

Projective Lag Quasi Synchronization of Fractional-Order Complex-Inertial Fuzzy Neural Networks With Time-Varying Delays

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Abstract. This study addresses the projective lag quasi-synchronization and quasi-projective synchronization challenge within delayed Caputo fractional-order complex inertial fuzzy neural networks. To facilitate the analysis, the framework introduced in this paper is decomposed into two subsystems through an appropriate transformation of variables. By employing fractional differential inequalities and suitable Lyapunov-based techniques, new synchronization criteria are established under a nonlinear control strategy. The proposed results extend previous work by introducing lag synchronization and fuzzy dynamics, providing a more general and effective control framework. An explicit estimation of the synchronization error bound is derived, and the theoretical findings are illustrated through additional numerical simulations, demonstrating the effectiveness of the proposed approach.

1 Introduction

Artificial neural networks are simplified mathematical models inspired by how the human brain processes information. By mimicking the structure of biological neural systems, they can handle and analyze complex data. In recent years, researchers have increasingly studied the behavior and dynamics of these networks to improve their performance in real-world tasks such as image and signal processing, memory storage, and communication systems [1, 2, 3].

Since Huygens' discovery that two pendulum clocks can synchronize their movements [4], synchronization has attracted considerable research interest [5, 6, 7, 8]. This concept helps explain various natural phenomena and has also been applied to artificial neural networks to model brain-like activities. It unifies several synchronization schemes, including complete synchronization, anti-synchronization, and projective synchronization, thereby offering a broader control framework [9, 10, 11, 12, 13].

In parallel, the development of fractional-order dynamical systems has provided powerful modeling tools to describe memory and hereditary behaviors observed in real-world processes. The incorporation of fractional derivatives into neural networks enriches their dynamical behavior and improves their realism and accuracy in modeling long-term dependencies [1, 2, 3].

Furthermore, complex-valued inertial neural networks introduce both complex-valued states and inertial terms (second-order dynamics), thereby enhancing convergence speed, stability, and representational capacity for oscillatory phenomena. Inertial models of fractional order are expressed through fractional differential equations that involve two distinct orders of state derivatives. Such formulations are widely regarded as capable of producing rich chaotic dynamics and bifurcations. Several researchers have explored the dynamics of inertial recurrent neural networks, including studies on stability, periodicity, and convergence properties such as [5, 6, 7, 14, 15].

Fuzzy neural networks provide an effective mathematical framework for describing nonlin-

ear systems with parameter uncertainty and imprecise information by integrating fuzzy inference mechanisms into neural dynamics. In such models, synaptic weights, activation functions, or system parameters are typically characterized through fuzzy rules, allowing the network to switch among multiple subsystems depending on the system state. When combined with fractional-order dynamics, fuzzy neural networks are capable of capturing both long-term memory effects and uncertain nonlinear behaviors in a unified formulation [1, 6, 16]. Moreover, the inclusion of inertial terms in complex-valued fractional-order fuzzy neural networks introduces second-order dynamics that significantly enrich the system behavior but also increase analytical complexity. Consequently, the study of synchronization for fractional-order complex-inertial fuzzy neural networks with time-varying delays requires the development of new mathematical tools and stability criteria, which motivates the analysis presented in this paper.

In practical applications, a variety of delay mechanisms, including constant, time-varying, and distributed delays, are often inevitable due to limited communication bandwidth, processing times, and network induced latencies. Each type of delay introduces distinct challenges: constant delays can shift system dynamics, time-varying delays may lead to instability or degraded synchronization, and distributed delays can significantly affect convergence behavior. Accurately modeling and compensating for these delays is therefore essential to ensure the robustness and performance of control and synchronization strategies [18, 5, 6, 7, 16, 17].

Projective synchronization is a specific form of synchronization in which the master–slave systems achieve synchronization with a certain scaling factor through appropriate control strategies. This type of synchronization has practical importance, particularly in extending binary communication to M-level digital communication, enabling faster transmission. Additionally, by adjusting the scaling factor, projective synchronization can be transformed into complete synchronization. However, due to the presence of inevitable disturbances in real systems, the concept of quasi-synchronization, where synchronization is achieved approximately, is considered more realistic and applicable. Hence, extending projective synchronization to quasi-projective synchronization becomes more meaningful. Given these advantages, studying projective synchronization in neural networks (NNs) holds both theoretical and practical significance. Numerous researchers have explored this topic [19, 20, 21, 22, 23, 24, 25]. For instance, [22] explored projective synchronization in complex-valued neural networks by developing an adaptive controller grounded in Lyapunov stability theory, while [23] examined the synchronization of delayed neural networks subject to nonidentical parameters, employing the Halanay inequality as a theoretical tool. In [24], the authors studied quasi-synchronization for fractional-order quaternion-valued memristive neural networks via an event-triggered control strategy.

Quasi-projective synchronization (QPS) is a generalized form of projective synchronization, where the response system synchronizes with a scaled and linearly transformed version of the drive system. Unlike complete synchronization or standard projective synchronization, which require either identical trajectories or a fixed scaling factor QPS allows different components of the system to synchronize with different scaling relationships. This added flexibility makes QPS particularly suitable for complex systems, such as chaotic or fractional-order neural networks, where strict synchronization may be difficult or undesirable. QPS can enhance control capabilities and broaden the range of applicable synchronization strategies in secure communications, signal processing, and other engineering applications [20, 21].

Projective Lag Quasi-Synchronization (PLQS) is a form of synchronization in coupled dynamical systems where the state of a response system follows a delayed and scaled version of the drive system. Unlike complete or traditional projective synchronization, PLQS introduces both a time lag and a scaling factor, allowing for more flexible alignment between the master and slave systems. This type of synchronization is particularly useful in practical applications where transmission delays or inherent system lags exist, such as in secure communications, control of delayed neural networks, and signal processing. By combining lag and projective factors, PLQS enables robust synchronization even in complex, time-delayed, or nonlinear systems [26, 27]. The introduction of projective lag quasi-synchronization (PLQS) is strongly motivated by both the mathematical structure and practical characteristics of the considered neural network model. In delayed Caputo fractional-order complex inertial fuzzy neural networks, exact synchronization is generally difficult to achieve because the coexistence of fractional memory effects, inertial dynamics, fuzzy uncertainties, and time-varying delays produces highly complex transient behaviors. The projective synchronization component allows the response system to track the drive

system up to a scaling factor, which is more flexible and suitable in applications such as secure communications and signal amplification where proportional state matching is sufficient. The lag synchronization component is naturally required because time-varying delays and inertial effects introduce unavoidable phase shifts and transmission lags between the master and slave systems. Meanwhile, the quasi-synchronization framework reflects the realistic situation in which exact asymptotic synchronization cannot be guaranteed due to approximation errors, parameter uncertainties, and nonlinear fuzzy perturbations. Therefore, combining projective, lag, and quasi synchronization into a unified PLQS scheme provides a more general, realistic, and practically meaningful synchronization framework for the studied neural network model than treating each synchronization type separately.

Although synchronization problems for neural networks have been extensively investigated in recent years, the existing literature mainly treats only partial aspects of the present model [20, 21, 22, 23, 25]. For example, some works focus exclusively on fractional-order neural networks without considering inertial effects, while others study complex-valued neural networks but neglect fuzzy dynamics or time-varying delays. Likewise, several papers address inertial neural networks or fuzzy neural networks separately, yet very few integrate these features into a unified framework. In particular, most available results on projective, quasi-projective, or lag synchronization concern either standard real-valued neural networks, fractional-order complex-valued neural networks without inertial terms, or delayed inertial systems without fuzzy rules. To the best of our knowledge, there are almost no published results dealing simultaneously with the combination of fractional-order dynamics, complex-valued states, inertial effects, fuzzy logic mechanisms, and time-varying delays in a single neural network model, especially under the projective lag quasi-synchronization (PLQS) scheme. Therefore, the present paper fills this gap by establishing PLQS criteria for delayed Caputo fractional-order complex inertial fuzzy neural networks, which constitutes a substantial extension of the current literature. Compared with previous studies, the proposed model is more general and realistic, since it captures memory effects, oscillatory inertial behavior, uncertainty represented by fuzzy rules, and delay-dependent interactions within one unified synchronization framework.

Building on the preceding discussions, this study focuses on the QPS of delayed fractional-order complex-inertial neural networks. By constructing appropriate Lyapunov functionals and employing fractional calculus, we derive sufficient conditions under which QPS can be achieved. The key contributions of this work are as follows:

- A novel delayed fractional-order complex-inertial fuzzy neural network model with time-varying delays is investigated.
- New sufficient conditions for PLQS are established via Lyapunov functional techniques and fractional differential inequalities.
- A non-decomposition analysis framework is developed to handle the complex-valued inertial fuzzy system directly.
- Suitable controllers are designed to guarantee synchronization, and explicit error bounds are derived.
- Numerical simulations are provided to verify the theoretical findings.

The structure of the paper is as follows: Section 2 introduce the necessary definitions, lemmas, and a detailed description of the proposed model. In Section 3, the QPS and quasi-synchronization of delayed fractional-order complex-inertial neural networks are analyzed, and the corresponding synchronization conditions are derived. Section 4 verifies the theoretical findings through numerical simulations. Finally, Section 5 ends the study with a recap of the main results.

This manuscript is an extended and significantly improved version of the paper entitled "Quasi-Projective Synchronization of Fractional-Order Complex-Inertial Neural Networks with Time-Varying Delays" presented at ICRAMI 2025 (Tunisia, November 2025). Compared to the conference version [28], this paper introduces projective lag quasi-synchronization and quasi-projective synchronization in the context of fractional-order complex inertial fuzzy neural networks, incorporates fuzzy dynamics, develops new Lyapunov-based synchronization criteria, provides a more explicit estimation of the synchronization error bound, and includes additional numerical simulations to further validate the proposed results.

2 Preliminaries

Definition 2.1. [29] The Caputo-type fractional derivative corresponding to $v(s)$ can be expressed as:

$$\mathcal{D}_s^\alpha v(s) = \frac{1}{\Gamma(1-\alpha)} \int_0^s (s-\ell)^{-\alpha} v'(\ell) d\ell, s > 0$$

where $0 < \alpha < 1$.

