

Projective synchronization analysis for non-identical quaternion fractional-order competitive neural networks with inconsistent orders

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This manuscript is an extended and significantly improved version of a full paper entitled "Synchronization problem for delayed quaternion-valued neural networks" presented at ICRAMI 2025 (Tunisia, November 2025). In the version presented at the conference, the considered model was based on the classical (integer-order) dynamics of inertial quaternion-valued neural networks with proportional delays. In contrast, this paper explores a more general framework by introducing fractional-order dynamics, which account for memory effects and provide a more accurate description of complex systems.

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Abstract This paper investigates the problem of projective synchronization ($\mathcal{P}$ - $\mathcal{S}$) for a class of non-identical quaternion-valued fractional-order competitive neural networks with inconsistent fractional orders. Unlike many existing works and to directly address the challenges posed by quaternion algebra, the non-commutativity of multiplication, and mismatched system parameters and derivative orders, the analysis is conducted directly in the quaternion domain without employing a separation method. A novel feedback control scheme is developed to guarantee the realization of $\mathcal{P}$ - $\mathcal{S}$ under these complex conditions. Sufficient conditions ensuring $\mathcal{P}$ - $\mathcal{S}$ are established using the fractional derivative properties and the Lyapunov direct method. Appropriate Lyapunov functions are constructed in the quaternion domain, and the stability analysis is carried out by exploiting key properties of fractional-order derivatives with non-uniform orders. Numerical simulations are provided to demonstrate the effectiveness and feasibility of the proposed synchronization scheme. The results confirm that the developed method can successfully realize $\mathcal{P}$ - $\mathcal{S}$ for non-identical quaternion fractional-order competitive neural networks with inconsistent orders, highlighting its potential applicability in complex-valued and high-dimensional neural network systems.

1 Introduction

Fractional calculus refers to a mathematical formalism that generalizes classical differentiation and integration to arbitrary, non-integer orders. One of its main advantages is its ability to represent memory effects, which makes it suitable for describing many real dynamical systems more realistically than classical calculus [1]. Because of this feature, fractional calculus has attracted growing interest and has been used in a wide range of applications, including biology [2], neural networks [3, 4], economics [5], and image encryption [6].

In neural network theory, the introduction of fractional-order operators has led to the development of fractional-order neural networks. These networks differ from integer-order models by their capability to preserve past information over time, allowing them to describe richer dynamic behaviors. Moreover, fractional-order models offer additional flexibility in system design, which makes them useful for handling complex engineering problems [7]. As a result, fractional-order neural networks have been successfully applied in areas such as circuit modeling, robotics, and

optimization methods, and their dynamic performance has been extensively studied [8, 9].

Neural network research has produced a wide variety of models, including Hopfield neural networks [10], Cohen–Grossberg neural networks [11], recurrent neural networks [12], bidirectional associative memory networks [13, 14], and competitive neural networks (ComNNs) [15], among others. Among these models, ComNNs were originally introduced by Cohen and Grossberg [16], and have since attracted significant interest due to their rich dynamic behavior. Later studies revealed that ComNNs can operate on multiple time scales and possess different memory mechanisms [17]. In particular, short-term memory (STM) is associated with fast neural responses, while long-term memory (LTM) reflects slow and gradual synaptic adaptations [18, 19]. Owing to these characteristics, ComNNs have been widely applied in practical areas such as optimization problems and image processing.

On the other hand, the concept of quaternions was first established by William Rowan Hamilton in 1843 [20]. A quaternion is a four-dimensional mathematical system that extends complex numbers. It is represented by one real component and three imaginary parts as $\mathbf{p} = p^R + ip^I + jp^J + kp^K$. A fundamental property of quaternions is their non-commutative multiplication, where the order of factors changes the product. This unique structure is highly important for modern technology, as it allows for the efficient representation of 3D rotations and multi-dimensional signals [21]. In the field of artificial intelligence, the traditional models often struggle to process high-dimensional data efficiently, as they are primarily designed for one or two-dimensional datasets. Consequently, it has become essential to investigate the dynamics of quaternion-valued neural networks and their potential applications directly within the quaternion domain [22].

Synchronization in dynamical systems occurs when two or more systems interact in a way that their behaviors become coordinated. Extensive research has been devoted to synchronization because of its importance in practical areas such as optimization, medicine, and automatic control [23, 24, 25]. Over the years, a variety of synchronization methods have been developed to coordinate the behavior of dynamical systems. Among these approaches are adaptive control [24], Event-triggered Control [26], predefined-time synchronization [27], and sliding mode control [28]. In parallel, different types of synchronization have been studied, including complete synchronization [29], lag synchronization [30], and $\mathcal{P}$ -S [31]. For example, [29] studied the complete synchronization of chaotic neural networks. They used an intermittent control method with two switches to stabilize the system. In [30], the authors focused on lag synchronization for networks with mixed delays. They designed a simple adaptive controller that works well even with external noise and perturbations. Finally, [31] investigated $\mathcal{P}$ -S in inertial quaternion-valued neural networks. Instead of simplifying the equations into real numbers, the authors used a direct approach to handle proportional delays and inertial terms.

$\mathcal{P}$ -S is a versatile form that encompasses complete synchronization and anti-synchronization as special cases. It describes a scenario in which the response system follows the drive system according to a scaling factor under the action of an appropriate controller. This type of synchronization has attracted significant interest due to its ability to maintain proportional relationships between systems and to enable faster signal transmission [32, 33]. For instance, [32] investigated adaptive $\mathcal{P}$ -S for fractional-order neural networks with delays. Similarly, [33] studied the $\mathcal{P}$ -S of fully quaternion-valued memristive networks. Moreover, synchronization of fractional-order quaternion-valued networks characterized by inconsistent orders is essential for constructing adaptable and selective frameworks for information processing [39]. From an application perspective, variations in parameters or non-ideal hardware conditions may lead to structural or functional inconsistencies between coupled systems. These discrepancies make the synchronization of non-identical fractional-order neural networks particularly challenging, requiring careful analysis and robust control strategies.

