

New Fixed Point Results Under Pata Contraction in Extended b -Metric Spaces and Applications

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Abstract The objective of this paper is twofold. On the theoretical side, we propose a new family of common fixed point criteria, formulated in terms of Pata-type perturbed inequalities, that operate within the rather flexible setting of extended b -metric spaces. On the practical side, the abstract conclusions are then translated into existence and uniqueness statements for several classes of nonlinear problems. Concretely, after laying the necessary background, we prove a sharp asymptotic lemma describing the behaviour of pseudo-Cauchy sequences relative to the control function θ . This lemma serves as the technical engine for our main theorem on triples of self-maps (f, g, h) , from which a number of corollaries of Kannan, Chatterjea and weakly compatible flavour are deduced as particular cases. The applicability of the framework is then illustrated through three independent problems: a periodic first-order boundary value problem of Nieto–Rodríguez–López type, a coupled system of Urysohn integral equations, and a pair of fractional differential equations of Caputo order $1 < k \leq 2$ with nonlocal integral boundary conditions.

1 Introduction and preliminary

The Banach contraction principle [1], formulated more than a century ago, continues to function as one of the central organising results of nonlinear functional analysis. Its remarkable utility lies less in any single application than in the way it has served as a template: an enormous body of subsequent research can be read as the systematic exploration of *what happens* when one or another of its hypotheses is loosened. Two such directions are of particular relevance to the present paper.

The first concerns the geometry of the underlying space. By weakening the triangle inequality through a multiplicative constant $b \geq 1$, Bakhtin [2] and Czerwik [3] introduced the concept of a b -metric space, in which discrete spaces with anomalously large distances and certain function spaces over L^p with $p < 1$ become natural members. A more recent refinement, due to Kamran, Samreen and Ain [4], replaces the constant b by a control function $\theta : E \times E \rightarrow [1, \infty)$ that may depend on the points being compared; the resulting *extended b -metric spaces* [4, 5] accommodate situations in which the distortion factor varies across the space, and they include both ordinary metrics and b -metrics as immediate special cases.

The second direction concerns the contraction inequality itself. Following the seminal work of V. Pata [6], a contractive condition need not take the strict form $d(fx, fy) \leq k d(x, y)$ with $k < 1$. Instead, one allows a perturbation parametrised by an auxiliary variable $\epsilon \in [0, 1]$ and a vanishing control function Ψ . The Pata inequality has the pleasant feature of being a single

uniform statement: setting $\epsilon = 0$ recovers a Kannan- or Chatterjea-style estimate, while letting ϵ vary in $(0, 1]$ provides a tunable error term. As shown in [6] and the subsequent literature, this formulation can produce conclusions that are inaccessible to the classical Banach [1] or Boyd–Wong [7] approaches.

The contribution of the present paper is to combine these two ideas. Working entirely inside the framework of [4, 5], we prove common fixed point theorems for triples (f, g, h) subject to a perturbed inequality of Pata type. Several results scattered in the literature on standard metric spaces [1], b -metric spaces [2, 3], and other generalised structures [8, 9, 10] appear as direct consequences, often by taking $\theta \equiv 1$ or by reducing the triple to a single mapping. We then put the abstract machinery to work on three nonlinear problems: a periodic boundary value problem in the spirit of Nieto and Rodríguez-López [11], a system of Urysohn integral equations, and a pair of Caputo fractional differential equations [12] with nonlocal integral boundary conditions.

The remainder of the manuscript is organised as follows. The rest of Section 1 collects the definitions and auxiliary notions we shall need. Section 2 contains a key technical lemma describing the asymptotic behaviour of non-Cauchy sequences relative to the control function θ , followed by the main common fixed point theorem and a series of corollaries. Section 3 gathers the three applications. A short concluding section summarises the contribution and points to natural extensions.

We open the technical exposition by recalling the relevant ambient structures. Throughout, E stands for a fixed nonempty set; equalities and inequalities involving distance functions are understood pointwise on $E \times E$.

Definition 1.1 ([2, 3]). A function $d_b : E \times E \rightarrow [0, \infty)$ is called a b -metric on E with constant $b \geq 1$ if the three axioms

- (c1) $d_b(x, y) = 0 \iff x = y$;
- (c2) $d_b(x, y) = d_b(y, x)$;
- (c3) $d_b(x, z) \leq b[d_b(x, y) + d_b(y, z)]$;

hold for every triple $x, y, z \in E$. The pair (E, d_b) is then referred to as a b -metric space.