Lemma 2.2. [30] Let $\xi(s)$ be a continuously analytic function over the complex field. Then, $\forall 0 < \alpha < 1$, we get:

$$\mathcal{D}_s^\alpha (\overline{\xi(s)} \xi(s)) \leq \overline{\xi(s)} \mathcal{D}_s^\alpha \xi(s) + \xi(s) \mathcal{D}_s^\alpha \overline{\xi(s)}.$$

Lemma 2.3. [30] For any $v_1, v_2 \in \mathbb{C}$, the inequality holds

$$v_1 \overline{v_2} + \overline{v_1} v_2 \leq \zeta v_1 \overline{v_1} + \frac{1}{\zeta} v_2 \overline{v_2}, \quad \text{where } \zeta > 0$$

Lemma 2.4. [31] Let $\tilde{\omega}(t) \in \mathbf{C}([-\rho, +\infty), \mathbb{R}_+)$, it satisfies

$$\mathcal{D}_t^\alpha \tilde{\omega}(t) \leq -\theta_1 \tilde{\omega}(t) + \theta_2,$$

for any $\theta_1 > 0$ and $\theta_2 > 0$, then there exist $N_1 \geq 1$ such that

$$\tilde{\omega}(t) \leq \theta_3 N_1 e^{-\theta_1 t} + \frac{\theta_2}{\theta_1} \quad \text{where } \theta_3 = \max_{t \in [-\rho, 0]} |\tilde{\omega}(t)|.$$

Remark 2.5. The exponential estimate derived in Lemma 2.4 is based on the Mittag-Leffler decay property for Caputo fractional differential inequalities. More precisely, we have the following Lemma:

Lemma 2.6. [32] Let $0 < \alpha < 1$, then for any positive constant θ_1 , there exist constant $N_1 \geq 1$, such that

$$\mathbb{E}(-\theta_1 t^\alpha) \leq N_1 e^{-\theta_1 t}.$$

We focus on the system described by:

$$\begin{aligned} \mathcal{D}_\zeta^{2\alpha} x_l(\zeta) = & -a_l \mathcal{D}_\zeta^\alpha x_l(\zeta) - b_l x_l(\zeta) + \sum_{\mathfrak{h}=1}^{\varpi} c_{l\mathfrak{h}} f_{\mathfrak{h}}(x_{\mathfrak{h}}(\zeta)) + \sum_{\mathfrak{h}=1}^{\varpi} d_{l\mathfrak{h}} f_{\mathfrak{h}}(x_{\mathfrak{h}}(\zeta - \rho_{\mathfrak{h}}(\zeta))) \\ & + \sum_{\mathfrak{h}=1}^{\varpi} \mathcal{W}_{l\mathfrak{h}} \varsigma_{\mathfrak{h}}(\zeta) + \bigwedge_{\mathfrak{h}=1}^{\varpi} \mathcal{Q}_{l\mathfrak{h}} f_{\mathfrak{h}}(x_{\mathfrak{h}}(\zeta - \rho_{\mathfrak{h}}(\zeta))) + \bigwedge_{\mathfrak{h}=1}^{\varpi} \mathcal{K}_{l\mathfrak{h}} \varsigma_{\mathfrak{h}}(\zeta) \\ & + \bigvee_{\mathfrak{h}=1}^{\varpi} \mathcal{P}_{l\mathfrak{h}} f_{\mathfrak{h}}(x_{\mathfrak{h}}(\zeta - \rho_{\mathfrak{h}}(\zeta))) + \bigvee_{\mathfrak{h}=1}^{\varpi} \mathcal{S}_{l\mathfrak{h}} \varsigma_{\mathfrak{h}}(\zeta) + I_l(\zeta). \end{aligned} \quad (2.1)$$

Where $0 < \alpha < 1$, $x_l(\cdot) \in \mathbb{C}$ stands for the state variables, ϖ is the nodes's number in the neural network, $a_l > 0$, $b_l > 0$ are constants, $f_{\mathfrak{h}}(\cdot)$ are the activations functions with $f_{\mathfrak{h}}(0) = 0$, $\mathcal{W}_{l\mathfrak{h}}$ denote the feed-forward template; $\mathcal{Q}_{l\mathfrak{h}}$, $\mathcal{P}_{l\mathfrak{h}}$, $\mathcal{K}_{l\mathfrak{h}}$ and $\mathcal{S}_{l\mathfrak{h}}$ stand elements of the fuzzy feedback MIN template, fuzzy feedback MAX template, fuzzy feed-forward MIN template and fuzzy feed-forward MAX template, respectively; $\bigwedge$ and $\bigvee$ denote the fuzzy AND and fuzzy OR operation, respectively; $\varsigma_{\mathfrak{h}}(\cdot)$ is the input with $\varsigma_{\mathfrak{h}}(0) = 0$ and $|\varsigma_{\mathfrak{h}}(\zeta)| \leq \varsigma_{\mathfrak{h}}^+$; $\rho_{\mathfrak{h}}(\cdot)$ is the time-varying delay satisfying $0 \leq \rho_{\mathfrak{h}}(\zeta) \leq \rho$, $\forall \zeta \geq 0$, $\mathfrak{h} = 1, 2, \dots, \varpi$, where $\rho_{\mathfrak{h}}(\cdot)$ is continuous on $[0, +\infty)$, and $\rho = \max_{1 \leq \mathfrak{h} \leq \varpi} \left\{ \sup_{\zeta \in \mathbb{R}_+} \rho_{\mathfrak{h}}(\zeta) \right\} < +\infty$. $c_{l\mathfrak{h}}, d_{l\mathfrak{h}} \in \mathbb{C}$ denote the inter-neuronal connection weights, $I_l(\cdot) \in \mathbb{C}$ the externally applied input to the neurons satisfying $|I_l(\zeta)| \leq I_l^+$.

The system (2.1) is defined with the initial conditions below:

$$\begin{cases} x_l(u) &= \varphi_l(u), \\ \mathcal{D}_\zeta^\alpha x_l(u) &= \psi_l(u), \quad -\rho \leq u \leq 0, \quad l = 1, \dots, \varpi \end{cases}$$

where $\varphi_l(\cdot)$ and $\psi_l(\cdot)$ are complex-valued continuous functions on $]-\rho, 0]$.

Remark 2.7. In the proposed fractional-order complex-inertial fuzzy neural network model, the inertial term introduces second-order dynamic effects that reflect acceleration, damping, and oscillatory transient behaviors, making the model more suitable for describing realistic neural processes than standard first-order systems. Meanwhile, the fuzzy rules represent uncertainty and nonlinear switching through AND/ OR mechanisms, allowing the network to combine multiple local subsystems according to state-dependent membership functions. The integration of inertial dynamics and fuzzy logic into a fractional-order complex-valued framework with time-varying delays provides a more general and realistic neural network model, while also increasing the complexity of the synchronization analysis.

Remark 2.8. In this paper, the Caputo fractional derivative is employed to describe the dynamics of the considered fuzzy neural network model. The choice of the Caputo operator is motivated by its well-established physical interpretability and its suitability for initial value problems, since it permits the use of classical initial conditions expressed in terms of integer-order derivatives.

The associated response model takes the form:

$$\begin{aligned} \mathcal{D}_{\zeta}^{2\alpha} y_l(\zeta) = & -a_l \mathcal{D}_{\zeta}^{\alpha} y_l(\zeta) - b_l y_l(\zeta) + \sum_{h=1}^{\varpi} c_{lh} f_h(y_h(\zeta)) + \sum_{h=1}^{\varpi} d_{lh} f_h(y_h(\zeta - \rho_h(\zeta))) \\ & + \sum_{h=1}^{\varpi} \mathcal{W}_{lh} s_h(\zeta) + \bigwedge_{h=1}^{\varpi} \mathcal{Q}_{lh} f_h(y_h(\zeta - \rho_h(\zeta))) + \bigwedge_{h=1}^{\varpi} \mathcal{K}_{lh} s_h(\zeta) \\ & + \bigvee_{h=1}^{\varpi} \mathcal{P}_{lh} f_h(y_h(\zeta - \rho_h(\zeta))) + \bigvee_{h=1}^{\varpi} \mathcal{S}_{lh} s_h(\zeta) + I_l(\zeta) + U_l(\zeta). \end{aligned} \quad (2.2)$$

with $U_l(\zeta)$ is the controller. For $l = 1, \dots, \varpi$, the system (2.2) is considered with the following initial conditions:

$$\begin{cases} y_l(u) &= \tilde{\varphi}_l(s), \\ \mathcal{D}_{\zeta}^{\alpha} y_l(u) &= \tilde{\psi}_l(u), \quad -\rho \leq u \leq 0, \end{cases}$$

where $\tilde{\varphi}_l(\cdot)$ and $\tilde{\psi}_l(\cdot)$ are complex-valued continuous functions on $]-\rho, 0]$.

Consider the variable substitution defined by : $s_l(\zeta) = \mathcal{D}_{\zeta}^{\alpha} x_l(\zeta) + x_l(\zeta)$, then equation (2.1) is equivalently written as:

$$\left\{ \begin{aligned} \mathcal{D}_{\zeta}^{\alpha} x_l(\zeta) &= s_l(\zeta) - x_l(\zeta) \\ \mathcal{D}_{\zeta}^{\alpha} s_l(\zeta) &= -(a_l - 1)s_l(\zeta) - \sigma_l x_l(\zeta) + \sum_{h=1}^{\varpi} c_{lh} f_h(x_h(\zeta)) \\ &+ \sum_{h=1}^{\varpi} d_{lh} f_h(x_h(\zeta - \rho_h(\zeta))) + \sum_{h=1}^{\varpi} \mathcal{W}_{lh} s_h(\zeta) + \bigwedge_{h=1}^{\varpi} \mathcal{Q}_{lh} f_h(x_h(\zeta - \rho_h(\zeta))) \\ &+ \bigwedge_{h=1}^{\varpi} \mathcal{K}_{lh} s_h(\zeta) + \bigvee_{h=1}^{\varpi} \mathcal{P}_{lh} f_h(x_h(\zeta - \rho_h(\zeta))) + \bigvee_{h=1}^{\varpi} \mathcal{S}_{lh} s_h(\zeta) + I_l(\zeta) \end{aligned} \right. \quad (2.3)$$

with $\sigma_l =: b_l - a_l + 1$

For the response system, let $h_l(\zeta) = \mathcal{D}_\zeta^\alpha y_l(\zeta) + y_l(\zeta)$. Hence, we get

$$\left\{ \begin{array}{l} \mathcal{D}_\zeta^\alpha y_l(\zeta) = h_l(\zeta) - y_l(\zeta) + \tilde{\mathcal{U}}_l(\zeta) \\ \mathcal{D}_\zeta^\alpha h_l(\zeta) = -(a_l - 1)h_l(\zeta) - \sigma_l y_l(\zeta) + \sum_{h=1}^{\varpi} c_{lh} f_h(y_h(\zeta)) \\ \quad + \sum_{h=1}^{\varpi} d_{lh} f_h(y_h(\zeta - \rho_h(\zeta))) + \bigwedge_{h=1}^{\varpi} \mathcal{Q}_{lh} f_h(y_h(\zeta - \rho_h(\zeta))) \\ \quad + \bigwedge_{h=1}^{\varpi} \mathcal{K}_{lh} s_h(\zeta) + \bigvee_{h=1}^{\varpi} \mathcal{P}_{lh} f_h(y_h(\zeta - \rho_h(\zeta))) + \sum_{h=1}^{\varpi} \mathcal{W}_{lh} s_h(\zeta) \\ \quad + \bigvee_{h=1}^{\varpi} \mathcal{S}_{lh} s_h(\zeta) + I_l(\zeta) + \mathcal{U}_l(\zeta) \end{array} \right. \quad (2.4)$$

with $\mathcal{U}_l(\zeta)$ and $\tilde{\mathcal{U}}_l(\zeta)$ are the controllers.