Although many studies on fractional-order quaternion-valued neural networks have focused on synchronization, they typically assume that the drive and response systems are identical [31, 32, 33]. In addition, several recent studies have examined synchronization and stability issues in complex-valued and quaternion-valued ComNNs. The authors in [34] developed a nonlinear control scheme to achieve quasi-synchronization and adaptive synchronization in fractional-order fully complex-valued ComNNs, based on recently established lemmas concerning inequalities with complex variables and fractional differential equations. In [35], the authors investigated the existence and uniqueness of equilibrium points, as well as the global

Mittag-Leffler stability of fully complex-valued, fractional-order competitive neural networks, using a fractional Lyapunov approach combined with inequality-based techniques. Furthermore, in [36], the projective synchronization of ComNNs with quaternion-valued inputs and multiple time scales was investigated using non-smooth analysis methods and quaternion inequalities. In [37], the problem of projective synchronization in uncertain ComNNs with identical fractional-order quaternion-valued inputs was addressed through the design of quaternion-valued feedback controllers. In practical situations, however, mismatches between these systems are almost inevitable. Such differences can prevent the synchronization error from converging to zero, destabilizing the synchronous state and degrade synchronization performance in practical applications. It should be noted that only a few studies, such as [37], have examined projective synchronization of quaternion-valued fractional-order ComNNs, and that these studies are primarily focused on identical systems with the same fractional order. Motivated by the preceding studies, this work examines the problem of $\mathcal{P}\text{-}\mathcal{S}$ for a specific class of non-identical quaternion-valued fractional-order ComNNs. A key focus of this study is handling systems with inconsistent fractional orders, which adds to the complexity of the synchronization process.

This manuscript is an extended and significantly improved version of a full paper entitled "Synchronization problem for delayed quaternion-valued neural networks" [38] presented at ICRAMI 2025 (Tunisia, November 2025). In the version presented at the conference, the considered model was based on the classical (integer-order) dynamics of inertial quaternion-valued neural networks with proportional delays. In contrast, this paper explores a more general framework by introducing fractional-order dynamics, which account for memory effects and provide a more accurate description of complex systems. Furthermore, this work addresses $\mathcal{P}\text{-}\mathcal{S}$ for non-identical quaternion-valued ComNNs with inconsistent fractional orders, proposes a new control strategy, and establishes more comprehensive theoretical results supported by extensive numerical simulations. The main contributions of this paper can be summarized as follows:

- This paper investigates, for the first time, the problem of $\mathcal{P}\text{-}\mathcal{S}$ for a class of non-identical quaternion-valued fractional-order ComNNs with inconsistent fractional orders, which extends existing works that mainly focus on identical systems with the same order [37].
- Unlike many existing approaches, the proposed analysis is carried out directly in the quaternion domain without using decomposition methods.
- A novel feedback control strategy is designed to achieve $\mathcal{P}\text{-}\mathcal{S}$, including parameter mismatches and different fractional orders between the drive and response systems.
- Unlike many studies that analyze fractional-order ComNNs dynamics using decomposition methods, this paper uses a non-decomposition approach based on quaternion properties.

The remainder of this article is organized as follows. In Section 2, we introduce the foundational mathematical tools regarding quaternion algebra and fractional derivatives. Section 3 is dedicated to the formulation of the network model and the definition of the synchronization problem. The core theoretical results and stability criteria are detailed in Section 4. To demonstrate the validity of these results, Section 5 presents illustrative numerical examples, while Section 6 provides a summary and closing remarks.

2 Preliminaries

We begin by defining the notation adopted in this work, followed by a brief overview of the Caputo fractional derivative and the statement of key preliminary lemmas.

We denote by $\mathbb{R}$ the real numbers and by $\mathbb{Q}$ the quaternion algebra. A quaternion $\mathbf{p} \in \mathbb{Q}$ has the form $\mathbf{p} = \mathbf{p}^R + i\mathbf{p}^I + j\mathbf{p}^J + k\mathbf{p}^K$, where $\mathbf{p}^R, \mathbf{p}^I, \mathbf{p}^J, \mathbf{p}^K \in \mathbb{R}$ and the basis elements i, j, k follow the non-commutative multiplication rules defined by Hamilton: $i^2 = j^2 = k^2 = ijk = -1$. From these, the pairwise products $ij = k, jk = i, \text{ and } ki = j$ follow, with anticommutation (e.g., $ij = -ji$). For a quaternion $\mathbf{p} = \mathbf{p}^R + i\mathbf{p}^I + j\mathbf{p}^J + k\mathbf{p}^K$, its conjugate is defined as $\bar{\mathbf{p}} = \mathbf{p}^R - i\mathbf{p}^I - j\mathbf{p}^J - k\mathbf{p}^K$. From this, two important norms are introduced: the ℓ_1 -norm, given by $\|\mathbf{p}\|_1 = |\mathbf{p}^R| + |\mathbf{p}^I| + |\mathbf{p}^J| + |\mathbf{p}^K|$, and the ℓ_2 -norm (or Euclidean norm), which can be expressed equivalently as $\|\mathbf{p}\|_2 = \sqrt{\mathbf{p}\bar{\mathbf{p}}} = \sqrt{\bar{\mathbf{p}}\mathbf{p}} = \sqrt{(\mathbf{p}^R)^2 + (\mathbf{p}^I)^2 + (\mathbf{p}^J)^2 + (\mathbf{p}^K)^2}$.

Definition 2.1. [3] For a continuously differentiable function $\theta(t)$, the Caputo fractional derivative with order σ where $0 < \sigma \leq 1$ is defined by

$${}^C_{t_0} \mathcal{D}_t^\sigma \theta(t) = \begin{cases} \frac{1}{\Gamma(1-\sigma)} \int_{t_0}^t (t-\vartheta)^{-\sigma} \theta'(\vartheta) d\vartheta, & 0 < \sigma < 1, \\ \theta'(t), & \sigma = 1. \end{cases}$$

Lemma 2.2. [32] Let $\theta(t) \in \mathbb{Q}$ be a differentiable quaternion-valued function. For any fractional order $0 < \sigma \leq 1$, the following inequality holds for the Caputo derivative

$${}^C_{t_0} \mathcal{D}_t^\sigma (\overline{\theta(t)} \theta(t)) \leq \left({}^C_{t_0} \mathcal{D}_t^\sigma \overline{\theta(t)} \right) \theta(t) + \overline{\theta(t)} {}^C_{t_0} \mathcal{D}_t^\sigma \theta(t), t \geq t_0.$$