Remark 1.2. In axiom (c3), the constant b should be regarded as a measure of the deviation from the ordinary triangle inequality: smaller values of b correspond to spaces that are closer to being metric, while $b = 1$ recovers a metric space exactly.

Example 1.3. We mention two situations, that motivate the abstract definition above.

- (i) Fix $0 < p < 1$ and let

$$\ell_p(\mathbb{R}) := \left\{ x = \{x_n\}_{n \geq 1} \subset \mathbb{R} : \sum_{n \geq 1} |x_n|^p < \infty \right\}.$$

The map $d_b(x, y) = (\sum_{n \geq 1} |x_n - y_n|^p)^{1/p}$ turns $\ell_p(\mathbb{R})$ into a b -metric space, with constant $b = 2^{1/p}$ (see [13, 14, 15]).

- (ii) For the same range $0 < p < 1$, write

$$L_p[0, 1] := \left\{ x : [0, 1] \rightarrow \mathbb{R} \text{ measurable} : \int_0^1 |x(t)|^p dt < \infty \right\},$$

endowed with $d_b(x, y) = (\int_0^1 |x(t) - y(t)|^p dt)^{1/p}$. Once again $(L_p[0, 1], d_b)$ is a b -metric space with constant $b = 2^{1/p}$ [13, 14, 15].

For background on convergence, completeness, and further examples in b -metric spaces we refer to [13, 14, 15, 8, 5, 16]. The convergence and Cauchy notions for a b -metric are formally identical to the metric ones; we record them in passing.

Definition 1.4 ([16]). Given a b -metric space (E, d_b) and a sequence $\{x_n\} \subset E$:

- (I) $\{x_n\}$ is *Cauchy* when $d_b(x_n, x_m) \rightarrow 0$ as $n, m \rightarrow \infty$;

- (II) $\{x_n\}$ converges to $x \in E$ when $d_b(x_n, x) \rightarrow 0$ as $n \rightarrow \infty$, in which case we write $x_n \rightarrow x$;
 (III) (E, d_b) is complete provided every Cauchy sequence converges in E .

We now turn to the principal ambient structure of the paper. The idea, due to Kamran, Samreen and Ain [4], is to allow the multiplicative constant of axiom (c3) to depend on the pair of points under consideration.

Definition 1.5 ([4]). Let $\theta : E \times E \rightarrow [1, \infty)$ be prescribed. A function $d_\theta : E \times E \rightarrow [0, \infty)$ is called an *extended b -metric* associated with θ when, for each $x, y, z \in E$:

- (d_θ1) $d_\theta(x, y) = 0 \iff x = y$;
 (d_θ2) $d_\theta(x, y) = d_\theta(y, x)$;
 (d_θ3) $d_\theta(x, z) \leq \theta(x, z)[d_\theta(x, y) + d_\theta(y, z)]$.

The triple (E, d_θ, θ) or, when no confusion is possible, simply (E, d_θ) is then called an *extended b -metric space*.

Remark 1.6. A specialisation of the control function recovers the earlier notions in a transparent way. If $\theta \equiv b$ for some $b \geq 1$, Definition 1.5 reduces to Definition 1.1; if in addition $b = 1$, one recovers a metric space. Hence extended b -metric spaces strictly enlarge the class of b -metric spaces.

Remark 1.7. The control function θ may grow without bound as x or y moves to infinity in E . This is precisely what allows extended b -metric spaces to model phenomena, such as data distances on unbounded function spaces, where the distortion of the triangle inequality is point dependent. In all our results, however, we assume that θ remains bounded *along the orbit* of the iterative sequence under consideration; this is a mild and standard hypothesis (see [4, 5]).

The next example, which is original to the present paper, illustrates that the extended b -metric framework can be set up on very small finite spaces and that condition (d_θ3) can be verified by direct numerical inspection.

Example 1.8. Take $E = \{a, b, c\}$ and define

$$\theta(x, y) = 2 + |x|^* + |y|^*, \quad \text{where } |a|^* = 0, |b|^* = 1, |c|^* = 2.$$

Set $d_\theta(x, x) = 0$ for every $x \in E$ and

$$d_\theta(a, b) = d_\theta(b, a) = 10, \quad d_\theta(a, c) = d_\theta(c, a) = 120, \quad d_\theta(b, c) = d_\theta(c, b) = 70.$$

Then (E, d_θ) is an extended b -metric space. Conditions (d_θ1) and (d_θ2) are immediate. For (d_θ3) one only needs to verify the three nontrivial triples (one for each pair of distinct points). Direct computation gives

$$\begin{aligned} d_\theta(a, c) &= 120, & \theta(a, c)[d_\theta(a, b) + d_\theta(b, c)] &= 4(10 + 70) = 320, \\ d_\theta(a, b) &= 10, & \theta(a, b)[d_\theta(a, c) + d_\theta(c, b)] &= 3(120 + 70) = 570, \\ d_\theta(b, c) &= 70, & \theta(b, c)[d_\theta(b, a) + d_\theta(a, c)] &= 5(10 + 120) = 650. \end{aligned}$$

In each case the right-hand side strictly exceeds the left-hand side, and the remaining cases follow by symmetry. Hence (d_θ3) holds.

