \beta < 1\}$.

Proof. For the proof of Theorem 3.3, we consider the same Lyapunov functional $\Phi_1(\cdot)$ in Theorem 3.1:

$$\begin{aligned}
 d\Phi_1(\rho) &= \mathcal{L}(\Phi_1(\rho))d\rho + 2\bar{q}^T(\rho)P\bar{\sigma}(\rho)d\omega(\rho) \\
 \mathcal{L}\Phi_1(\rho) &\leq 2\bar{q}^T(\rho)P[-\bar{\delta}\bar{q}(\rho) + \bar{W}^1 g(\bar{q}(\rho)) + \bar{W}^2 g(\bar{q}(\rho - \ell(\rho))) + \bar{W}^3 \bar{\theta}(\rho) + \bar{W}^4 \bar{\theta}(\rho) \\
 &\quad + \bar{W}^5 g(\bar{q}(\rho - \ell(\rho))) + \bar{W}^6 g(\bar{q}(\rho - \ell(\rho))) + \bar{W}^7 \bar{\theta}(\rho) + \bar{u}(\rho)] + \text{trace}(\bar{\sigma}^T(\rho)Q\bar{\sigma}(\rho)) \\
 &\quad + \bar{q}^T(\rho)R_1\bar{q}(\rho) - (1 - \ell(\rho))\bar{q}^T(\rho - \ell(\rho))R_1\bar{q}(\rho - \ell(\rho)) + g^T(\bar{q}(\rho))R_2g(\bar{q}(\rho)) \\
 &\quad - (1 - \ell(\rho))g^T(\bar{q}(\rho - \ell(\rho)))R_2g(\bar{q}(\rho - \ell(\rho))) \\
 &\leq -(1 - \zeta)\bar{q}^T(\rho)[P\bar{\delta} + \bar{\delta}^T P]\bar{q}(\rho) + 2\bar{q}^T(\rho)P\bar{W}^1 g(\bar{q}(\rho)) + 2\bar{q}^T(\rho)P\bar{W}^2 g(\bar{q}(\rho - \ell(\rho))) \\
 &\quad + 2\bar{q}^T(\rho)P\bar{W}^3 \bar{\theta}(\rho) + 2\bar{q}^T(\rho)P\bar{W}^4 \bar{\theta}(\rho) + 2\bar{q}^T(\rho)P\bar{W}^5 g(\bar{q}(\rho - \ell(\rho))) \\
 &\quad + 2\bar{q}^T(\rho)P\bar{W}^6 g(\bar{q}(\rho - \ell(\rho))) + 2\bar{q}^T(\rho)P\bar{W}^7 \bar{\theta}(\rho) + 2\bar{q}^T(\rho)P\bar{q}(\rho)] + \text{trace}(\bar{\sigma}^T(\rho)Q\bar{\sigma}(\rho)) \\
 &\quad + \bar{q}^T(\rho)R_1\bar{q}(\rho) - (1 - \bar{\ell})\bar{q}^T(\rho - \ell(\rho))R_1\bar{q}(\rho - \ell(\rho)) + g^T(\bar{q}(\rho))R_2g(\bar{q}(\rho)) \\
 &\quad - (1 - \bar{\ell})g^T(\bar{q}(\rho - \ell(\rho)))R_2g(\bar{q}(\rho - \ell(\rho))) - \zeta\bar{q}^T(\rho)[P\bar{\delta} + \bar{\delta}^T P]\bar{q}(\rho) \\
 &\quad + \alpha[\bar{q}^T(\rho)(\bar{L}^+)^T \bar{L}^+ \bar{q}(\rho) - g^T(\bar{q}(\rho))g(\bar{q}(\rho))] \\
 &= -(1 - \zeta)\bar{q}^T(\rho)[P\bar{\delta} + \bar{\delta}^T P]\bar{q}(\rho) + 2\bar{q}^T(\rho)P\bar{W}^1 g(\bar{q}(\rho)) + 2\bar{q}^T(\rho)P\Pi g(\bar{q}(\rho - \ell(\rho))) \\
 &\quad + 2\bar{q}^T(\rho)P\kappa\bar{\theta}(\rho) + 2\bar{q}^T(\rho)P\bar{u}(\rho) + \text{trace}(\bar{\sigma}^T(\rho)Q\bar{\sigma}(\rho)) \\
 &\quad + \bar{q}^T(\rho)R_1\bar{q}(\rho) - (1 - \bar{\ell})\bar{q}^T(\rho - \ell(\rho))R_1\bar{q}(\rho - \ell(\rho)) + g^T(\bar{q}(\rho))R_2g(\bar{q}(\rho)) \\
 &\quad - (1 - \bar{\ell})g^T(\bar{q}(\rho - \ell(\rho)))R_2g(\bar{q}(\rho - \ell(\rho))) - \zeta\bar{q}^T(\rho)[P\bar{\delta} + \bar{\delta}^T P]\bar{q}(\rho) \\
 &\quad + \alpha[\bar{q}^T(\rho)(\bar{L}^+)^T \bar{L}^+ \bar{q}(\rho) - g^T(\bar{q}(\rho))g(\bar{q}(\rho))] \\
 &\leq -(1 - \zeta)\bar{q}^T(\rho)[P\bar{\delta} + \bar{\delta}^T P]\bar{q}(\rho) + 2\bar{q}^T(\rho)P\bar{W}^1 g(\bar{q}(\rho)) + 2\bar{q}^T(\rho)P\Pi g(\bar{q}(\rho - \ell(\rho))) \\
 &\quad + 2\bar{q}^T(\rho)P\kappa\bar{\theta} + 2\bar{q}^T(\rho)P\bar{u} + \text{trace}(\bar{\sigma}^T(\rho)Q\bar{\sigma}(\rho)) \\
 &\quad + \bar{q}^T(\rho)R_1\bar{q}(\rho) - (1 - \bar{\ell})\bar{q}^T(\rho - \ell(\rho))R_1\bar{q}(\rho - \ell(\rho)) + g^T(\bar{q}(\rho))R_2g(\bar{q}(\rho)) \\
 &\quad - (1 - \bar{\ell})g^T(\bar{q}(\rho - \ell(\rho)))R_2g(\bar{q}(\rho - \ell(\rho))) - \zeta\bar{q}^T(\rho)[P\bar{\delta} + \bar{\delta}^T P]\bar{q}(\rho) \\
 &\quad + \alpha[\bar{q}^T(\rho)(\bar{L}^+)^T \bar{L}^+ \bar{q}(\rho) - g^T(\bar{q}(\rho))g(\bar{q}(\rho))]. \tag{3.7}
 \end{aligned}$$