Lemma 2.3. [32] For any two quaternions $\theta_1, \theta_2 \in \mathbb{Q}$ and a given positive scalar $\epsilon > 0$, the following inequality is satisfied:

$$\overline{\theta_1} \theta_2 + \overline{\theta_2} \theta_1 \leq \epsilon \overline{\theta_1} \theta_1 + \epsilon^{-1} \overline{\theta_2} \theta_2.$$

Lemma 2.4. [40] Consider the Mittag-Leffler function $E_\sigma(\cdot)$ with an order $0 < \sigma \leq 1$. For any $t \geq t_0$ and $\kappa \leq 0$, the function $E_\sigma(\kappa(t-t_0)^\sigma)$ is bounded and non-increasing. Specifically, it satisfies the following property

$$0 \leq E_\sigma(\kappa(t-t_0)^\sigma) \leq 1.$$

Lemma 2.5. [41] Consider a differentiable scalar function $\mathfrak{L}(t)$ defined on $[t_0, \infty)$. If $\mathfrak{L}(t)$ satisfies the following fractional differential inequality for $t \geq t_0$,

$${}^C_{t_0} \mathcal{D}_t^\sigma \mathfrak{L}(t) \leq -\aleph \mathfrak{L}(t) + \tilde{\aleph},$$

where $\aleph > 0$, $\tilde{\aleph} \geq 0$, then the function $\mathfrak{L}(t)$ is bounded by:

$$\mathfrak{L}(t) \leq \left(\mathfrak{L}(t_0) - \frac{\tilde{\aleph}}{\aleph} \right) E_\sigma(-\aleph(t-t_0)^\sigma) + \frac{\tilde{\aleph}}{\aleph}.$$

Lemma 2.6. [3] Let $\theta_1, \theta_2 \in \mathbb{Q}$ be any two quaternions. The conjugation operator satisfies the following algebraic relations:

$$(i) \quad \overline{\overline{\theta_1} + \overline{\theta_2}} = \bar{\theta}_1 + \bar{\theta}_2.$$

$$(ii) \quad \overline{\theta_1 \theta_2} = \bar{\theta}_2 \bar{\theta}_1.$$

Remark 2.7. Unlike integer-order systems, which consider only the present moment, fractional-order systems take into account memory and history. This allows them to better describe real-world phenomena such as strange diffusion, or biological and neural processes, where past events continue to influence current behavior [42]. In fact, any real-world system with some form of memory can be modeled using fractional derivatives. Overall, fractional-order models offer us greater flexibility and precision in capturing the dynamics of concrete systems. In this paper, we use these advantages to study projective synchronization in non-identical quaternion fractional-order ComNNs with inconsistent orders.

3 Problem Formulation

This study investigates a system of fractional-order quaternion-valued ComNNs, which is defined by the equations below

$$\begin{cases} \text{STM} : {}^C_{t_0} \mathcal{D}_t^\sigma x_i(t) = -a_i x_i(t) + \sum_{j=1}^n b_{ij} f_j(x_j(t)) + B_i \sum_{l=1}^m y_{il}(t) \omega_l + I_i(t) \\ \text{LTM} : {}^C_{t_0} \mathcal{D}_t^\sigma y_{il}(t) = -c_i y_{il}(t) + d_i \omega_l f_i(x_i(t)), \end{cases} \quad (3.1)$$

In model (3.1), the indices i, j range from 1 to n , and l varies from 1 to m . The strength of the external input is quantified by B_i . For each neuron, $x_i(t) \in \mathbb{Q}$ represents its current activity level, which changes according to a fractional order $0 < \sigma \leq 1$. The internal self-inhibition of the neuron is given by $a_i > 0$. b_{ij} represents the connection between neurons i and j . The activation functions are determined by $f_j(\cdot) \in \mathbb{Q}$. y_{il} denotes the synaptic efficacy. ω_l is the constant external stimulus. The constants c_i and d_i represent adjustable scaling parameters, with $c_i > 0$. I_i is the external input vector.

To simplify the analysis of system (3.1), we define $z_i(t) = \sum_{l=1}^m y_{il}(t)\omega_l$. Substituting this expression into system (3.1) yields

$$\begin{cases} \text{STM} : {}^C_{t_0}\mathcal{D}_t^\sigma x_i(t) = -a_i x_i(t) + \sum_{j=1}^n b_{ij} f_j(x_j(t)) + B_i z_i(t) + I_i(t) \\ \text{LTM} : {}^C_{t_0}\mathcal{D}_t^\sigma z_i(t) = -c_i z_i(t) + d_i \|\omega\| f_i(x_i(t)), \end{cases} \quad (3.2)$$

where $\|\omega\|^2 = \sum_{l=1}^m \omega_l^2$ is a constant. Without loss of generality, we assume that the external input vector $\omega = (\omega_1, \omega_2, \dots, \omega_m)^T$ is normalized, i.e., $\|\omega\|^2 = 1$. As a result, system (3.2) can be reformulated as

$$\begin{cases} \text{STM} : {}^C_{t_0}\mathcal{D}_t^\sigma x_i(t) = -a_i x_i(t) + \sum_{j=1}^n b_{ij} f_j(x_j(t)) + B_i z_i(t) + I_i(t) \\ \text{LTM} : {}^C_{t_0}\mathcal{D}_t^\sigma z_i(t) = -c_i z_i(t) + d_i f_i(x_i(t)), \end{cases} \quad (3.3)$$

The drive-response synchronization pair is constructed by taking Eq. (3.3) as the drive system and defining the response system as

$$\begin{cases} \text{STM} : {}^C_{t_0}\mathcal{D}_t^\rho u_i(t) = -\hat{a}_i u_i(t) + \sum_{j=1}^n \hat{b}_{ij} g_j(u_j(t)) + \hat{B}_i v_i(t) + \hat{I}_i(t) + U_i(t) \\ \text{LTM} : {}^C_{t_0}\mathcal{D}_t^\rho v_i(t) = -\hat{c}_i v_i(t) + \hat{d}_i g_i(u_i(t)) + V_i(t), \end{cases} \quad (3.4)$$

In (3.4), $u_i(t)$ and $v_i(t)$ represent the state variables for the Short-Term Memory (STM) and Long-Term Memory (LTM) layers, respectively. The parameter ρ denotes the order of the Caputo fractional derivative, satisfying $0 < \rho \leq 1$. The internal characteristics of the response system are defined by the coefficients $\hat{a}_i, \hat{b}_{ij}, \hat{B}_i, \hat{c}_i$, and $\hat{d}_i$, which may differ from the drive system's parameters. Additionally, $\hat{I}_i(t)$ corresponds to the external input vector, while $g_j(\cdot)$ acts as the neuronal activation function. To ensure the system achieves the desired synchronization objectives, the control signals $U_i(t)$ and $V_i(t)$ are strategically implemented as external inputs for both the STM and LTM components.