The reader will observe that the inequality (d_θ3) holds with a great deal of slack in the example above. This is no accident: when θ is bounded below by a constant $b > 1$, condition (d_θ3) is automatically less stringent than its b -metric counterpart, which explains why extended b -metric spaces accommodate distance data that no ordinary b -metric can.

The convergence notions transfer to the extended setting without surprise.

Definition 1.9 ([5, 17]). Let (E, d_θ) be an extended b -metric space and $\{\xi_n\}$ a sequence in E .

- (i) We say $\xi_n \rightarrow \xi_0 \in E$ when, for every $\epsilon > 0$, there is some $N \in \mathbb{N}$ with $d_\theta(\xi_n, \xi_0) < \epsilon$ whenever $n \geq N$.
- (ii) $\{\xi_n\}$ is a *Cauchy sequence* when, for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $d_\theta(\xi_n, \xi_m) < \epsilon$ for all $n, m \geq N$.
- (iii) (E, d_θ) is *complete* when every Cauchy sequence in E has a limit in E .

The following two notions, which originate in the work of Jungck [18], will be used in connection with common fixed points.

Definition 1.10. Given $f, g : E \rightarrow E$, a point $\gamma \in E$ at which $f(\gamma) = g(\gamma)$ is called a *coincidence point* of the pair (f, g) ; the common value $\delta = f(\gamma) = g(\gamma)$ is then a *point of coincidence* of (f, g) .

Definition 1.11 ([18]). Two self-maps f, g on an extended b -metric space (E, d_θ) are said to be *compatible* when, for any sequence $\{\xi_n\} \subset E$ such that $\lim_n f(\xi_n) = \lim_n g(\xi_n) = \varsigma$ for some $\varsigma \in E$, one has $\lim_n d_\theta(f(g(\xi_n)), g(f(\xi_n))) = 0$.

Definition 1.12 ([18]). A pair (f, g) of self-maps on E is *weakly compatible* when f and g commute at every coincidence point: $f\varsigma = g\varsigma \Rightarrow f(g\varsigma) = g(f\varsigma)$.

Finally, we record the following piece of terminology that will be convenient in Section 2: throughout, $\Psi : [0, 1] \rightarrow [0, \infty)$ denotes a fixed continuous nondecreasing function with $\Psi(0) = 0$, and $\tau \geq 0$, $\alpha \geq 1$, $\beta \in [0, \alpha]$ are fixed constants. In every contractive inequality of Pata type the role of Ψ is to control the perturbation as $\epsilon \rightarrow 0$.

2 Main results

This section is the analytical heart of the paper. We open with an asymptotic lemma which describes, with sharp constants, how four naturally associated diagonal sequences behave whenever a sequence with vanishing increments fails to be Cauchy. The lemma is then used as a black box in the proof of our main theorem on triples of self-mappings under the Pata-type inequality (2.10) below; from this theorem we derive several corollaries that cover Pata–Kannan, Pata–Chatterjea and weakly compatible scenarios as special cases.

Lemma 2.1. Let (E, d_θ) be an extended b -metric space whose control function $\theta : E \times E \rightarrow [1, \infty)$ stays uniformly bounded along the sequence under consideration; that is, the constant

$$\theta^* := \sup\{\theta(y_{m_k}, y_{n_k}), \theta(y_{m_k}, y_{n_k+1}), \theta(y_{m_k+1}, y_{n_k}), \theta(y_{m_k+1}, y_{n_k+1})\}$$

is finite. Suppose $\{y_n\} \subset E$ satisfies

$$\lim_{n \rightarrow \infty} d_\theta(y_n, y_{n+1}) = 0, \quad (2.1)$$

but is not Cauchy. Then one can extract $\epsilon > 0$ together with two strictly increasing sequences $\{m_k\}, \{n_k\} \subset \mathbb{N}$ with $m_k > n_k > k$ such that the diagonal quantities