Let the projective lag synchronization error between the systems (2.3) and (2.4) be $e_l(\zeta) = y_l(\zeta) - \lambda x_l(\zeta - \gamma)$ and $r_l(\zeta) = h_l(\zeta) - \lambda s_l(\zeta - \gamma)$ where γ is the lag delay and λ is the projective coefficient. Noted that the variable substitution defined by : $s_l(\zeta - \gamma) = \mathcal{D}_\zeta^\alpha x_l(\zeta - \gamma) + x_l(\zeta - \gamma)$, then equation (2.1) is equivalently written as:

$$\left\{ \begin{array}{l} \mathcal{D}_\zeta^\alpha x_l(\zeta - \gamma) = s_l(\zeta - \gamma) - x_l(\zeta - \gamma) \\ \mathcal{D}_\zeta^\alpha s_l(\zeta - \gamma) = -(a_l - 1)s_l(\zeta - \gamma) - \sigma_l x_l(\zeta - \gamma) + \sum_{h=1}^{\varpi} c_{lh} f_h(x_h(\zeta)) \\ \quad + \sum_{h=1}^{\varpi} d_{lh} f_h(x_h(\zeta - \rho_h(\zeta))) + \sum_{h=1}^{\varpi} \mathcal{W}_{lh} s_h(\zeta) + \bigwedge_{h=1}^{\varpi} \mathcal{Q}_{lh} f_h(x_h(\zeta - \rho_h(\zeta))) \\ \quad + \bigwedge_{h=1}^{\varpi} \mathcal{K}_{lh} s_h(\zeta) + \bigvee_{h=1}^{\varpi} \mathcal{P}_{lh} f_h(x_h(\zeta - \rho_h(\zeta))) + \bigvee_{h=1}^{\varpi} \mathcal{S}_{lh} s_h(\zeta) + I_l(\zeta) \end{array} \right. \quad (2.5)$$

Consequently, the corresponding error dynamics is given by:

$$\left\{ \begin{array}{l} \mathcal{D}_\zeta^\alpha r_l(\zeta) = -(a_l - 1)r_l(\zeta) - \sigma_l e_l(\zeta) + \sum_{h=1}^{\varpi} c_{lh} F_h + \sum_{h=1}^{\varpi} d_{lh} \tilde{F}_h + \bigwedge_{h=1}^{\varpi} \mathcal{Q}_{lh} \tilde{F}_h \\ \quad + \bigvee_{h=1}^{\varpi} \mathcal{P}_{lh} \tilde{F}_h + \mathcal{U}_l(\zeta) + (1 - \lambda)\mathfrak{I}_l(\zeta) \\ \mathcal{D}_\zeta^\alpha e_l(\zeta) = -e_l(\zeta) + r_l(\zeta) + \tilde{\mathcal{U}}_l(\zeta). \end{array} \right. \quad (2.6)$$

with

$$\begin{aligned} F_h &= f_h(y_h(\zeta)) - \lambda f_h(x_h(\zeta - \gamma)) \\ \tilde{F}_h &= f_h(y_h(\zeta - \rho_h(\zeta))) - \lambda f_h(x_h(\zeta - \rho_h(\zeta) - \gamma)) \\ \mathfrak{I}_l(\zeta) &= \sum_{h=1}^{\varpi} \mathcal{W}_{lh} s_h(\zeta) + \bigwedge_{h=1}^{\varpi} \mathcal{K}_{lh} s_h(\zeta) + \bigvee_{h=1}^{\varpi} \mathcal{S}_{lh} s_h(\zeta) + I_l(\zeta) \end{aligned}$$

Definition 2.9. [33] The systems (2.1) and (2.2) achieve *Lag quasi-projective synchronization*, if there exists a constant time delay $\gamma > 0$ such that

$$\lim_{\zeta \rightarrow +\infty} \|\Theta(\zeta)\| \leq \varepsilon$$

is satisfied for some $\varepsilon > 0$, with

$$\Theta(\zeta) = (r_1(\zeta), \dots, r_\varpi(\zeta), e_1(\zeta), \dots, e_\varpi(\zeta))^T \text{ and } \|\Theta(\zeta)\|^2 = \sum_{l=1}^{\varpi} \overline{r_l(\zeta)} r_l(\zeta) + \sum_{l=1}^{\varpi} \overline{e_l(\zeta)} e_l(\zeta).$$

Definition 2.10. [34] The systems (2.1) and (2.2) achieve *quasi projective synchronization* provided that

$$\lim_{\zeta \rightarrow +\infty} \|y(\zeta) - \lambda x_l(\zeta)\| < \varepsilon \text{ and } \lim_{\zeta \rightarrow +\infty} \|h_l(\zeta) - \lambda s_l(\zeta)\| < \varepsilon$$

for some $\varepsilon > 0$.

Remark 2.11. Quasi-projective synchronization may find practical applications. In secure communication systems, two dynamical systems (master and slave) can be employed to evolve under quasi-projective synchronization. The message is embedded within the dynamics of the master system and recovered on the slave side through quasi-projective synchronization. The fact that synchronization is "projective" (with a fixed scaling factor) or "quasi-projective" (with a varying factor) increases the difficulty for an attacker to decode the signal, thereby enhancing security.

Remark 2.12. According to Definition 2.10, when the aforementioned projective coefficient is set to $\lambda = 1$, then we obtain quasi-synchronization between systems (2.1) and (2.2). Moreover, when $\varepsilon = 0$, the complete synchronization between (2.1) and (2.2) is attained.

Assumption:

($\mathcal{A}_1$) For $\hbar, \aleph \in \mathbb{C}, \exists L_{\hbar}, B_{\hbar}^+ > 0$ satisfying

$$|f_{\hbar}(\hbar(\zeta)) - f_{\hbar}(\aleph(\zeta))| \leq L_{\hbar}|\hbar - \aleph| \quad \text{and} \quad |f_{\hbar}(\hbar(\zeta))| \leq B_{\hbar}^+.$$

Lemma 2.13. [1] Let $x_{\hbar}, y_{\hbar}, \mathcal{P}_{\mathfrak{l}\hbar}, \mathcal{Q}_{\mathfrak{l}\hbar} \in \mathbb{C}, f_{\hbar} : \mathbb{C} \rightarrow \mathbb{R}$ be continuous functions, and $\hbar, \mathfrak{l} = 1, 2, \dots, \varpi$, then the above inequalities hold

$$\begin{aligned} \left| \bigwedge_{\hbar=1}^{\varpi} \mathcal{Q}_{\mathfrak{l}\hbar} f_{\hbar}(x_{\hbar}(\zeta)) - \bigwedge_{\hbar=1}^{\varpi} \mathcal{Q}_{\mathfrak{l}\hbar} f_{\hbar}(y_{\hbar}(\zeta)) \right| &\leq \sum_{\hbar=1}^{\varpi} |\mathcal{Q}_{\mathfrak{l}\hbar}| |f_{\hbar}(x_{\hbar}(\zeta)) - f_{\hbar}(y_{\hbar}(\zeta))|, \\ \left| \bigvee_{\hbar=1}^{\varpi} \mathcal{P}_{\mathfrak{l}\hbar} f_{\hbar}(x_{\hbar}(\zeta)) - \bigvee_{\hbar=1}^{\varpi} \mathcal{P}_{\mathfrak{l}\hbar} f_{\hbar}(y_{\hbar}(\zeta)) \right| &\leq \sum_{\hbar=1}^{\varpi} |\mathcal{P}_{\mathfrak{l}\hbar}| |f_{\hbar}(x_{\hbar}(\zeta)) - f_{\hbar}(y_{\hbar}(\zeta))|, \end{aligned}$$

3 Main results

To fulfill the quasi-projective synchronization, the controllers are selected as:

$$\begin{cases} \mathcal{U}_{\mathfrak{l}}(\zeta) &= -\mu_{\mathfrak{l}} r_{\mathfrak{l}}(\zeta) \\ \tilde{\mathcal{U}}_{\mathfrak{l}}(\zeta) &= -\tilde{\mu}_{\mathfrak{l}} e_{\mathfrak{l}}(\zeta) \end{cases} \quad (3.1)$$

where $\mu_{\mathfrak{l}}$ and $\tilde{\mu}_{\mathfrak{l}}$ are positive constants.

Theorem 3.1. Systems (2.1) and (2.2) realise the projective Lag quasi-synchronization, under Assumption ($\mathcal{A}_1$), the controller (3.1) and, if the following inequalities hold

$$\begin{aligned} \beta_1 &= \min_{1 \leq \mathfrak{l} \leq \varpi} \left\{ 2|a_{\mathfrak{l}} - 1| + \sigma_{\mathfrak{l}}^2 - 2 \sum_{\hbar=1}^{\varpi} |c_{\mathfrak{l}\hbar}|^2 - 3\varpi - 1 - |1 - \lambda|^2 + 2\mu_{\mathfrak{l}} \right\} > 0 \\ \beta_2 &= \min_{1 \leq \mathfrak{l} \leq \varpi} \left\{ 2 - \varpi L_{\mathfrak{l}}^2 + 2\tilde{\mu}_{\mathfrak{l}} \right\} > 0, \\ \beta &= \min\{\beta_1, \beta_2\}, \\ \chi &= 2\varpi \sum_{\hbar=1}^{\varpi} (1 + \bar{\lambda}\lambda) B_{\hbar}^{+2} + \sum_{\mathfrak{l}=1}^{\varpi} \mathfrak{I}_{\mathfrak{l}}^{+2} + 2 \sum_{\mathfrak{l}=1}^{\varpi} \sum_{\hbar=1}^{\varpi} |d_{\mathfrak{l}\hbar}|^2 B_{\hbar}^{+2} + \sum_{\mathfrak{l}=1}^{\varpi} \sum_{\hbar=1}^{\varpi} |d_{\mathfrak{l}\hbar}|^2 |B_{\hbar}^+ + \lambda B_{\hbar}^+|^2 \\ &\quad + 2 \sum_{\mathfrak{l}=1}^{\varpi} \sum_{\hbar=1}^{\varpi} |\mathcal{Q}_{\mathfrak{l}\hbar}|^2 B_{\hbar}^{+2} + \sum_{\mathfrak{l}=1}^{\varpi} \sum_{\hbar=1}^{\varpi} |\mathcal{Q}_{\mathfrak{l}\hbar}|^2 |B_{\hbar}^+ + \lambda B_{\hbar}^+|^2 + 2 \sum_{\mathfrak{l}=1}^{\varpi} \sum_{\hbar=1}^{\varpi} |\mathcal{P}_{\mathfrak{l}\hbar}|^2 B_{\hbar}^{+2} \\ &\quad + \sum_{\mathfrak{l}=1}^{\varpi} \sum_{\hbar=1}^{\varpi} |\mathcal{P}_{\mathfrak{l}\hbar}|^2 |B_{\hbar}^+ + \lambda B_{\hbar}^+|^2. \end{aligned} \quad (3.2)$$

Proof. The Lyapunov function is chosen as:

$$V(\zeta) = \sum_{\mathfrak{l}=1}^{\varpi} \overline{r_{\mathfrak{l}}(\zeta)} r_{\mathfrak{l}}(\zeta) + \sum_{\mathfrak{l}=1}^{\varpi} \overline{e_{\mathfrak{l}}(\zeta)} e_{\mathfrak{l}}(\zeta). \quad (3.3)$$