By using the Lemma 2.9, we get

$$\begin{aligned}
 2\bar{q}^T(\rho)P\bar{u} &\leq \varepsilon_1\bar{q}^T(\rho)\bar{q}(\rho) + \varepsilon_1^{-1}\bar{u}^T P^T P\bar{u}, \\
 2\bar{q}^T(\rho)P\kappa\bar{\theta} &\leq \varepsilon_2\bar{q}^T(\rho)\bar{q}(\rho) + \varepsilon_2^{-1}\bar{\theta}^T \kappa^T P^T P\kappa\bar{\theta}.
 \end{aligned}$$

So

$$\begin{aligned}
 \mathcal{L}\Phi_1(\rho) &\leq -(1 - \zeta)\bar{q}^T(\rho)[P\bar{\delta} + \bar{\delta}^T P]\bar{q}(\rho) + (\varepsilon_1 + \varepsilon_2)\bar{q}^T(\rho)\bar{q}(\rho) + \varepsilon_1^{-1}\bar{u}^T P^T P\bar{u} \\
 &\quad + \varepsilon_2^{-1}\bar{\theta}^T \kappa^T P^T P\kappa\bar{\theta} + \xi^T(\rho)\Sigma_2\xi(\rho) \\
 &\leq -\bar{q}^T(\rho)\Theta\bar{q}(\rho) + \varepsilon_1^{-1}|P\bar{u}|^2 + \varepsilon_2^{-1}|P\kappa\bar{\theta}|^2. \tag{3.8}
 \end{aligned}$$