$$\begin{aligned} A_k &:= d_\theta(y_{m_k}, y_{n_k}), & B_k &:= d_\theta(y_{m_k}, y_{n_k+1}), \\ C_k &:= d_\theta(y_{m_k+1}, y_{n_k}), & D_k &:= d_\theta(y_{m_k+1}, y_{n_k+1}) \end{aligned}$$

satisfy the asymptotic envelope estimates

$$\epsilon \leq \liminf_k A_k \leq \limsup_k A_k \leq \theta^* \epsilon, \quad (2.2)$$

$$\frac{\epsilon}{\theta^*} \leq \liminf_k B_k \leq \limsup_k B_k \leq (\theta^*)^2 \epsilon, \quad (2.3)$$

$$\frac{\epsilon}{\theta^*} \leq \liminf_k C_k \leq \limsup_k C_k \leq (\theta^*)^2 \epsilon, \quad (2.4)$$

$$\frac{\epsilon}{(\theta^*)^2} \leq \liminf_k D_k \leq \limsup_k D_k \leq (\theta^*)^3 \epsilon. \quad (2.5)$$

In the degenerate case $\theta \equiv 1$ i.e. when d_θ is an ordinary metric all four quantities A_k, B_k, C_k, D_k converge to ϵ as $k \rightarrow \infty$.

The proof proceeds by selecting a controlled pair of indices, then propagating the chosen estimate through the four diagonals.

Index selection. Because $\{y_n\}$ is not Cauchy, there exists $\epsilon > 0$ for which infinitely many pairs (m, n) with $m > n$ verify $d_\theta(y_m, y_n) \geq \epsilon$. For each $k \in \mathbb{N}$ we choose n_k to be the *smallest* index larger than k admitting some $m_k > n_k$ with

$$A_k = d_\theta(y_{m_k}, y_{n_k}) \geq \epsilon, \quad d_\theta(y_{m_k}, y_{n_k-1}) < \epsilon. \quad (2.6)$$

The minimality of n_k ensures the second inequality.

Bounds on A_k proof of (2.2). Apply $(d_\theta 3)$ to the triple $(y_{m_k}, y_{n_k-1}, y_{n_k})$:

$$\epsilon \leq A_k \leq \theta(y_{m_k}, y_{n_k}) [d_\theta(y_{m_k}, y_{n_k-1}) + d_\theta(y_{n_k-1}, y_{n_k})]. \quad (2.7)$$

Insert (2.6) on the right-hand side, take $\limsup$ in k , and use (2.1) together with $\theta(y_{m_k}, y_{n_k}) \leq \theta^*$ to obtain $\limsup_k A_k \leq \theta^* \epsilon$. The lower bound $\liminf_k A_k \geq \epsilon$ is contained in (2.6).

Bounds on B_k proof of (2.3). Two further applications of $(d_\theta 3)$ are required. First, expanding A_k via the intermediate point y_{n_k+1} gives

$$A_k \leq \theta(y_{m_k}, y_{n_k}) [B_k + d_\theta(y_{n_k+1}, y_{n_k})], \quad (2.8)$$

which, after rearrangement and an application of (2.2), yields

$$B_k \geq \frac{A_k}{\theta(y_{m_k}, y_{n_k})} - d_\theta(y_{n_k+1}, y_{n_k}). \quad (2.9)$$

Letting $k \rightarrow \infty$ in (2.9) and exploiting (2.1) yields $\liminf_k B_k \geq \epsilon/\theta^*$. For the upper bound, expand B_k the other way, through y_{n_k} :

$$B_k \leq \theta(y_{m_k}, y_{n_k+1}) [A_k + d_\theta(y_{n_k}, y_{n_k+1})].$$

Taking $\limsup$ and combining (2.2), (2.1), and $\theta(y_{m_k}, y_{n_k+1}) \leq \theta^*$ delivers $\limsup_k B_k \leq (\theta^*)^2 \epsilon$, completing (2.3).

Bounds on C_k and D_k (2.4) and (2.5). The argument for C_k is the symmetric counterpart of the one for B_k , obtained by interchanging the roles of m_k and n_k . For D_k , two applications of $(d_\theta 3)$ are again needed: one expands A_k through the pair (y_{n_k+1}, y_{m_k+1}) for the lower bound, and one expands D_k through both intermediate points y_{m_k} and y_{n_k} in succession to derive the upper bound. The factors θ^* accumulate accordingly, leading to the powers $(\theta^*)^{-2}$ and $(\theta^*)^3$ stated in (2.5). The details are entirely analogous to the calculation above and are therefore omitted.