To examine the quaternion $\mathcal{P}$ -S between systems (3.3) and (3.4), the error states are defined as $\alpha_i(t) = u_i(t) - \lambda x_i(t)$ and $\beta_i(t) = v_i(t) - \lambda z_i(t)$, where $\lambda \in \mathbb{R}$ represents the projective coefficient.

From Eqs. (3.3) and (3.4), it follows that

$$\begin{aligned}
 {}^C_{t_0}\mathcal{D}_t^\rho u_i(t) - \lambda {}^C_{t_0}\mathcal{D}_t^\sigma x_i(t) &= -\widehat{a}_i u_i(t) + \sum_{j=1}^n \widehat{b}_{ij} g_j(u_j(t)) + \widehat{B}_i v_i(t) + \widehat{I}_i(t) + U_i(t) \\
 &\quad - \lambda \left(-a_i x_i(t) + \sum_{j=1}^n b_{ij} f_j(x_j(t)) + B_i z_i(t) + I_i(t) \right) \\
 &= -\widehat{a}_i \alpha_i(t) + \lambda (a_i - \widehat{a}_i) x_i(t) + \sum_{j=1}^n \widehat{b}_{ij} \left(g_j(u_j(t)) - g_j(\lambda x_j(t)) \right) \\
 &\quad + \sum_{j=1}^n \widehat{b}_{ij} g_j(\lambda x_j(t)) - \sum_{j=1}^n \lambda b_{ij} f_j(x_j(t)) \\
 &\quad + \widehat{B}_i v_i(t) - B_i \lambda z_i(t) + \widehat{I}_i(t) - \lambda I_i(t) + U_i(t)
 \end{aligned} \tag{3.5}$$

and

$$\begin{aligned}
 {}^C_{t_0}\mathcal{D}_t^\rho v_i(t) - \lambda {}^C_{t_0}\mathcal{D}_t^\sigma z_i(t) &= -\widehat{c}_i v_i(t) + \widehat{d}_i g_i(u_i(t)) + V_i(t) \\
 &\quad - \lambda \left(-c_i z_i(t) + d_i f_i(x_i(t)) \right) \\
 &= -\widehat{c}_i \beta_i(t) + \lambda (c_i - \widehat{c}_i) z_i(t) + \widehat{d}_i \left(g_i(u_i(t)) - g_i(\lambda x_i(t)) \right) \\
 &\quad + \widehat{d}_i g_i(\lambda x_i(t)) - \lambda d_i f_i(x_i(t)) + V_i(t)
 \end{aligned} \tag{3.6}$$

To establish the primary results, we assume that the functions f and g satisfy the Lipschitz condition.

(A1) We assume that the functions f and g satisfy the Lipschitz condition:

$$\|f_i(x) - f_i(y)\| \leq L_i^f \|x - y\|,$$

and

$$\|g_i(x) - g_i(y)\| \leq L_i^g \|x - y\|,$$

for all $x, y \in \mathbb{Q}$, where L_i^f and L_i^g are positive constants.

Definition 3.1. Systems (3.3) and (3.4) achieve $\mathcal{P}$ - $\mathcal{S}$ if there exists a constant $\lambda \in \mathbb{R}$ such that the synchronization error

$$\lim_{t \rightarrow +\infty} \left(\|u_i(t) - \lambda x_i(t)\| + \|v_i(t) - \lambda z_i(t)\| \right) = 0.$$

for all $i = 1, 2, \dots, n$. Here, λ represents the projective coefficient.

Remark 3.2. The synchronization behavior defined above is versatile, as the scaling factor λ determines the specific type of collective dynamics:

- When $\lambda = 1$, the synchronization error reduces to a standard state-matching problem, known as complete synchronization.
- In the case where $\lambda = -1$, the states of the drive and response networks evolve in opposite directions, a phenomenon referred to as anti-synchronization.

It is worth noting that $\mathcal{P}$ - $\mathcal{S}$ is a more general framework, as it allows for an arbitrary proportional relationship between the coupled systems.

Remark 3.3. In this paper, we go beyond complex-valued models by using quaternion-type fractional derivatives (Caputo-type). Because quaternions have four dimensions, they allow us to

naturally treat elements such as rotation, amplitude, and phase together, which proves very useful for 3D dynamics, color images, or neural networks exhibiting competitive behavior. The addition of Caputo derivatives introduces memory and long-range effects into the model. For our study on the projective synchronization of non-identical quaternion-based fractional-order ComNNs with inconsistent orders, this configuration offers greater flexibility and a more realistic way to model real-world systems where space and time are closely linked.

4 Main results

This section focuses on establishing a sufficient condition for the $\mathcal{P}$ - $\mathcal{S}$ of quaternion-valued ComNNs by introducing an appropriate control strategy.

The $\mathcal{P}$ - $\mathcal{S}$ objective is achieved by designing the controllers in the following form:

$$U_i(t) = U_{1i}(t) + U_{2i}(t) + U_{3i}(t) \quad (4.1)$$

where

$$\begin{cases} U_{1i}(t) = \lambda_{t_0}^C \mathcal{D}_t^\rho x_i(t) - \lambda_{t_0}^C \mathcal{D}_t^\sigma x_i(t) \\ U_{2i}(t) = -\lambda(a_i - \hat{a}_i)x_i(t) - \sum_{j=1}^n \hat{b}_{ij}g_j(\lambda x_j(t)) + \sum_{j=1}^n \lambda b_{ij}f_j(x_j(t)) - \hat{B}_i v_i(t) + B_i \lambda z_i(t) \\ \quad - \hat{I}_i(t) + \lambda I_i(t) \\ U_{3i}(t) = -\varsigma_i \alpha_i(t) \end{cases}$$

and

$$V_i(t) = V_{1i}(t) + V_{2i}(t) + V_{3i}(t) \quad (4.2)$$

where

$$\begin{cases} V_{1i}(t) = \lambda_{t_0}^C \mathcal{D}_t^\rho z_i(t) - \lambda_{t_0}^C \mathcal{D}_t^\sigma z_i(t) \\ V_{2i}(t) = -\lambda(c_i - \hat{c}_i)z_i(t) - \hat{d}_i g_i(\lambda x_i(t)) + \lambda d_i f_i(x_i(t)) \\ V_{3i}(t) = -\hat{\varsigma}_i \beta_i(t) \end{cases}$$

where $\varsigma_i > 0$ and $\hat{\varsigma}_i > 0$.