From Lemma 2.2, it follow that

$$\begin{aligned}
 \mathcal{D}_\zeta^\alpha V(\zeta) &\leq \sum_{l=1}^{\bar{w}} \left[\overline{r_l(\zeta)} \mathcal{D}_\zeta^\alpha r_l(\zeta) + r_l(\zeta) \mathcal{D}_\zeta^\alpha \overline{r_l(\zeta)} \right] + \sum_{l=1}^{\bar{w}} \left[\overline{e_l(\zeta)} \mathcal{D}_\zeta^\alpha e_l(\zeta) + e_l(\zeta) \mathcal{D}_\zeta^\alpha \overline{e_l(\zeta)} \right] \\
 &\leq \sum_{l=1}^{\bar{w}} \overline{r_l(\zeta)} \left[-(a_l - 1)r_l(\zeta) - \sigma_l e_l(\zeta) + \sum_{h=1}^{\bar{w}} c_{lh} F_h + \sum_{h=1}^{\bar{w}} d_{lh} \tilde{F}_h + \bigwedge_{h=1}^{\bar{w}} \mathcal{Q}_{lh} \tilde{F}_h \right. \\
 &\quad \left. + \bigvee_{h=1}^{\bar{w}} \mathcal{P}_{lh} \tilde{F}_h + \mathcal{U}_l(\zeta) + (1 - \lambda) \mathfrak{J}_l(\zeta) \right] + \sum_{l=1}^{\bar{w}} r_l(\zeta) \left[-(-1 + a_l) \overline{r_l(\zeta)} \right. \\
 &\quad \left. - \sigma_l \overline{e_l(\zeta)} + \sum_{h=1}^{\bar{w}} \overline{c_{lh} F_h} + \sum_{h=1}^{\bar{w}} \overline{d_{lh} \tilde{F}_h} + \bigwedge_{h=1}^{\bar{w}} \overline{\mathcal{Q}_{lh} \tilde{F}_h} + \bigvee_{h=1}^{\bar{w}} \overline{\mathcal{P}_{lh} \tilde{F}_h} + \overline{\mathcal{U}_l(\zeta)} + \overline{(1 - \lambda) \mathfrak{J}_l(\zeta)} \right] \\
 &\quad + \sum_{l=1}^{\bar{w}} \overline{e_l(\zeta)} \left[r_l(\zeta) - e_l(\zeta) + \tilde{\mathcal{U}}_l(\zeta) \right] + \sum_{l=1}^{\bar{w}} e_l(\zeta) \left[-\overline{e_l(\zeta)} + \overline{r_l(\zeta)} + \overline{\tilde{\mathcal{U}}_l(\zeta)} \right] \\
 &\leq - \sum_{l=1}^{\bar{w}} (a_l - 1) \overline{r_l(\zeta)} r_l(\zeta) - \sum_{l=1}^{\bar{w}} \sigma_l \overline{r_l(\zeta)} e_l(\zeta) + \sum_{l=1}^{\bar{w}} \sum_{h=1}^{\bar{w}} c_{lh} \overline{r_l(\zeta)} F_h + \sum_{l=1}^{\bar{w}} \sum_{h=1}^{\bar{w}} d_{lh} \overline{r_l(\zeta)} \tilde{F}_h \\
 &\quad + \sum_{l=1}^{\bar{w}} \bigwedge_{h=1}^{\bar{w}} \mathcal{Q}_{lh} \overline{r_l(\zeta)} \tilde{F}_h + \sum_{l=1}^{\bar{w}} \bigvee_{h=1}^{\bar{w}} \mathcal{P}_{lh} \overline{r_l(\zeta)} \tilde{F}_h + \sum_{l=1}^{\bar{w}} \overline{r_l(\zeta)} (1 - \lambda) \mathfrak{J}_l(\zeta) + \sum_{l=1}^{\bar{w}} \overline{r_l(\zeta)} \mathcal{U}_l(\zeta) \\
 &\quad + \sum_{l=1}^{\bar{w}} -(a_l - 1) r_l(\zeta) \overline{r_l(\zeta)} - \sum_{l=1}^{\bar{w}} \sigma_l r_l(\zeta) \overline{e_l(\zeta)} + \sum_{l=1}^{\bar{w}} \sum_{h=1}^{\bar{w}} \overline{c_{lh} r_l(\zeta)} F_h + \sum_{l=1}^{\bar{w}} \sum_{h=1}^{\bar{w}} \overline{d_{lh} r_l(\zeta)} \tilde{F}_h \\
 &\quad + \sum_{l=1}^{\bar{w}} \bigwedge_{h=1}^{\bar{w}} \overline{\mathcal{Q}_{lh} r_l(\zeta)} \tilde{F}_h + \sum_{l=1}^{\bar{w}} \bigvee_{h=1}^{\bar{w}} \overline{\mathcal{P}_{lh} r_l(\zeta)} \tilde{F}_h + \sum_{l=1}^{\bar{w}} r_l(\zeta) \overline{(1 - \lambda) \mathfrak{J}_l(\zeta)} + \sum_{l=1}^{\bar{w}} r_l(\zeta) \overline{\mathcal{U}_l(\zeta)} \\
 &\quad - \sum_{l=1}^{\bar{w}} \overline{e_l(\zeta)} e_l(\zeta) + \sum_{l=1}^{\bar{w}} \overline{e_l(\zeta)} r_l(\zeta) + \sum_{l=1}^{\bar{w}} \overline{e_l(\zeta)} \tilde{\mathcal{U}}_l(\zeta) - \sum_{l=1}^{\bar{w}} e_l(\zeta) \overline{e_l(\zeta)} + \sum_{l=1}^{\bar{w}} e_l(\zeta) \overline{r_l(\zeta)} \\
 &\quad + \sum_{l=1}^{\bar{w}} e_l(\zeta) \overline{\tilde{\mathcal{U}}_l(\zeta)}. \tag{3.4}
 \end{aligned}$$

Using Lemma 2.3, we get

$$\sum_{l=1}^{\bar{w}} \sigma_l \left[\overline{r_l(\zeta)} e_l(\zeta) + r_l(\zeta) \overline{e_l(\zeta)} \right] \leq \sum_{l=1}^{\bar{w}} \left[\sigma_l^2 \overline{r_l(\zeta)} r_l(\zeta) + e_l(\zeta) \overline{e_l(\zeta)} \right]. \tag{3.5}$$

From Assumption $(\mathcal{A}_1)$ and Lemma 2.3, we obtain

$$\begin{aligned}
 \sum_{l=1}^{\bar{w}} \sum_{h=1}^{\bar{w}} \left[c_{lh} \overline{r_l(\zeta)} F_h + \overline{c_{lh} r_l(\zeta)} \tilde{F}_h \right] &= \sum_{l=1}^{\bar{w}} \sum_{h=1}^{\bar{w}} c_{lh} \overline{r_l(\zeta)} \left[f_h(y_h(\zeta)) - f_h(\lambda x_h(\zeta - \gamma)) \right] \\
 &\quad + \sum_{l=1}^{\bar{w}} \sum_{h=1}^{\bar{w}} c_{lh} \overline{r_l(\zeta)} \left[f_h(\lambda x_h(\zeta - \gamma)) - \lambda f_h(x_h(\zeta - \gamma)) \right] \\
 &\quad + \sum_{l=1}^{\bar{w}} \sum_{h=1}^{\bar{w}} \overline{c_{lh} r_l(\zeta)} \left[\overline{f_h(y_h(\zeta)) - f_h(\lambda x_h(\zeta - \gamma))} \right] \\
 &\quad + \sum_{l=1}^{\bar{w}} \sum_{h=1}^{\bar{w}} \overline{c_{lh} r_l(\zeta)} \left[\overline{f_h(\lambda x_h(\zeta - \gamma)) - \lambda f_h(x_h(\zeta - \gamma))} \right] \\
 &\leq \sum_{l=1}^{\bar{w}} \sum_{h=1}^{\bar{w}} |c_{lh}|^2 \overline{r_l(\zeta)} r_l(\zeta) + \sum_{l=1}^{\bar{w}} \sum_{h=1}^{\bar{w}} L_h^2 \overline{e_h(\zeta)} e_h(\zeta)
 \end{aligned}$$

$$\begin{aligned}
& + \sum_{l=1}^{\varpi} \sum_{h=1}^{\varpi} |c_{lh}|^2 \overline{r_l(\zeta)} r_l(\zeta) \\
& + 2 \sum_{l=1}^{\varpi} \sum_{h=1}^{\varpi} \left[\overline{f_h(\lambda x_h(\zeta - \gamma))} f_h(\lambda x_h(\zeta - \gamma)) \right. \\
& \quad \left. + \overline{\lambda \lambda f_h(x_h(\zeta - \gamma))} f_h(x_h(\zeta - \gamma)) \right] \\
& \leq 2 \sum_{l=1}^{\varpi} \sum_{h=1}^{\varpi} |c_{lh}|^2 \overline{r_l(\zeta)} r_l(\zeta) + \varpi \sum_{l=1}^{\varpi} L_l^2 \overline{e_l(\zeta)} e_l(\zeta) \\
& + 2\varpi \sum_{h=1}^{\varpi} (1 + \overline{\lambda} \lambda) B_h^{+2},
\end{aligned}$$

and

$$\begin{aligned}
\sum_{l=1}^{\varpi} \sum_{h=1}^{\varpi} \left[d_{lh} \overline{r_l(\zeta)} \tilde{F}_h + \overline{d_{lh} r_l(\zeta)} \tilde{F}_h \right] & \leq 2 \sum_{l=1}^{\varpi} \sum_{h=1}^{\varpi} |d_{lh}|^2 B_h^{+2} + \varpi \sum_{l=1}^{\varpi} \overline{r_l(\zeta)} r_l(\zeta) \\
& + \sum_{l=1}^{\varpi} \sum_{h=1}^{\varpi} |d_{lh}|^2 |B_h^+ + \lambda B_h^+|^2.
\end{aligned} \tag{3.6}$$

on the other hand, from Lemma 2.13 we have

$$\begin{aligned}
\sum_{l=1}^{\varpi} \left| \bigwedge_{h=1}^{\varpi} \mathcal{Q}_{lh} \overline{r_l(\zeta)} \tilde{F}_h \right| & \leq \sum_{l=1}^{\varpi} \sum_{h=1}^{\varpi} \left| \mathcal{Q}_{lh} \overline{r_l(\zeta)} \tilde{F}_h \right|, \\
\sum_{l=1}^{\varpi} \left| \bigvee_{h=1}^{\varpi} \mathcal{P}_{lh} \overline{r_l(\zeta)} \tilde{F}_h \right| & \leq \sum_{l=1}^{\varpi} \sum_{h=1}^{\varpi} \left| \mathcal{P}_{lh} \overline{r_l(\zeta)} \tilde{F}_h \right|
\end{aligned}$$

and

$$\begin{aligned}
\sum_{l=1}^{\varpi} \left| \bigwedge_{h=1}^{\varpi} \overline{\mathcal{Q}_{lh} r_l(\zeta)} \tilde{F}_h \right| & \leq \sum_{l=1}^{\varpi} \sum_{h=1}^{\varpi} \left| \overline{\mathcal{Q}_{lh} r_l(\zeta)} \tilde{F}_h \right|, \\
\sum_{l=1}^{\varpi} \left| \bigvee_{h=1}^{\varpi} \overline{\mathcal{P}_{lh} r_l(\zeta)} \tilde{F}_h \right| & \leq \sum_{l=1}^{\varpi} \sum_{h=1}^{\varpi} \left| \overline{\mathcal{P}_{lh} r_l(\zeta)} \tilde{F}_h \right|
\end{aligned}$$

then,

$$\begin{aligned}
\sum_{l=1}^{\varpi} \sum_{h=1}^{\varpi} \left| \mathcal{Q}_{lh} \overline{r_l(\zeta)} \tilde{F}_h \right| + \sum_{l=1}^{\varpi} \sum_{h=1}^{\varpi} \left| \overline{\mathcal{Q}_{lh} r_l(\zeta)} \tilde{F}_h \right| & \leq 2 \sum_{l=1}^{\varpi} \sum_{h=1}^{\varpi} |\mathcal{Q}_{lh}|^2 B_h^{+2} + \varpi \sum_{l=1}^{\varpi} \overline{r_l(\zeta)} r_l(\zeta) \\
& + \sum_{l=1}^{\varpi} \sum_{h=1}^{\varpi} |\mathcal{Q}_{lh}|^2 |B_h^+ + \lambda B_h^+|^2
\end{aligned}$$

and

$$\begin{aligned}
\sum_{l=1}^{\varpi} \sum_{h=1}^{\varpi} \left| \mathcal{P}_{lh} \overline{r_l(\zeta)} \tilde{F}_h \right| + \sum_{l=1}^{\varpi} \sum_{h=1}^{\varpi} \left| \overline{\mathcal{P}_{lh} r_l(\zeta)} \tilde{F}_h \right| & \leq 2 \sum_{l=1}^{\varpi} \sum_{h=1}^{\varpi} |\mathcal{P}_{lh}|^2 B_h^{+2} + \varpi \sum_{l=1}^{\varpi} \overline{r_l(\zeta)} r_l(\zeta) \\
& + \sum_{l=1}^{\varpi} \sum_{h=1}^{\varpi} |\mathcal{P}_{lh}|^2 |B_h^+ + \lambda B_h^+|^2
\end{aligned}$$