By the property of expectation operation and Jensen's inequality, we have

$$\begin{aligned}
 d\mathbb{E}\Phi_1(\rho, \bar{q}(\rho)) &= \mathbb{E}\mathcal{L}\Phi_1(\rho, \bar{q}(\rho))d\rho \\
 &\leq \left(-\lambda_{\min}(\Theta)\mathbb{E}|\bar{q}|^2 + \varepsilon_1^{-1}|P\bar{u}|^2 + \varepsilon_2^{-1}|P\kappa\bar{\theta}|^2 \right) d\rho \\
 &\leq \left(-\lambda_{\min}(\Theta)\mathbb{E}|\bar{q}|^2 + \varepsilon_1^{-1}|P\bar{u}|^2 + \varepsilon_2^{-1}|P\kappa\bar{\theta}|^2 \right) d\rho < 0, \tag{3.9}
 \end{aligned}$$

where, $\mathbb{E}\bar{q} \in \mathbb{Q}^n \setminus \Xi_2$. It follows from Itô's formula, we have

$$\mathbb{E}\Phi_1(\rho, \bar{q}(\rho)) = \mathbb{E}\Phi_1(\rho_0, \bar{q}_0) + \int_{\rho_0}^{\rho} \mathbb{E}\mathcal{L}\Phi_1(v, \bar{q}(v))dv.$$

On the other hand,

$$\mathbb{E}\Phi_1(\rho, \bar{q}(\rho)) \geq \lambda_{\min}(\Theta)\mathbb{E}|\bar{q}(\rho)|^2 \geq \lambda_{\min}(\Theta)|\mathbb{E}\bar{q}(\rho)|^2. \quad (3.10)$$

So that there exists a $\mathcal{T}(\rho_0, \bar{\aleph})$ such that $\mathbb{E}\bar{q}(\rho) \in \Xi_2$ for $\rho > \rho_0 + \mathcal{T}(\rho_0, \bar{\aleph})$. By Definition 2.7, the SDEs (1.2) is globally dissipative in mean and Ξ_2 is an attractive set in mean of it. $\square$

Corollary 3.4. *In Theorem 3.3, let $P = I$, $R_1 = \frac{H_2}{(1-\bar{\ell})}$ and $R_2 = I$, then the SDEs (1.2) is globally dissipative in mean if there exists a scalar $\alpha > 1$ such that*

$$\Sigma_3 = \begin{pmatrix} \psi_{11} & \bar{W}^1 & \Pi \\ \star & \psi_{22} & 0 \\ \star & \star & \psi_{33} \end{pmatrix} < 0, \quad (3.11)$$

where

$$\begin{aligned} \psi_{11} &= -2\bar{\delta} + \varrho H_1 + \frac{H_2}{(1-\bar{\ell})} + \alpha(\bar{L}^+)^T \bar{L}^+, \quad \psi_{22} = -(\alpha - 1)I, \\ \psi_{33} &= -(1 - \bar{\ell})I. \end{aligned}$$

From Lemma 2.10, (3.11) is equivalent to

$$-2\bar{\delta} + \varrho H_1 + \frac{H_2}{(1-\bar{\ell})} + \alpha(\bar{L}^+)^T \bar{L}^+ + \frac{1}{\alpha-1} \bar{W}^1 (\bar{W}^1)^T + \Pi \Pi^T.$$

Then the SDEs (1.2) is a global dissipative in mean, and the set

$$\Xi_3 \triangleq \left\{ \bar{q} \in \mathbb{Q}^n \setminus |\bar{q}| \leq \frac{\sqrt{\varepsilon_2 |\check{u}|^2 + \varepsilon_1 |\kappa \check{\theta}|^2}}{\sqrt{\varepsilon_1 \varepsilon_2 \lambda_{\min}(\Theta)}} \right\}$$

is globally attractive in mean of (1.2), where $\Theta = 2(1 - \zeta)\bar{\delta} - (\varepsilon_1 + \varepsilon_2)I$, $\zeta \triangleq \inf\{\beta | \Sigma(\beta) = -2\bar{\delta} + \varrho H_1 + \frac{H_2}{(1-\bar{\ell})} + \alpha(\bar{L}^+)^T \bar{L}^+ + \frac{1}{\alpha-1} \bar{W}^1 (\bar{W}^1)^T + \Pi \Pi^T \leq 0\}$ and $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ are selected for $\Theta > 0$.

Remark 3.5. It is worth emphasizing that most existing results on dissipativity or stability of neural networks consider only one or two of the following aspects: stochastic perturbations, fuzzy logic, or quaternion-valued dynamics, but not their simultaneous interaction.

For instance, stochastic neural networks without fuzzy or quaternion structures were studied in [30, 31], where classical Lyapunov-based techniques were employed. Similarly, fuzzy neural networks were investigated in [32, 33], but without considering stochastic effects or quaternion algebra. On the other hand, quaternion-valued neural networks were analyzed in [34, 35] under deterministic settings or without fuzzy components.