By substituting the controllers defined in (4.1) and (4.2) into systems (3.5) and (3.6), respectively, the resulting closed-loop dynamics can be expressed as

$$\begin{cases} \text{STM} : {}_{t_0}^C \mathcal{D}_t^\rho \alpha_i(t) = -(\hat{a}_i + \varsigma_i)\alpha_i(t) + \sum_{j=1}^n \hat{b}_{ij} \left(g_j(u_j(t)) - g_j(\lambda x_j(t)) \right) \\ \text{LTM} : {}_{t_0}^C \mathcal{D}_t^\rho \beta_i(t) = -(\hat{c}_i + \hat{\varsigma}_i)\beta_i(t) + \hat{d}_i \left(g_i(u_i(t)) - g_i(\lambda x_i(t)) \right). \end{cases} \quad (4.3)$$

Theorem 4.1. *Under the control strategy specified in (4.1) and (4.2), systems (3.3) and (3.4) are able to realize $\mathcal{P}$ - $\mathcal{S}$, assuming that Assumption (A1) holds and the following condition is satisfied*

$$\aleph = \min\{\aleph_1, \aleph_2\} > 0, \quad (4.4)$$

where

$$\begin{aligned} \aleph_1 &= \min_{1 \leq i \leq n} \left\{ -(\hat{a}_i + \varsigma_i) + \frac{n+1}{2} (L_i^g)^2 + \frac{1}{2} \sum_{j=1}^n \|\hat{b}_{ij}\|^2 \right\}, \\ \aleph_2 &= \min_{1 \leq i \leq n} \left\{ -(\hat{c}_i + \hat{\varsigma}_i) + \frac{1}{2} \hat{d}_i^2 \right\}. \end{aligned}$$

Proof. To establish the $\mathcal{P}$ - $\mathcal{S}$ between systems (3.3) and (3.4), we construct an appropriate Lyapunov function based on the synchronization error.

Consider the following Lyapunov candidate function:

$$\mathfrak{L}(t) = \frac{1}{2} \sum_{i=1}^n \left(\overline{\alpha_i(t)} \alpha_i(t) + \overline{\beta_i(t)} \beta_i(t) \right). \quad (4.5)$$

According to the fractional Lyapunov lemma for quaternion-valued systems (see, Lemma 2.2), the ρ -order Caputo derivative (with $0 < \rho < 1$) of $\mathfrak{L}(t)$ along the trajectories of the error system satisfies:

$${}_{t_0}^C \mathcal{D}_t^\rho \mathfrak{L}(t) \leq \frac{1}{2} \sum_{i=1}^n \left(({}_{t_0}^C \mathcal{D}_t^\rho \overline{\alpha_i(t)}) \alpha_i(t) + \overline{\alpha_i(t)} ({}_{t_0}^C \mathcal{D}_t^\rho \alpha_i(t)) \right) \quad (4.6)$$

$$+ ({}_{t_0}^C \mathcal{D}_t^\rho \overline{\beta_i(t)}) \beta_i(t) + \overline{\beta_i(t)} ({}_{t_0}^C \mathcal{D}_t^\rho \beta_i(t)). \quad (4.7)$$

Substituting the error dynamics (4.3) into the above inequality, we obtain

$$\begin{aligned} {}_{t_0}^C \mathcal{D}_t^\rho \mathfrak{L}(t) &\leq \frac{1}{2} \sum_{i=1}^n \left[\overline{-(\hat{a}_i + \varsigma_i) \alpha_i(t) + \sum_{j=1}^n \hat{b}_{ij} \left(g_j(u_j(t)) - g_j(\lambda x_j(t)) \right)} \right] \alpha_i(t) \\ &+ \frac{1}{2} \sum_{i=1}^n \overline{\alpha_i(t)} \left[-(\hat{a}_i + \varsigma_i) \alpha_i(t) + \sum_{j=1}^n \hat{b}_{ij} \left(g_j(u_j(t)) - g_j(\lambda x_j(t)) \right) \right] \\ &+ \frac{1}{2} \sum_{i=1}^n \left[\overline{-(\hat{c}_i + \hat{\varsigma}_i) \beta_i(t) + \hat{d}_i \left(g_i(u_i(t)) - g_i(\lambda x_i(t)) \right)} \right] \beta_i(t) \\ &+ \frac{1}{2} \sum_{i=1}^n \overline{\beta_i(t)} \left[-(\hat{c}_i + \hat{\varsigma}_i) \beta_i(t) + \hat{d}_i \left(g_i(u_i(t)) - g_i(\lambda x_i(t)) \right) \right] \\ &= \frac{1}{2} \sum_{i=1}^n \left[-(\hat{a}_i + \varsigma_i) \overline{\alpha_i(t)} \alpha_i(t) + \sum_{j=1}^n \overline{\hat{b}_{ij} \left(g_j(u_j(t)) - g_j(\lambda x_j(t)) \right)} \alpha_i(t) \right] \\ &+ \frac{1}{2} \sum_{i=1}^n \left[-(\hat{a}_i + \varsigma_i) \overline{\alpha_i(t)} \alpha_i(t) + \sum_{j=1}^n \overline{\alpha_i(t)} \hat{b}_{ij} \left(g_j(u_j(t)) - g_j(\lambda x_j(t)) \right) \right] \\ &+ \frac{1}{2} \sum_{i=1}^n \left[-(\hat{c}_i + \hat{\varsigma}_i) \overline{\beta_i(t)} \beta_i(t) + \overline{\hat{d}_i \left(g_i(u_i(t)) - g_i(\lambda x_i(t)) \right)} \beta_i(t) \right] \\ &+ \frac{1}{2} \sum_{i=1}^n \left[-(\hat{c}_i + \hat{\varsigma}_i) \overline{\beta_i(t)} \beta_i(t) + \overline{\hat{d}_i \beta_i(t)} \left(g_i(u_i(t)) - g_i(\lambda x_i(t)) \right) \right]. \quad (4.8) \end{aligned}$$