For the external input, from $(\mathcal{A}_2)$ and Lemma 2.3, we have

$$\sum_{l=1}^{\varpi} \overline{r_l(\zeta)}(1-\lambda)\mathfrak{I}_l(\zeta) + \sum_{l=1}^{\varpi} r_l(\zeta) \overline{(1-\lambda)\mathfrak{I}_l(\zeta)} \leq \sum_{l=1}^{\varpi} |1-\lambda|^2 \overline{r_l(\zeta)} r_l(\zeta) + \sum_{l=1}^{\varpi} \mathfrak{I}_l^{+2} \quad (3.7)$$

$$\text{with } \mathfrak{I}_l^+ = \sum_{h=1}^{\varpi} |\mathcal{W}_{lh}| \varsigma_h^+ + \sum_{h=1}^{\varpi} |\mathcal{K}_{lh}| \varsigma_h^+ + \sum_{h=1}^{\varpi} |\mathcal{S}_{lh}| \varsigma_h^+ + I_l^+$$

For the controller, we have

$$\begin{aligned} & \sum_{l=1}^{\varpi} \overline{r_l(\zeta)} \mathcal{U}_l(\zeta) + \sum_{l=1}^{\varpi} r_l(\zeta) \overline{\mathcal{U}_l(\zeta)} + \sum_{l=1}^{\varpi} \overline{e_l(\zeta)} \tilde{\mathcal{U}}_l(\zeta) + \sum_{h=1}^{\varpi} e_l(\zeta) \overline{\tilde{\mathcal{U}}_l(\zeta)} \\ &= -2 \sum_{l=1}^{\varpi} \mu_l \overline{r_l(\zeta)} r_l(\zeta) - 2 \sum_{l=1}^{\varpi} \tilde{\mu}_l \overline{e_l(\zeta)} e_l(\zeta) \end{aligned} \quad (3.8)$$

Combining (3.4)-(3.8), we can derive

$$\begin{aligned} \mathcal{D}_\zeta^\alpha V(\zeta) \leq & -2 \sum_{l=1}^{\varpi} |a_l - 1| \overline{r_l(\zeta)} r_l(\zeta) - \sigma_l^2 \sum_{l=1}^{\varpi} \overline{r_l(\zeta)} r_l(\zeta) - \sum_{l=1}^{\varpi} \overline{e_l(\zeta)} e_l(\zeta) \\ & + 2 \sum_{l=1}^{\varpi} \sum_{h=1}^{\varpi} |c_{lh}|^2 \overline{r_l(\zeta)} r_l(\zeta) + \varpi \sum_{l=1}^{\varpi} L_l^2 \overline{e_l(\zeta)} e_l(\zeta) + 2\varpi \sum_{h=1}^{\varpi} (1 + \bar{\lambda}\lambda) B_h^{+2} \\ & + 2 \sum_{l=1}^{\varpi} \sum_{h=1}^{\varpi} |d_{lh}|^2 B_h^{+2} + \varpi \sum_{l=1}^{\varpi} \overline{r_l(\zeta)} r_l(\zeta) + \sum_{l=1}^{\varpi} \sum_{h=1}^{\varpi} |d_{lh}|^2 |B_h^+ + \lambda B_h^+|^2 \\ & + 2 \sum_{l=1}^{\varpi} \sum_{h=1}^{\varpi} |Q_{lh}|^2 B_h^{+2} + \varpi \sum_{l=1}^{\varpi} \overline{r_l(\zeta)} r_l(\zeta) + \sum_{l=1}^{\varpi} \sum_{h=1}^{\varpi} |Q_{lh}|^2 |B_h^+ + \lambda B_h^+|^2 \\ & + 2 \sum_{l=1}^{\varpi} \sum_{h=1}^{\varpi} |P_{lh}|^2 B_h^{+2} + \varpi \sum_{l=1}^{\varpi} \overline{r_l(\zeta)} r_l(\zeta) + \sum_{l=1}^{\varpi} \sum_{h=1}^{\varpi} |P_{lh}|^2 |B_h^+ + \lambda B_h^+|^2 \\ & + \sum_{l=1}^{\varpi} |1-\lambda|^2 \overline{r_l(\zeta)} r_l(\zeta) + \sum_{l=1}^{\varpi} \mathfrak{I}_l^{+2} + \sum_{l=1}^{\varpi} \overline{r_l(\zeta)} r_l(\zeta) + \sum_{l=1}^{\varpi} \overline{e_l(\zeta)} e_l(\zeta) \\ & - 2 \sum_{l=1}^{\varpi} \mu_l \overline{r_l(\zeta)} r_l(\zeta) - 2 \sum_{l=1}^{\varpi} \tilde{\mu}_l \overline{e_l(\zeta)} e_l(\zeta) - \sum_{l=1}^{\varpi} \overline{e_l(\zeta)} \overline{e_l(\zeta)} - \sum_{l=1}^{\varpi} e_l(\zeta) \overline{e_l(\zeta)}. \end{aligned}$$

Which leads to

$$\begin{aligned} \mathcal{D}_\zeta^\alpha V(\zeta) \leq & - \min_{1 \leq l \leq \varpi} \left\{ 2|a_l - 1| + \sigma_l^2 - 2 \sum_{h=1}^{\varpi} |c_{lh}|^2 - 3\varpi - 1 - |1-\lambda|^2 + 2\mu_l \right\} \sum_{l=1}^{\varpi} \overline{r_l(\zeta)} r_l(\zeta) \\ & - \min_{1 \leq l \leq \varpi} \left\{ 2 - \varpi L_l^2 + 2\tilde{\mu}_l \right\} \sum_{l=1}^{\varpi} \overline{e_l(\zeta)} e_l(\zeta) + 2\varpi \sum_{h=1}^{\varpi} (1 + \bar{\lambda}\lambda) B_h^{+2} + \sum_{l=1}^{\varpi} \mathfrak{I}_l^{+2} \\ & + 2 \sum_{l=1}^{\varpi} \sum_{h=1}^{\varpi} |d_{lh}|^2 B_h^{+2} + \sum_{l=1}^{\varpi} \sum_{h=1}^{\varpi} |d_{lh}|^2 |B_h^+ + \lambda B_h^+|^2 + 2 \sum_{l=1}^{\varpi} \sum_{h=1}^{\varpi} |Q_{lh}|^2 B_h^{+2} \\ & + \sum_{l=1}^{\varpi} \sum_{h=1}^{\varpi} |Q_{lh}|^2 |B_h^+ + \lambda B_h^+|^2 + 2 \sum_{l=1}^{\varpi} \sum_{h=1}^{\varpi} |P_{lh}|^2 B_h^{+2} + \sum_{l=1}^{\varpi} \sum_{h=1}^{\varpi} |P_{lh}|^2 |B_h^+ + \lambda B_h^+|^2 \\ & := -\beta V(\zeta) + \chi. \end{aligned} \quad (3.9)$$

From (3.2), we have $\beta > 0$ and $\chi > 0$, then from Lemma 2.4, it follow that there exists $N_1 \geq 1$ such that

$$V(\zeta) \leq \eta N_1 e^{-\beta\zeta} + \frac{\chi}{\beta}$$

where $\eta = \max_{\zeta \in [-\rho, 0]} |V(\zeta)|$.

Since we have $\Theta(\zeta) = (r_1(\zeta), \dots, r_\varpi(\zeta), e_1(\zeta), \dots, e_\varpi(\zeta))^T$, then $\|\Theta(\zeta)\|^2 = V(\zeta)$. Consequently,

$$\|\Theta(\zeta)\| \leq \sqrt{\eta N_1 e^{-\beta\zeta} + \frac{\chi}{\beta}}.$$

Therefore,

$$\lim_{\zeta \rightarrow +\infty} \|\Theta(\zeta)\| \leq \lim_{\zeta \rightarrow +\infty} \sqrt{\eta N_1 e^{-\beta\zeta} + \frac{\chi}{\beta}} = \sqrt{\frac{\chi}{\beta}}.$$

That's mean that, systems (2.1)–(2.2) can attain the PLQS. $\square$

In particular, if $\gamma = 0$ we get

Corollary 3.2. *Systems (2.1) and (2.2) achieve the quasi-projective synchronization, under Assumption $(\mathcal{A}_1)$, the controller (3.1), and if the following conditions hold*

$$\begin{aligned} \beta_1 &= \min_{1 \leq l \leq \varpi} \left\{ 2|a_l - 1| + \sigma_l^2 - 2 \sum_{h=1}^{\varpi} |c_{lh}|^2 - 3\varpi - 1 - |1 - \lambda|^2 + 2\mu_l \right\} > 0 \\ \beta_2 &= \min_{1 \leq l \leq \varpi} \left\{ 2 - \varpi L_l^2 + 2\tilde{\mu}_l \right\} > 0, \quad \beta = \min\{\beta_1, \beta_2\}, \\ \chi &= 2\varpi \sum_{h=1}^{\varpi} (1 + \bar{\lambda}\lambda) B_h^{+2} + \sum_{l=1}^{\varpi} \mathfrak{I}_l^{+2} + 2 \sum_{l=1}^{\varpi} \sum_{h=1}^{\varpi} |d_{lh}|^2 B_h^{+2} + \sum_{l=1}^{\varpi} \sum_{h=1}^{\varpi} |d_{lh}|^2 |B_h^+ + \lambda B_h^+|^2 \\ &\quad + 2 \sum_{l=1}^{\varpi} \sum_{h=1}^{\varpi} |Q_{lh}|^2 B_h^{+2} + \sum_{l=1}^{\varpi} \sum_{h=1}^{\varpi} |Q_{lh}|^2 |B_h^+ + \lambda B_h^+|^2 + 2 \sum_{l=1}^{\varpi} \sum_{h=1}^{\varpi} |P_{lh}|^2 B_h^{+2} \\ &\quad + \sum_{l=1}^{\varpi} \sum_{h=1}^{\varpi} |P_{lh}|^2 |B_h^+ + \lambda B_h^+|^2. \end{aligned}$$

Now, if $\gamma = 0$ and $\lambda = 1$, the following result holds:

Corollary 3.3. *Systems (2.1) and (2.2) achieve the quasi synchronization, under Assumption $(\mathcal{A}_1)$ and the controller (3.1).*

$$\begin{aligned} \beta_1 &= \min_{1 \leq l \leq \varpi} \left\{ 2|a_l - 1| + \sigma_l^2 - 2 \sum_{h=1}^{\varpi} |c_{lh}|^2 - 3\varpi - 1 + 2\mu_l \right\} > 0 \\ \beta_2 &= \min_{1 \leq l \leq \varpi} \left\{ 2 - \varpi L_l^2 + 2\tilde{\mu}_l \right\} > 0, \quad \beta = \min\{\beta_1, \beta_2\}, \\ \chi &= 4\varpi \sum_{h=1}^{\varpi} B_h^{+2} + \sum_{l=1}^{\varpi} \mathfrak{I}_l^{+2} + 6 \sum_{l=1}^{\varpi} \sum_{h=1}^{\varpi} |d_{lh}|^2 B_h^{+2} \\ &\quad + 6 \sum_{l=1}^{\varpi} \sum_{h=1}^{\varpi} |Q_{lh}|^2 B_h^{+2} + 6 \sum_{l=1}^{\varpi} \sum_{h=1}^{\varpi} |P_{lh}|^2 B_h^{+2}. \end{aligned}$$

Remark 3.4. In [35, 36], hybrid controllers were employed, which may pose practical limitations and are often undesirable in real-world applications. In contrast, this work utilizes nonlinear feedback controllers to investigate the QPS problem in Caputo fractional-order complex inertial neural networks with time-varying delays. The proposed approach offers a more practical and readily implementable framework for achieving synchronization.