In contrast, the present work establishes a unified framework that simultaneously incorporates stochastic disturbances, fuzzy logic operations, and quaternion-valued states. This combination introduces additional coupling effects and analytical difficulties that are not addressed in the above-mentioned works.

Moreover, it is easy to verify that when stochastic terms are removed, the obtained conditions reduce to those in [32, 34], and when fuzzy or quaternion components are omitted, they recover several existing results on stochastic neural networks such as [30]. Therefore, the proposed results can be viewed as a generalization and unification of these earlier contributions.

4 Numerical example

This section considers a reduced-order stochastic neural network with $n = 2$ neurons:
The system is described as follows:

$$\begin{aligned}
 dq_1(\rho) = & \left(-13q_1(\rho) + \sum_{z=1}^2 w_{1z}^1 \phi_z(q_z(\rho)) + \sum_{z=1}^2 w_{1z}^2 \phi_z(q_z(\rho - \ell(\rho))) \right. \\
 & + \sum_{z=1}^2 w_{1z}^3 \theta_z(\rho) + \bigwedge_{z=1}^2 w_{1z}^4 \theta_z(\rho) + \bigwedge_{z=1}^2 w_{1z}^5 \phi_z(q_z(\rho - \ell(\rho))) \\
 & + \bigvee_{z=1}^2 w_{1z}^6 \phi_z(q_z(\rho - \ell(\rho))) + \bigvee_{z=1}^2 w_{1z}^7 \theta_z(\rho) + u_1(\rho) \Big) d\rho \\
 & + \sum_{z=1}^2 \sigma_{1z}(\rho, q_z(\rho), q_z(\rho - \ell(\rho))) d\omega_z(\rho), \tag{4.1}
 \end{aligned}$$

$$\begin{aligned}
 dq_2(\rho) = & \left(-13q_2(\rho) + \sum_{z=1}^2 w_{2z}^1 \phi_z(q_z(\rho)) + \sum_{z=1}^2 w_{2z}^2 \phi_z(q_z(\rho - \ell(\rho))) \right. \\
 & + \sum_{z=1}^2 w_{2z}^3 \theta_z(\rho) + \bigwedge_{z=1}^2 w_{2z}^4 \theta_z(\rho) + \bigwedge_{z=1}^2 w_{2z}^5 \phi_z(q_z(\rho - \ell(\rho))) \\
 & + \bigvee_{z=1}^2 w_{2z}^6 \phi_z(q_z(\rho - \ell(\rho))) + \bigvee_{z=1}^2 w_{2z}^7 \theta_z(\rho) + u_2(\rho) \Big) d\rho \\
 & + \sum_{z=1}^2 \sigma_{2z}(\rho, q_z(\rho), q_z(\rho - \ell(\rho))) d\omega_z(\rho). \tag{4.2}
 \end{aligned}$$

The quaternion-valued state is defined by

$$q(\rho) = q^R(\rho) + iq^I(\rho) + jq^J(\rho) + kq^K(\rho) = (q_1(\rho), q_2(\rho))^T.$$

The weight matrices are given as:

$$\begin{aligned}
 W^1 &= \begin{pmatrix} 0.7 + 0.4i & 0.5 - 0.3j \\ 1.2 + i & 1.1 - 1.4j \end{pmatrix}, & W^2 &= \begin{pmatrix} 1.5 + 0.5i & 0.7 - 0.7j \\ 0.3 + 0.1i & 0.2 + 0.4k \end{pmatrix}, \\
 W^3 &= \begin{pmatrix} 0.004 + 0.003k & 0.003i \\ 0.006k - 0.005i & 0.004 + 0.003j \end{pmatrix}, & W^4 &= \begin{pmatrix} 0.5i & 0.01i - 0.1j \\ 0.01 + 0.3j & 0.7 - i \end{pmatrix}, \\
 W^5 &= \begin{pmatrix} 0.5 + 0.5j & 0.5 - 0.6k \\ 1.1 + 0.2i & 0.2 - 0.5k \end{pmatrix}, & W^6 &= \begin{pmatrix} 0.2 + 0.3i & 0.3 - 0.3k \\ 0.5i & -0.5j \end{pmatrix}, \\
 W^7 &= \begin{pmatrix} 0.01 + 0.1i & 0.02 - 0.01i \\ 0.02 - 0.01j & 0.01i + 0.01j \end{pmatrix}.
 \end{aligned}$$