According to Assumption A1, Lemma 2.3 and Lemma 2.6, we have

$$\begin{aligned} &\sum_{i=1}^n \sum_{j=1}^n \left[\overline{\left(g_j(u_j(t)) - g_j(\lambda x_j(t)) \right)} \hat{b}_{ij} \alpha_i(t) + \overline{\alpha_i(t)} \hat{b}_{ij} \left(g_j(u_j(t)) - g_j(\lambda x_j(t)) \right) \right] \\ &\leq \sum_{i=1}^n \sum_{j=1}^n \left(\overline{g_j(u_j(t)) - g_j(\lambda x_j(t))} \right) \left(g_j(u_j(t)) - g_j(\lambda x_j(t)) \right) \\ &+ \sum_{i=1}^n \sum_{j=1}^n \overline{\alpha_i(t)} \hat{b}_{ij} \overline{\hat{b}_{ij}} \alpha_i(t) \\ &\leq n \sum_{i=1}^n (L_i^g)^2 \overline{\alpha_i(t)} \alpha_i(t) + \sum_{i=1}^n \sum_{j=1}^n \|\hat{b}_{ij}\|^2 \overline{\alpha_i(t)} \alpha_i(t). \quad (4.9) \end{aligned}$$

Similarly, we have

$$\begin{aligned} & \sum_{i=1}^n \left(\overline{g_i(u_i(t)) - g_i(\lambda x_i(t))} \right) \widehat{d}_i \beta_i(t) + \widehat{d}_i \overline{\beta_i(t)} \left(g_i(u_i(t)) - g_i(\lambda x_i(t)) \right) \\ & \leq \sum_{i=1}^n (L_i^g)^2 \overline{\alpha_i(t)} \alpha_i(t) + \sum_{i=1}^n \widehat{d}_i^2 \overline{\beta_i(t)} \beta_i(t). \end{aligned} \quad (4.10)$$

By combining inequalities (4.9) and (4.10), we obtain

$$\begin{aligned} {}^C_{t_0} \mathcal{D}_t^\rho \mathfrak{L}(t) & \leq \sum_{i=1}^n \left(-(\widehat{a}_i + \varsigma_i) + \frac{n+1}{2} (L_i^g)^2 + \frac{1}{2} \sum_{j=1}^n \|\widehat{b}_{ij}\|^2 \right) \overline{\alpha_i(t)} \alpha_i(t) \\ & \quad + \sum_{i=1}^n \left(-(\widehat{c}_i + \widehat{\varsigma}_i) + \frac{1}{2} \widehat{d}_i^2 \right) \overline{\beta_i(t)} \beta_i(t) \\ & \leq -\aleph_1 \sum_{i=1}^n \overline{\alpha_i(t)} \alpha_i(t) - \aleph_2 \sum_{i=1}^n \overline{\beta_i(t)} \beta_i(t), \end{aligned} \quad (4.11)$$

therefore

$${}^C_{t_0} \mathcal{D}_t^\rho \mathfrak{L}(t) \leq -2\aleph \mathfrak{L}(t). \quad (4.12)$$

On the basis of Lemma 2.5 with $\widetilde{\aleph} = 0$ together with Eq. (4.12), it can be concluded that

$$\mathfrak{L}(t) \leq \mathfrak{L}(t_0) E_\rho(-2\aleph(t-t_0)^\rho), t \geq t_0 \geq 0. \quad (4.13)$$

Because the Mittag-Leffler function satisfies $\lim_{t \rightarrow +\infty} E_\rho(-\aleph(t-t_0)^\rho) = 0$, it follows that $\mathfrak{L}(t) \rightarrow 0$ as $t \rightarrow \infty$. Consequently,

$$\|\alpha(t)\| + \|\beta(t)\| \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

Therefore, under the controller (4.1) and (4.2), the drive system (3.3) achieves $\mathcal{P}$ -S with the response system (3.4). $\square$

Corollary 4.2. Consider the drive system (3.3) and the response system (3.4) of identical fractional order $\sigma = \rho$ and with the same coefficients. The systems can achieve $\mathcal{P}$ -S under the linear control law

$$U_i(t) = U_{2i}(t) + U_{3i}(t) \quad (4.14)$$

where

$$\begin{cases} U_{2i}(t) = -\lambda(a_i - \widehat{a}_i)x_i(t) - \sum_{j=1}^n \widehat{b}_{ij}g_j(\lambda x_j(t)) + \sum_{j=1}^n \lambda b_{ij}f_j(x_j(t)) - \widehat{B}_i v_i(t) + B_i \lambda z_i(t) \\ \quad - \widehat{I}_i(t) + \lambda I_i(t) \\ U_{3i}(t) = -\varsigma_i \alpha_i(t) \end{cases}$$

and

$$V_i(t) = V_{2i}(t) + V_{3i}(t) \quad (4.15)$$

where

$$\begin{cases} V_{2i}(t) = -\lambda(c_i - \widehat{c}_i)z_i(t) - \widehat{d}_i g_i(\lambda x_i(t)) + \lambda d_i f_i(x_i(t)) \\ V_{3i}(t) = -\widehat{\varsigma}_i \beta_i(t) \end{cases}$$

if

$$\aleph = \min\{\aleph_1, \aleph_2\} > 0. \quad (4.16)$$

Remark 4.3. Most existing studies on ComNNs have been conducted in real-valued ($\mathbb{R}$) and complex-valued ($\mathbb{C}$) settings under integer-order dynamics, where various synchronization results have been established [43, 44, 45]. Moreover, the work in [36] is restricted to integer-order quaternion-valued neural networks, where several results on $\mathcal{P}$ -S were obtained. In contrast, the present paper considers a more general and challenging framework by introducing fractional-order dynamics, which capture memory and hereditary effects and provide greater modeling flexibility. Based on this, we investigate $\mathcal{P}$ -S for non-identical quaternion-valued ComNNs with inconsistent fractional orders, thereby extending and generalizing existing results.

5 Numerical simulations

The effectiveness of the derived theoretical results is examined through numerical simulations, which are reported in this section.