Remark 3.5. In many previous studies, complex-valued systems are typically separated into two real-valued subsystems for analysis, as seen in [37, 38, 39]. In contrast, the present work treats the complex-valued system as an integrated entity, which significantly simplifies the theoretical analysis and reduces computational complexity. Consequently, the approach adopted here holds greater research significance and practical value.

Remark 3.6. In previous studies [34, 40, 41, 42], to remove the order of the inequalities, the results were shown in the form of Mittag-Leffler functions. To obtain the error bound, its monotonicity is usually used to realize QPS. However, the Mittag-Leffler function is complex and unfamiliar to us. Inspired by *Hu et al.* [32], Lemma 2.4 is proposed based on the classical exponential function. It can be seen that the constructed lemma is very flexible and useful.

4 Numerical examples

To evaluate the validity of the obtained results, we introduce the following examples:

Example 4.1. Let the following drive system

$$\begin{aligned} \mathcal{D}_{\zeta}^{2\alpha} x_l(\zeta) = & -a_l \mathcal{D}_{\zeta}^{\alpha} x_l(\zeta) - b_l x_l(\zeta) + \sum_{h=1}^{\varpi} c_{lh} f_h(x_h(\zeta)) + \sum_{h=1}^{\varpi} d_{lh} f_h(x_h(\zeta - \rho_h(\zeta))) \\ & + \sum_{h=1}^{\varpi} \mathcal{W}_{lh} \varsigma_h(\zeta) + \bigwedge_{h=1}^{\varpi} \mathcal{Q}_{lh} f_h(x_h(\zeta - \rho_h(\zeta))) + \bigwedge_{h=1}^{\varpi} \mathcal{K}_{lh} \varsigma_h(\zeta) \\ & + \bigvee_{h=1}^{\varpi} \mathcal{P}_{lh} f_h(x_h(\zeta - \rho_h(\zeta))) + \bigvee_{h=1}^{\varpi} \mathcal{S}_{lh} \varsigma_h(\zeta) + I_l(\zeta). \end{aligned} \quad (4.1)$$

with $\alpha = 0.9$, $\varpi = 2$, $\rho(\zeta) = 0.1$, $a_1 = a_2 = 1.2$, $b_1 = b_2 = 1.5$, $\mathfrak{I}_l(\zeta) = 0$, $\lambda = 0.1 + 0.3i$, $\gamma = 0.1$,

$$\begin{aligned} (c_{lh})_{1 \leq l, h \leq 2} &= \begin{bmatrix} 0.5 - 0.3i & -0.1 + 0.1i \\ 0.5 + 0.1i & 0.9 - 0.4i \end{bmatrix}; \\ (d_{lh})_{1 \leq l, h \leq 2} &= \begin{bmatrix} 0.3 - 1.2i & -1 + 1.2i \\ 1.4 + 1.2i & 1 - 1.2i \end{bmatrix}; \\ (\mathcal{P}_{lh})_{1 \leq l, h \leq 2} &= \begin{bmatrix} -1.5 - 1.2i & -0.12 + 1.14i \\ 0.15 + 1.2i & 0.94 - 1.1i \end{bmatrix}; \\ (\mathcal{Q}_{lh})_{1 \leq l, h \leq 2} &= \begin{bmatrix} 0.2 - 1.1i & -1 + 0.3i \\ 1.14 + 0.22i & 0.1 - 0.1i \end{bmatrix}. \end{aligned}$$

Hence $x_l(\zeta) = x_l^R(\zeta) + ix_l^I(\zeta)$, then f_h is defined as

$$f_h(x_l(\zeta)) = \tanh(x_l^R(\zeta)) + i \tanh(x_l^I(\zeta)).$$

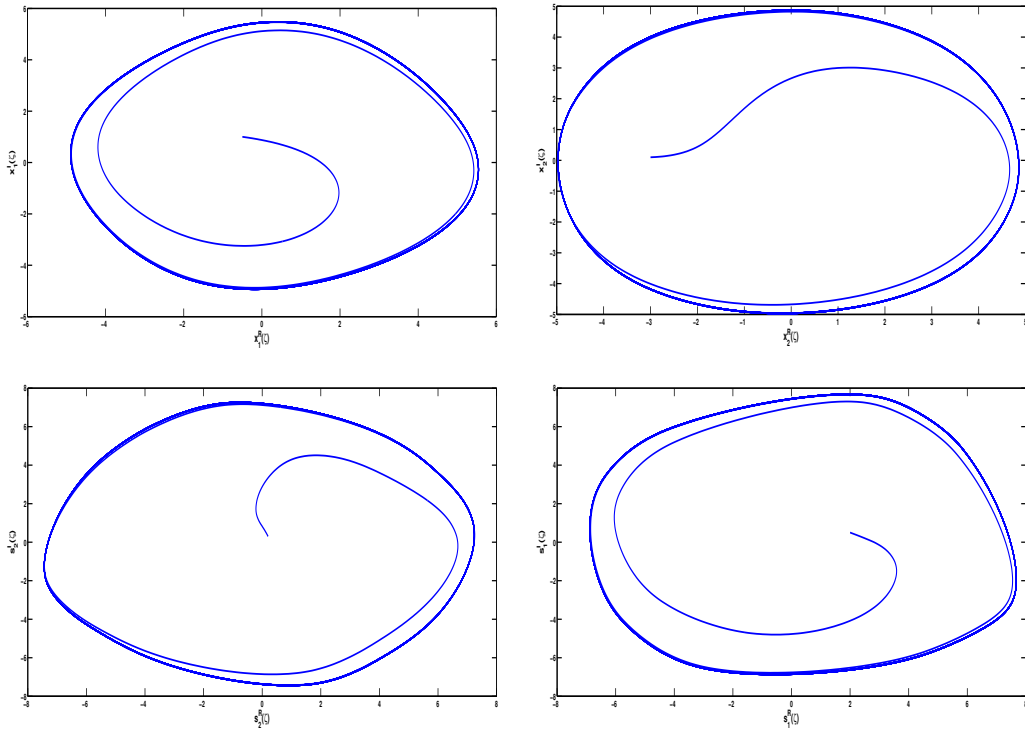


Figure 1. The trajectory of master system (4.1).

The response model is formulated as

$$\begin{aligned}
 \mathcal{D}_{\zeta}^{2\alpha} y_l(\zeta) = & -a_l \mathcal{D}_{\zeta}^{\alpha} y_l(\zeta) - b_l y_l(\zeta) + \sum_{h=1}^{\varpi} c_{lh} f_h(y_h(\zeta)) + \sum_{h=1}^{\varpi} d_{lh} f_h(y_h(\zeta - \rho_h(\zeta))) \\
 & + \sum_{h=1}^{\varpi} \mathcal{W}_{lh} s_h(\zeta) + \bigwedge_{h=1}^{\varpi} \mathcal{Q}_{lh} f_h(y_h(\zeta - \rho_h(\zeta))) + \bigwedge_{h=1}^{\varpi} \mathcal{K}_{lh} s_h(\zeta) \\
 & + \bigvee_{h=1}^{\varpi} \mathcal{P}_{lh} f_h(y_h(\zeta - \rho_h(\zeta))) + \bigvee_{h=1}^{\varpi} \mathcal{S}_{lh} s_h(\zeta) + I_l(\zeta) + U_l(\zeta).
 \end{aligned} \quad (4.2)$$

Figure 1 presents the trajectory of master system (4.1). By calculation, we obtain $L_h = B_h^+ = 1$.

$$\begin{aligned}
 \beta_1 &= \min_{1 \leq l \leq 2} \left\{ 2|a_l - 1| + \sigma_l^2 - 2 \sum_{h=1}^2 |c_{lh}|^2 - 6 - 1 - |1 - \lambda|^2 + 2\mu_l \right\} \\
 &= \min\{13.72, 11.98\} = 11.98 > 0 \\
 \beta_2 &= \min_{1 \leq l \leq 2} \left\{ 2 - 2L_l^2 + 2\tilde{\mu}_l \right\} = 20 > 0, \\
 \chi &= 22 \sum_{h=1}^2 (1 + \bar{\lambda}\lambda) B_h^{+2} + \sum_{l=1}^2 \mathfrak{J}_l^{+2} + 2 \sum_{l=1}^2 \sum_{h=1}^2 |d_{lh}|^2 B_h^{+2} + \sum_{l=1}^2 \sum_{h=1}^2 |d_{lh}|^2 |B_h^+ + \lambda B_h^+|^2 \\
 &\quad + 2 \sum_{l=1}^2 \sum_{h=1}^2 |\mathcal{Q}_{lh}|^2 B_h^{+2} + \sum_{l=1}^2 \sum_{h=1}^2 |\mathcal{Q}_{lh}|^2 |B_h^+ + \lambda B_h^+|^2 + 2 \sum_{l=1}^2 \sum_{h=1}^2 |\mathcal{P}_{lh}|^2 B_h^{+2} \\
 &\quad + \sum_{l=1}^2 \sum_{h=1}^2 |\mathcal{P}_{lh}|^2 |B_h^+ + \lambda B_h^+|^2 = 84.6.
 \end{aligned}$$

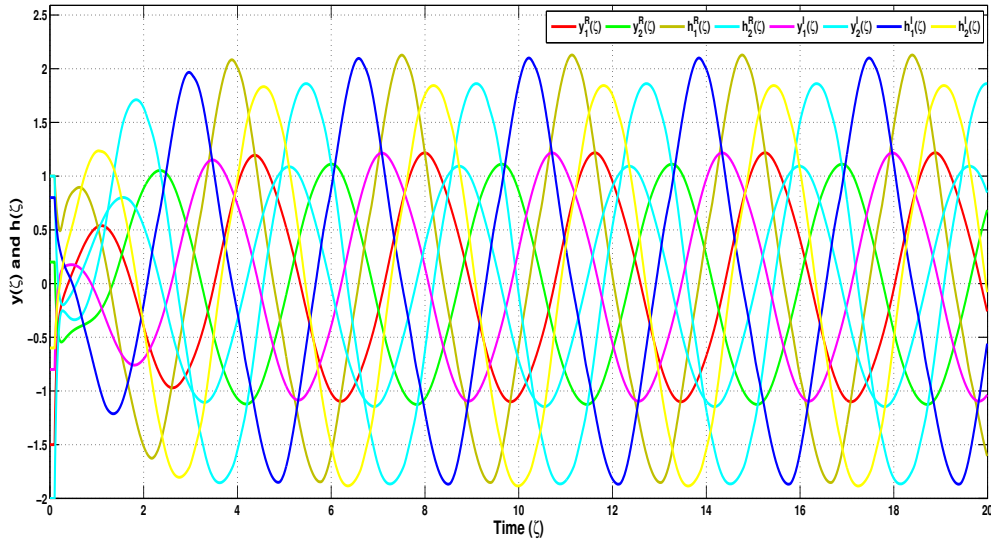


Figure 2. The evolutions of node states $y(\zeta)$ and $h(\zeta)$.