The stochastic term is defined as:

$$\sigma(\rho, q(\rho), q(\rho - \ell(\rho))) = \text{diag} \left\{ 0.25q_1(\rho) + 0.2q_1(\rho - \ell(\rho)), 0.3q_2(\rho) \right\}.$$

It is easy to verify that

$$\text{trace}(\sigma^T \sigma) \leq q^T(\rho) H_1 q(\rho) + q^T(\rho - \ell(\rho)) H_2 q(\rho - \ell(\rho)),$$

with

$$H_1 = 0.15I, \quad H_2 = 0.1I.$$

The external inputs are:

$$u_1(\rho) = 0.7 \sin(\rho) + i0.9 \sin(\rho), \quad u_2(\rho) = 0.6 \cos(\rho) + i0.3 \cos(\rho).$$

The fuzzy logic inputs are:

$$\theta_1(\rho) = \theta_2(\rho) = 1 + i + j + k.$$

The time-varying delay is given by:

$$\ell(\rho) = 1 - 0.2 \sin(\rho), \quad \tilde{\ell} = 0.2.$$

The activation functions are defined as:

$$\phi_z(q_z(\rho)) = \tanh(q_z^R(\rho)) + i \tanh(q_z^I(\rho)) + j \tanh(q_z^J(\rho)) + k \tanh(q_z^K(\rho)).$$

By selecting $H_1 = 0.15I$, $H_2 = 0.1I$, and $W = I$, the conditions of the main theorem are satisfied. Solving the corresponding LMI using YALMIP yields feasible solutions, ensuring that the system is globally dissipative in the mean sense.

$$\varrho = 207.2633, \quad P = \text{diag}\{28.6398, 28.8684, 28.0675, 29.1381, 28.3254, 28.7979, 28.2689, 29.3150\},$$

$$R_1 = \text{diag}\{207.8130, 3209.9529, 3204.5785, 3196.0183, 3208.0918, 3208.2207, 3205.5111, 397.7156\},$$

$$R_2 = \text{diag}\{139.1092, 123.3823, 135.3758, 117.2573, 137.6138, 124.4974, 136.5543, 119.6461\},$$

$$\tilde{\chi} = \text{diag}\{311.9214, 290.9439, 303.6989, 293.9593, 308.2418, 291.5231, 306.7761, 296.5784\}.$$

Therefore, all conditions of the main theorem are satisfied. Hence, the considered stochastic quaternion-valued neural network is globally dissipative in the mean sense. By estimating the bounded inputs, we obtain

$$\check{u} = (0.7, 0.9, 0, 0, 0.6, 0.3, 0, 0)^T, \quad \check{\theta} = (1, 1, 1, 1, 1, 1, 1, 1)^T.$$

Then,

$$\gamma = \check{u}^T P \check{u} + \check{\theta}^T P \check{\theta} \approx 279.6284.$$

Thus, we obtain the attractive set

$$\Xi = \{\bar{q} \in \mathbb{Q}^2 \mid \bar{q}_1^2 + \bar{q}_2^2 \leq 279.6284\}$$

which is globally attractive.

Remark 4.1. The system parameters used in the numerical examples are selected to satisfy the sufficient conditions of the proposed LMIs and to validate the dissipativity analysis, noting that these choices are not unique.

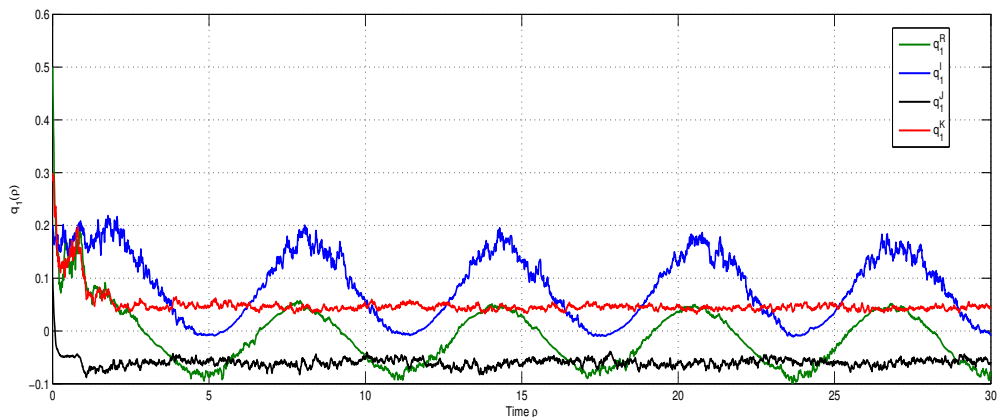


Figure 1. Time evolution of $q_1(\rho)$.