Consider a fractional-order quaternion-valued ComNNs system described by the following equations.

$$\begin{cases} \text{STM} : {}^C_{t_0}\mathcal{D}_t^\sigma x_i(t) = -a_i x_i(t) + \sum_{j=1}^2 b_{ij} f_j(x_j(t)) + B_i z_i(t) + I_i(t) \\ \text{LTM} : {}^C_{t_0}\mathcal{D}_t^\sigma z_i(t) = -c_i z_i(t) + d_i f_i(x_i(t)), \end{cases} \quad (5.1)$$

where $i = 1, 2$, $x_i(t) = x_i^R(t) + ix_i^I(t) + jx_i^J(t) + kx_i^K(t)$ with $x_i^R(t), x_i^I(t), x_i^J(t), x_i^K(t) \in \mathbb{R}$, $z_i(t) = z_i^R(t) + iz_i^I(t) + jz_i^J(t) + kz_i^K(t)$ with $z_i^R(t), z_i^I(t), z_i^J(t), z_i^K(t) \in \mathbb{R}$. $\sigma = 0.86, a_1 = a_2 = 1.9, c_1 = c_2 = 0.9, d_1 = 1, d_2 = 2.3, B_1 = 1.2, B_2 = 1.1, f_i(x_i(t)) = \tanh(x_i^R(t)) + i \tanh(x_i^I(t)) + j \tanh(x_i^J(t)) + k \tanh(x_i^K(t)), I_1(t) = I_2(t) = 0$,

$$(b_{ij})_{2 \times 2} = \begin{pmatrix} 1.9 + 2.2i + 2.1j + 2.1k & -0.2 - 0.2i - 0.2j - 0.2k \\ -2 + 2.9i - 2.1j - 2.9k & 1 + 0.9i + 0.9j + 0.9k \end{pmatrix}$$

The response model is given by

$$\begin{cases} \text{STM} : {}^C_{t_0}\mathcal{D}_t^\rho u_i(t) = -\hat{a}_i u_i(t) + \sum_{j=1}^n \hat{b}_{ij} g_j(u_j(t)) + \hat{B}_i v_i(t) + \hat{I}_i(t) + U_i(t) \\ \text{LTM} : {}^C_{t_0}\mathcal{D}_t^\rho v_i(t) = -\hat{c}_i v_i(t) + \hat{d}_i g_i(u_i(t)) + V_i(t), \end{cases} \quad (5.2)$$

where $i = 1, 2$, $u_i(t) = u_i^R(t) + iu_i^I(t) + ju_i^J(t) + ku_i^K(t)$ with $u_i^R(t), u_i^I(t), u_i^J(t), u_i^K(t) \in \mathbb{R}$, $v_i(t) = v_i^R(t) + iv_i^I(t) + jv_i^J(t) + kv_i^K(t)$ with $v_i^R(t), v_i^I(t), v_i^J(t), v_i^K(t) \in \mathbb{R}$. $\rho = 0.96, \hat{a}_1 = \hat{a}_2 = 0.9, \hat{c}_1 = \hat{c}_2 = 1.9, \hat{d}_1 = 4, \hat{d}_2 = 3.3, \hat{B}_1 = 0.2, \hat{B}_2 = 0.1, g_i(u_i(t)) = \sin(u_i^R(t)) + i \sin(u_i^I(t)) + j \sin(u_i^J(t)) + k \sin(u_i^K(t)), \hat{I}_1(t) = \hat{I}_2(t) = 0$,

$$(\hat{b}_{ij})_{2 \times 2} = \begin{pmatrix} 2.8 + 1.2i + 1.1j + 1.1k & -0.4 - 0.4i - 0.4j - 0.4k \\ -1 + 1.9i - 1.1j - 1.9k & 2 + 0.5i + 0.5j + 0.5k \end{pmatrix}$$

The initial values of the system (5.1) are $x_1(0) = -2 + 0.9i + 0.4j + 1.3k, x_2(0) = 0.2 - 2.8i - 1.5j - 0.1k, z_1(0) = 3.3 - 2.1i - 0.1j - 3.2k, z_2(0) = -2.1 + 2.1i - 0.7j + 0.2k$ and initial conditions of the system (5.2) are $u_1(0) = 4.2 - 1.1i - 2.9j - 1.9k, u_2(0) = -0.1 + 0.9i + 2.7j + 0.4k, v_1(0) = 1.8 - 0.7i + 0.4j + 1.2k, v_2(0) = 2.6 - i + 3.5j - 2.7k$.

The linear feedback controllers $U_i(t)$ and $V_i(t)$ are designed according to equations (4.1) and (4.2), respectively. The parameters are chosen as $\lambda = 1, \varsigma_1 = 2, \varsigma_2 = 3, \hat{\varsigma}_1 = 1$ and $\hat{\varsigma}_2 = 1$. By

simple calculation, we obtain

$$\begin{aligned}\aleph_1 &= \min_{1 \leq i \leq 2} \left\{ -(\hat{a}_i + \varsigma_i) + \frac{n+1}{2} (L_i^g)^2 + \frac{1}{2} \sum_{j=1}^2 \|\hat{b}_{ij}\|^2 \right\} \\ &= \min \left\{ \underbrace{-(0.9 + 2) + \frac{3}{2}}_{\text{for } i=1} + \frac{1}{2}(12.34), \underbrace{-(0.9 + 3) + \frac{3}{2} + \frac{1}{2}(14.18)}_{\text{for } i=2} \right\} \\ &= \min\{4.77, 4.69\} = 4.69\end{aligned}$$

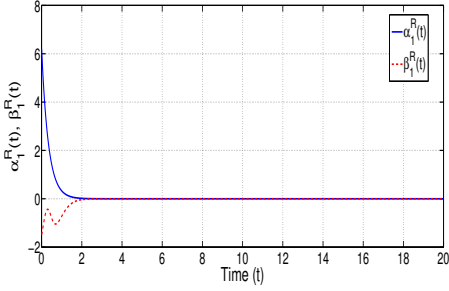
and

$$\begin{aligned}\aleph_2 &= \min_{1 \leq i \leq 2} \left\{ -(\hat{c}_i + \hat{\varsigma}_i) + \frac{1}{2} \hat{d}_i^2 \right\} \\ &= \min \left\{ \underbrace{-(1.9 + 1) + 8}_{\text{for } i=1}, \underbrace{-(1.9 + 1) + \frac{1}{2}(3.3)^2}_{\text{for } i=2} \right\} \\ &= \min\{5.1, 2.545\} = 2.545,\end{aligned}$$

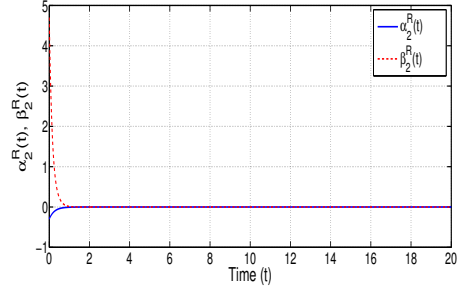
therefore

$$\aleph = \min\{\aleph_1, \aleph_2\} = \min\{4.69, 2.545\} = 2.545 > 0.$$

Hence, the conditions imposed by Theorem 4.1 are verified. As a result, synchronization is achieved between the systems, and the synchronization errors are illustrated in Figs. 1-4.