To fulfill the quasi-projective synchronization, the controllers parameters are selected as: $\mu_1 = \mu_2 = \tilde{\mu}_1 = \tilde{\mu}_2 = 10$, the initial conditions are given as: $x_1(s) = -\frac{1}{2} + i$, $s_1(s) = -3 + 0.1i$, $y_1(s) = -1.5 - \frac{8}{10}i$, $h_1(s) = 1 + \frac{8}{10}i$, $x_2(s) = -3 + \frac{1}{10}i$, $s_2(s) = -\frac{2}{10} + \frac{3}{10}i$, $y_2(s) = \frac{2}{10} + i$, and $h_2(s) = -2 - \frac{6}{10}i$. Given that the above results meet the requirements of Theorem 3.1, systems (4.1)–(4.2) reach the quasi-projective synchronization. The error bound is further approximated by:

$$\sqrt{\frac{\chi}{\beta}} = 2.66.$$

Figures 2 and 3 illustrate the evolution of the state variables.

Remark 4.2. It is worth noting that, compared with the recent works in [43, 44], which address fractional-order reaction-diffusion systems and finite-time synchronization of discrete neural models, respectively, the present work introduces a significantly more general framework by considering projective lag quasi synchronization of fractional-order complex-inertial fuzzy neural networks with time-varying delays. This setting simultaneously incorporates fractional dynamics, inertial effects, fuzzy uncertainty, and time delays, leading to a richer and more realistic class of complex dynamical networks.

Figure 4 depicts the time evolution of $e(\zeta)$ and $r(\zeta)$. It can be observed that the error does not converge to zero, which is consistent with the projective lag quasi synchronization objective, where the trajectories approach a scaled and delayed synchronization manifold rather than exact synchronization.

Figure 5 presents the trajectory of $\|\Theta(\zeta)\|$ under the controller (3.1), from which it is evident that the error norm exhibits minor oscillations and tends toward bounded behavior as time progresses.

Remark 4.3. In addition to the theoretical contributions, the obtained results also have important practical implications. The proposed projective lag quasi-synchronization and quasi-projective synchronization schemes can be applied to a broad class of complex systems modeled by delayed fractional-order complex inertial fuzzy neural networks, which are relevant in many real-world applications involving memory effects, nonlinearities, and uncertainties. In particular, the introduction of lag synchronization provides greater flexibility in practical implementations, especially in secure communication and signal transmission systems where time delays naturally

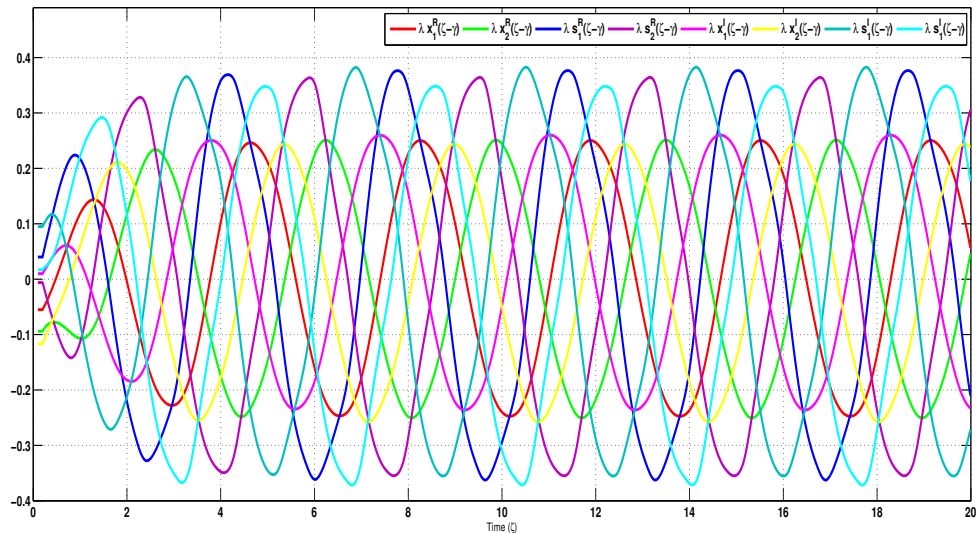


Figure 3. The evolutions of node states $\lambda x(\zeta - \gamma)$ and $\lambda s(\zeta - \gamma)$.

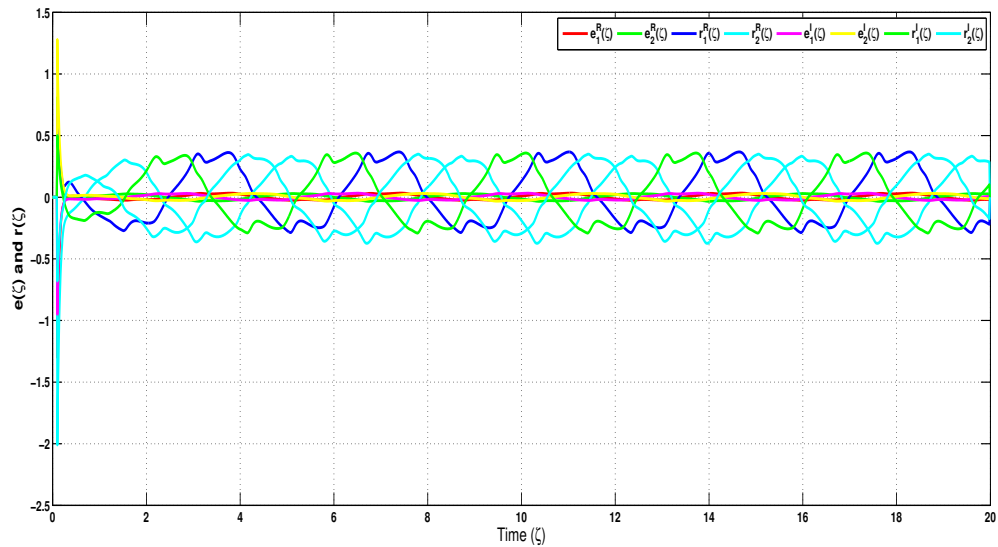


Figure 4. The time evolution of $e(\zeta)$ and $r(\zeta)$.

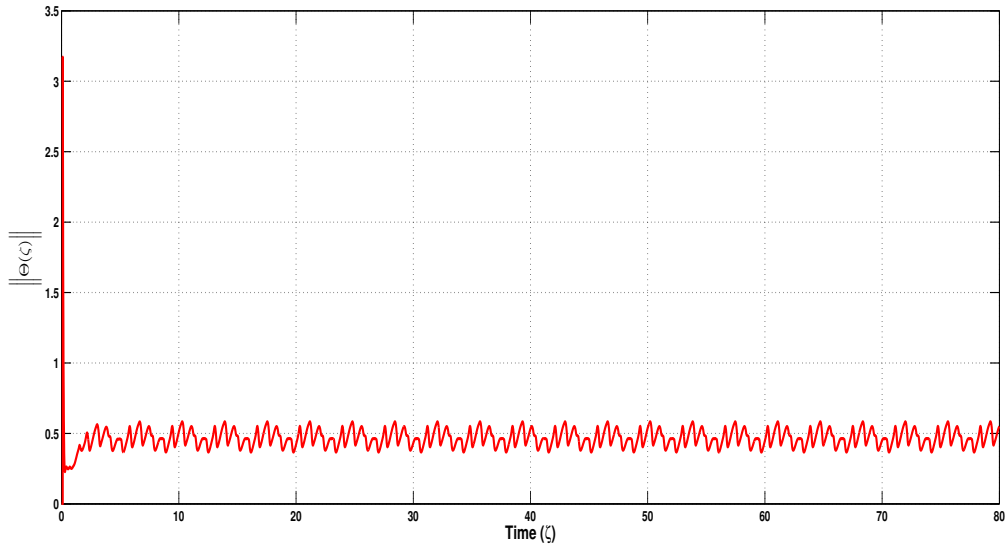


Figure 5. The behavior of $\|\Theta(\zeta)\|$ in the presence of controller (3.1).

occur. Furthermore, the incorporation of fuzzy dynamics enhances the robustness of the model against uncertainties and imprecise information. The derived synchronization criteria, along with the explicit estimation of the synchronization error bound, offer useful guidelines for the design of efficient nonlinear control strategies in engineering applications such as intelligent control systems and neural computation.

Example 4.4. Let the following drive system

$$\mathcal{D}_{\zeta}^{2\alpha} x_l(\zeta) = -a_l \mathcal{D}_{\zeta}^{\alpha} x_l(\zeta) - b_l x_l(\zeta) + \sum_{h=1}^{\varpi} c_{lh} f_h(x_h(\zeta)) + \sum_{h=1}^{\varpi} d_{lh} f_h(x_h(\zeta - \rho_h(\zeta))) + I_l(\zeta). \quad (4.3)$$

with $\alpha = 0.95$, $\varpi = 2$, $\rho(\zeta) = 0.1$, $a_1 = a_2 = 2.5$, $b_1 = b_2 = 1.5$, $I_l(\zeta) = 0$, $\lambda = 0.1 + 0.3i$

$$(c_{lh})_{1 \leq l, h \leq 2} = \begin{bmatrix} 1.5 - 1.3i & -1.1 + 1.1i \\ 1.5 + 1.1i & 0.9 - 1.4i \end{bmatrix}; \quad (d_{lh})_{1 \leq l, h \leq 2} = \begin{bmatrix} 1.3 - 1.2i & -1 + 1.2i \\ 1.4 + 1.2i & 1 - 1.2i \end{bmatrix}.$$

Hence $x_l(\zeta) = x_l^R(\zeta) + ix_l^I(\zeta)$, then f_h is defined as

$$f_h(x_l(\zeta)) = \frac{-e^{-x_l^R(\zeta)} + 1}{e^{-x_l^R(\zeta)} + 1} + i \frac{1}{e^{-x_l^I(\zeta)} + 1}.$$

The response model is formulated as

$$\mathcal{D}_{\zeta}^{2\alpha} y_l(\zeta) = -a_l \mathcal{D}_{\zeta}^{\alpha} y_l(\zeta) - b_l y_l(\zeta) + \sum_{h=1}^{\varpi} c_{lh} f_h(y_h(\zeta)) + \sum_{h=1}^{\varpi} d_{lh} f_h(y_h(\zeta - \rho_h(\zeta))) + I_l(\zeta) + U_l(\zeta). \quad (4.4)$$

Figure 6 presents the trajectory of master system (4.3).