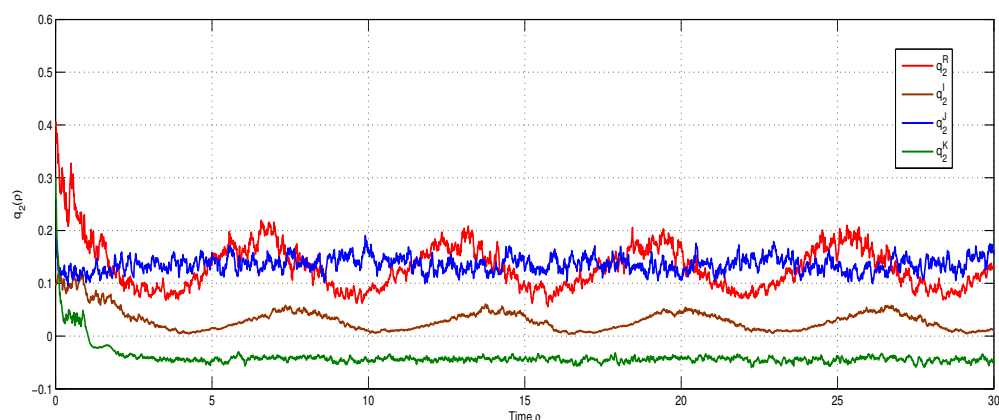


Figure 2. Time evolution of $q_2(\rho)$.

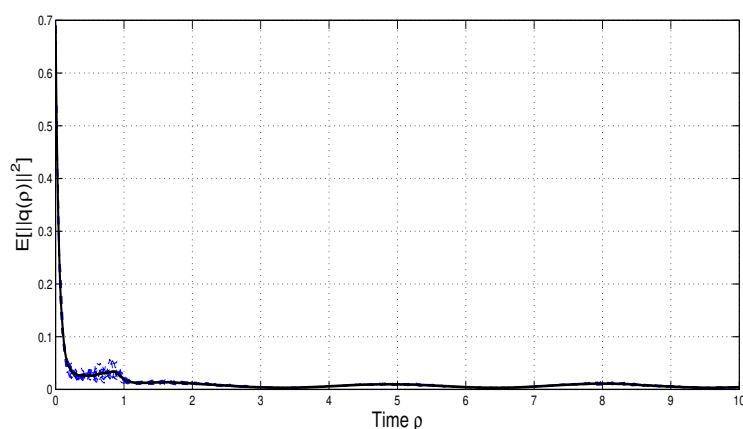


Figure 3. Time evolution of $q_3(\rho)$.

Figures 1–3 illustrate the dynamic behavior of the considered system. Figures 1 and 2 present the time evolution of the quaternion-valued states $q_1(\rho)$ and $q_2(\rho)$, including their real and imaginary components, while Figure 3 depicts the evolution of the mean-square norm $\mathbb{E}[\|q(\rho)\|^2]$. The simulation results clearly demonstrate that all state trajectories remain bounded and converge toward a compact set. The evolution of $\mathbb{E}[\|q(\rho)\|^2]$ confirms the mean-square dissipative behavior of the system. Furthermore, the phase portrait shows that the trajectories are ultimately confined within the attractive set Ξ , which is consistent with and validates the theoretical results.

5 Conclusion and perspective

This study investigates the problem of GD in mean of SNNs. By employing quaternion multiplication, the SNNs are decomposed into four real-valued components, leading to an equivalent real-valued model. With the aid of Lyapunov functionals and LMI tools, we develop new dissipativity conditions that ensure the model's mean GD behavior. The results obtained here extend and generalize those previously established in both the real and complex domains. To support the theoretical developments, a numerical example together with simulation results were provided, showing the practicality and efficiency of the proposed approach. Future research may explore mean GD in neural networks under more general stochastic processes, such as Lévy jump processes or Markov jump processes.

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