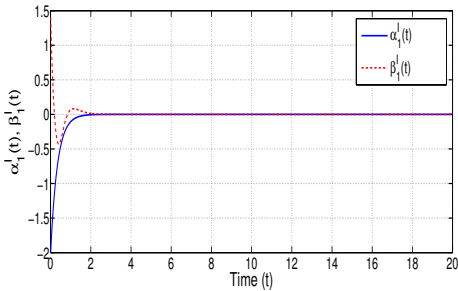


(a) Evolution of the real error $\alpha_1^R(t)$ and $\beta_1^R(t)$.

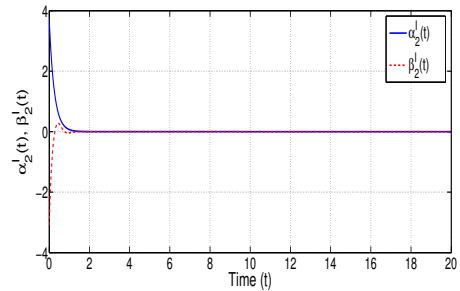


(b) Evolution of the real error $\alpha_2^R(t)$ and $\beta_2^R(t)$.

Figure 1: Time evolution of the synchronization errors for the real parts α^R, β^R .



(a) Evolution of the imaginary error $\alpha_1^I(t)$ and $\beta_1^I(t)$.

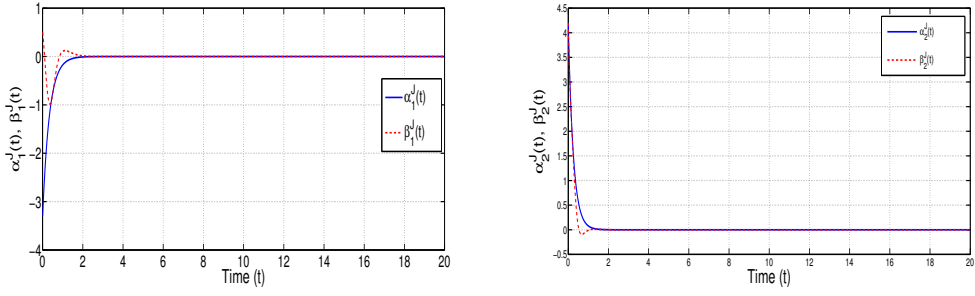


(b) Evolution of the imaginary error $\alpha_2^I(t)$ and $\beta_2^I(t)$.

Figure 2: Time evolution of the synchronization errors for the imaginary parts α^I, β^I .

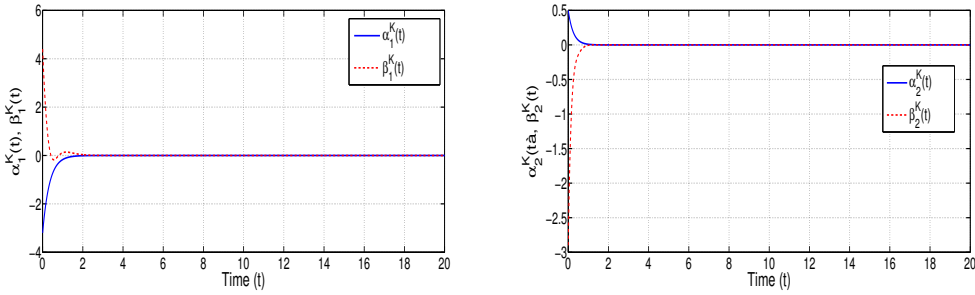
6 Conclusion and Future Work

In this research, we have addressed the challenge of $\mathcal{P}\text{-}\mathcal{S}$ for a specific class of non-identical quaternion-valued ComNNs. The study successfully incorporated fractional-order dynamics



(a) Evolution of the imaginary error $\alpha_1^J(t)$ and $\beta_1^J(t)$. (b) Evolution of the imaginary error $\alpha_2^J(t)$ and $\beta_2^J(t)$.

Figure 3: Time evolution of the synchronization errors for the imaginary parts α^J, β^J .



(a) Evolution of the imaginary error $\alpha_1^K(t)$ and $\beta_1^K(t)$. (b) Evolution of the imaginary error $\alpha_2^K(t)$ and $\beta_2^K(t)$.

Figure 4: Time evolution of the synchronization errors for the imaginary parts α^K, β^K .

where the fractional orders are inconsistent, reflecting more realistic and complex dynamical behaviors. By staying within the quaternion field and avoiding the traditional decomposition into real-valued systems, we preserved the algebraic integrity of the model.

Based on Lyapunov stability theory and the design of an effective control law, we derived several new criteria to ensure that the response system accurately synchronizes with the drive system. The theoretical findings were further validated through numerical simulations, which confirmed the robustness and efficiency of the proposed synchronization scheme. In future work, we aim to extend the present study toward practical applications of quaternion-valued neural networks, particularly in areas such as color image processing, 3D signal representation, and secure information transmission. In [35], the authors studied the dynamical analysis of fractional-order fully complex-valued uncertain competitive neural networks with an application to image privacy protection, which highlights the practical potential of such models. This motivates the investigation of quaternion-valued frameworks for handling higher-dimensional data. In addition, the current model does not consider external disturbances or parameter uncertainties. Therefore, an important direction for future research is to study the robustness of the proposed control strategy by incorporating disturbances and uncertainties, and to develop more general synchronization criteria under realistic conditions.

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