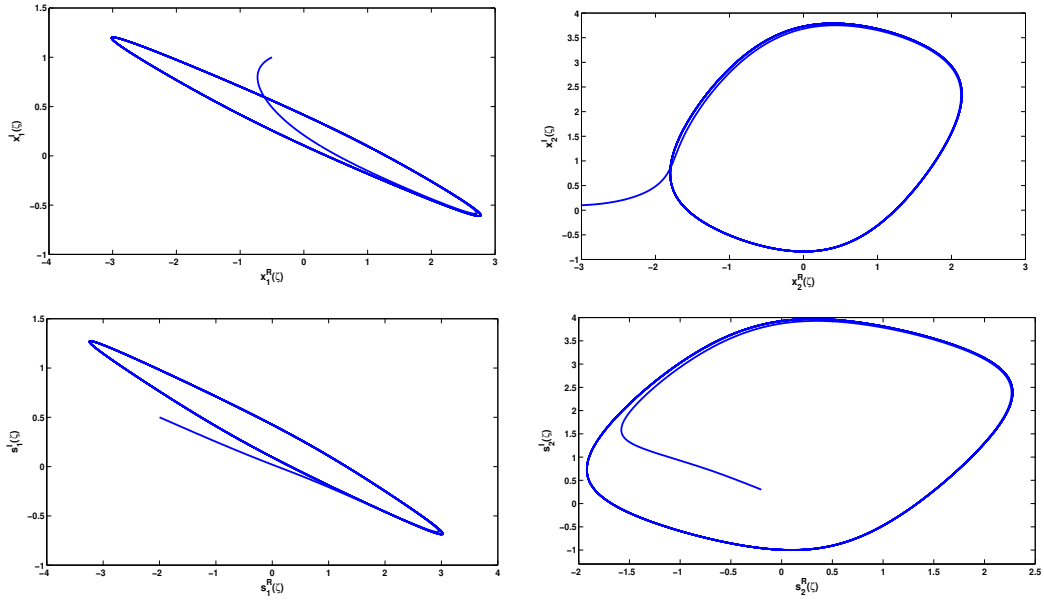


Figure 6. The trajectory of master system (4.3).

By calculation, we obtain $L_{\mathfrak{h}} = 1$ and $B_{\mathfrak{h}}^+ = \sqrt{2}$.

$$\beta_1 = \min_{1 \leq l \leq 2} \left\{ 2|a_l - 1| - \sigma_l^2 - 2 \sum_{\mathfrak{h}=1}^2 |c_{l\mathfrak{h}}|^2 - 2\varpi - |1 - \lambda|^2 + 2\mu_l - 2 \right\} = 39.84$$

$$\beta_2 = \min_{1 \leq l \leq 2} \left\{ -1 - \varpi \sum_{\mathfrak{h}=1}^2 L_{\mathfrak{h}}^2 + 2\tilde{\mu}_l - 2 \right\} = 49$$

$$\begin{aligned} \chi &= 2 \sum_{l=1}^2 \sum_{\mathfrak{h}=1}^2 |d_{l\mathfrak{h}}|^2 B_{\mathfrak{h}}^{+2} + 2\varpi \sum_{\mathfrak{h}=1}^2 (1 + \bar{\lambda}\lambda) B_{\mathfrak{h}}^{+2} + \sum_{l=1}^2 \sum_{\mathfrak{h}=1}^2 |d_{l\mathfrak{h}}|^2 |B_{\mathfrak{h}}^+ + \lambda B_{\mathfrak{h}}^+|^2 + \sum_{l=1}^2 I_l^{+2} \\ &+ \sum_{l=1}^2 \xi_l^2 + \sum_{l=1}^2 \tilde{\xi}_l^2 = 93.8. \end{aligned}$$

To fulfill the quasi-projective synchronization, the controllers parameters are selected as: $\mu_1 = \mu_2 = \tilde{\mu}_1 = \tilde{\mu}_2 = 28$, $\xi_1 = \xi_2 = 0.3$, $\tilde{\xi}_1 = \tilde{\xi}_2 = 0.6$, the initial conditions are given as: $x_1(s) = -\frac{1}{2} + i$, $s_1(s) = -2 + \frac{1}{2}i$, $y_1(s) = -1.5 - \frac{8}{10}i$, $h_1(s) = 1 + \frac{8}{10}i$, $x_2(s) = -3 + \frac{1}{10}i$, $s_2(s) = -\frac{2}{10} + \frac{3}{10}i$, $y_2(s) = \frac{2}{10} + i$, and $h_2(s) = -2 - \frac{6}{10}i$. Given that the above results meet the requirements of Corollary 3.2, systems (4.3)-(4.4) reach the quasi-projective synchronization. The error bound is further approximated by:

$$\lim_{\zeta \rightarrow +\infty} \|\Theta(\zeta)\| \leq \sqrt{\frac{\chi}{\beta}} = 1.54.$$

Figures 7 and 8 illustrate the evolution of the state variables. Figure 9 illustrate the trajectories of $e(\zeta)$ and $r(\zeta)$ demonstrating that systems (4.3)-(4.4) realize the QPS under controller (3.1).

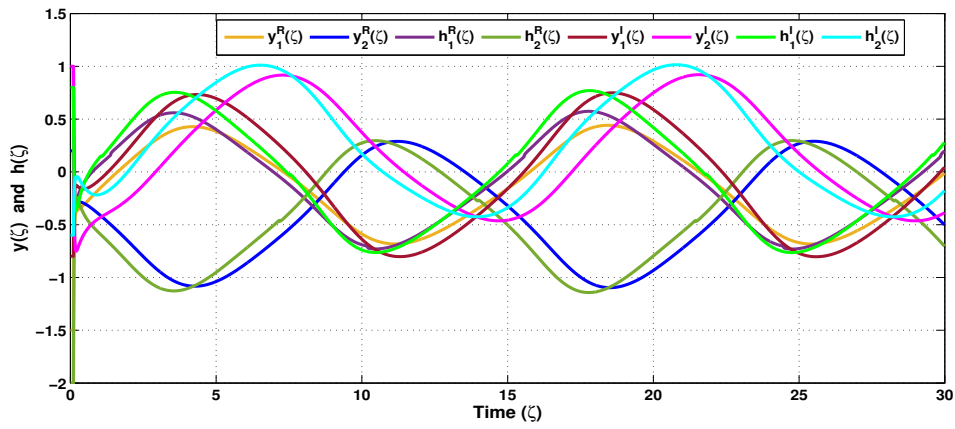


Figure 7. The evolutions of node states $y(\zeta)$ and $h(\zeta)$.

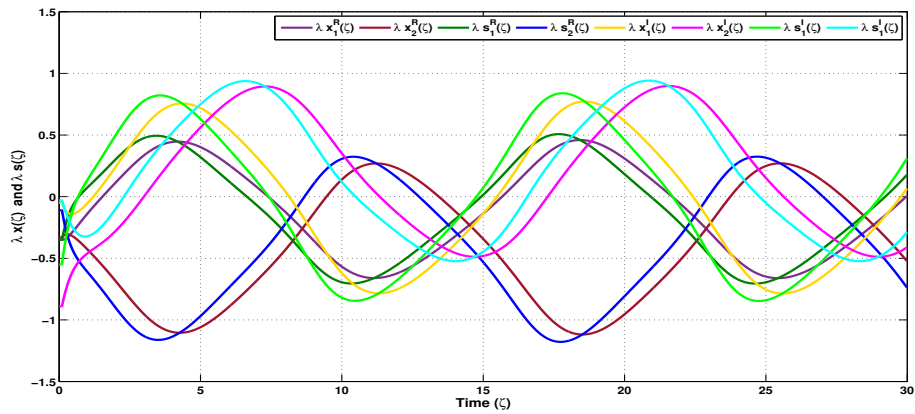


Figure 8. The evolutions of node states $\lambda x(\zeta)$ and $\lambda s(\zeta)$.

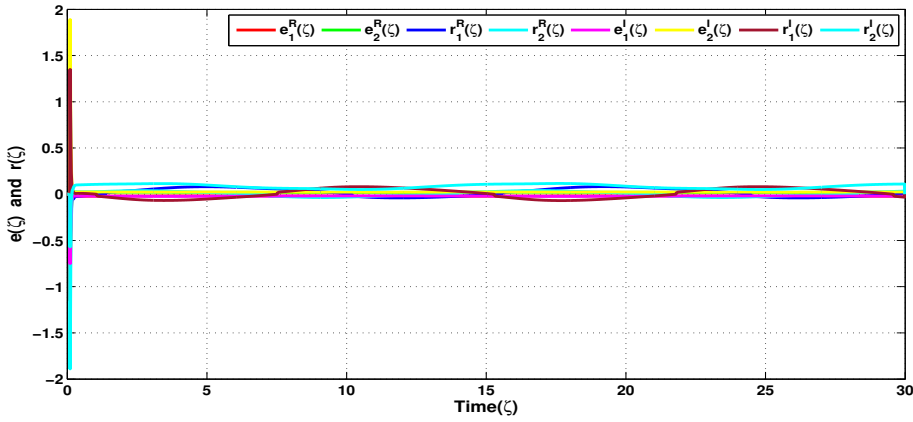


Figure 9. The time evolution of $e(\zeta)$ and $r(\zeta)$.

Figure 10 presents the trajectory of $\|\Theta(\zeta)\|$ under the controller (3.1), from which it is evident that the error norm exhibits minor oscillations and tends toward bounded behavior as time progresses.

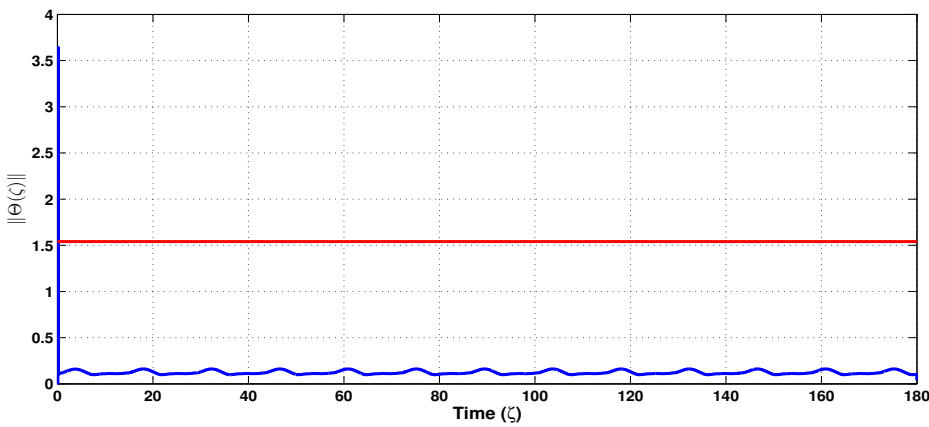


Figure 10. The behavior of $\|\Theta(\zeta)\|$ in the presence of controller (3.1).

Remark 4.5. In Theorems 3.1 and Corollary 3.2, sufficient conditions for LPQS and QPS have been established. The simulation results show that the error exhibits bounded fluctuations, which further confirms the validity of the theoretical findings. Moreover, unlike the results reported in [34, 40, 41, 42], which are expressed in terms of the Mittag-Leffler function and are often difficult to verify in practice, we employ Lemma 2.4, which leads to estimates exhibiting the characteristics of the classical exponential function. This makes the obtained results more tractable and easier to implement in practical applications.

5 Conclusion

This study focuses on the investigation of LPQS as well as quasi-projective synchronization of delayed Caputo fractional-order complex-inertial fuzzy neural networks. A decomposition method was employed to overcome the computational complexity in the synchronization analysis. Through the formulation of appropriate Lyapunov functions combined with fractional calculus techniques, sufficient conditions ensuring QPS and quasi-synchronization have been established. The theoretical findings were further validated through numerical simulations, demonstrating the robustness and applicability of the proposed approach. These results provide valuable insights for the synchronization control of complex delayed neural systems and may serve as a

foundation for future research involving more general network architectures or control strategies. Recent contributions have also explored related aspects of nonlinear systems, such as actuator fault reconstruction [45], modeling of complex interconnected systems [46], and observer-based control strategies [47], which provide complementary perspectives relevant to our framework.

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