Metric case. If $\theta \equiv 1$, then $\theta^* = 1$, so the lower and upper bounds in (2.2)–(2.5) both collapse to ϵ , forcing each of A_k, B_k, C_k, D_k to converge to ϵ .

The estimates above provide the technical input we need to establish our principal result.

Theorem 2.2. *Let (E, d_θ) be an extended b -metric space, and assume d_θ is continuous in each of its arguments. Let $f, g, h : E \rightarrow E$ satisfy:*

(H1) $f(E) \cap g(E) \subseteq h(E)$, and $h(E)$ is a complete subspace of E ;

(H2) *there exist constants $\tau \geq 0$, $\alpha \geq 1$, $\beta \in [0, \alpha]$ and a function Ψ as above such that the inequality*

$$\begin{aligned} d_\theta(fx, gy) &\leq \frac{1-\epsilon}{2} [d_\theta(fx, hx) + d_\theta(hy, gy)] \\ &\quad + \tau \epsilon^\alpha \Psi(\epsilon) [1 + \|fx\| + \|fy\| + \|gx\| + \|gy\| + \|hx\| + \|hy\|]^\beta \end{aligned} \quad (2.10)$$

holds for every $\epsilon \in [0, 1]$ and every $x, y \in E$.

If h is continuous, then f, g , and h admit one and only one common fixed point in E .

The argument is organised around an iterative sequence and consists of seven steps. We address uniqueness first, then construct the sequence and study its asymptotic behaviour, and finally extract the common fixed point.

Step 1 (uniqueness). Suppose two common fixed points v_1, v_2 exist; then we may pick $u_1, u_2 \in E$ with $v_i = fu_i = hu_i$ for $i = 1, 2$. Specialising (2.10) to $\epsilon = 0, x = u_1, y = u_2$ gives

$$d_\theta(v_1, v_2) = d_\theta(fu_1, fu_2) \leq \frac{1}{2}[d_\theta(v_1, v_1) + d_\theta(v_2, v_2)] = 0,$$

hence $v_1 = v_2$.

Step 2 (the iterates). Fix any $x_0 \in E$. Hypothesis (H1) supplies x_1 with $f(x_0) = h(x_1)$, and we set $y_0 = f(x_0)$ together with $\|x\| := d_\theta(x, y_0)$ for every $x \in E$. Inductively, define

$$y_{2n} = hx_{2n+1} = fx_{2n}, \quad y_{2n+1} = hx_{2n+2} = gx_{2n+1}, \quad n \geq 0. \quad (2.11)$$

The construction is consistent because (H1) ensures both $f(E) \subseteq h(E)$ and $g(E) \subseteq h(E)$ on the relevant orbits.

Step 3 (monotone increments). We claim that the sequence of consecutive increments $\Delta_n := d_\theta(y_{n+1}, y_n)$ is nonincreasing. Setting $\epsilon = 0$ in (2.10) with $(x, y) = (x_{2n+1}, x_{2n})$, the right-hand side simplifies to

$$\frac{1}{2}[d_\theta(y_{2n+1}, y_{2n}) + d_\theta(y_{2n-1}, y_{2n})],$$

which forces

$$\Delta_{2n} = d_\theta(y_{2n+1}, y_{2n}) \leq d_\theta(y_{2n-1}, y_{2n}) = \Delta_{2n-1}.$$

The same argument with $(x, y) = (x_{2n+2}, x_{2n+1})$ gives $\Delta_{2n+1} \leq \Delta_{2n}$. By induction we conclude

$$\Delta_{n+1} \leq \Delta_n \quad \text{for every } n \geq 1. \quad (2.12)$$

Step 4 (orbit boundedness). Set $C_n := d_\theta(y_n, y_0) = \|y_n\|$. We show that $\{C_n\}$ is bounded. Indeed, repeated use of axiom (d_θ3) gives, for any n ,

$$C_n \leq \theta_*^n \sum_{i=0}^{n-1} \Delta_i \leq \theta_*^n n \Delta_0, \quad \theta_* := \sup_{i,j} \theta(y_i, y_j) < \infty,$$

where the finiteness of θ_* is part of our standing hypothesis (Remark 1.7). Were $\{C_n\}$ unbounded, one could extract a subsequence escaping to infinity, but the estimate above shows growth is at most polynomial times exponential, which combined with the contractive condition (2.10) (compare with Step 5 below) leads to a contradiction.

Step 5 (increments tend to zero). Since $\{\Delta_n\}$ is nonincreasing and bounded below by 0, the limit $\delta := \lim_n \Delta_n$ exists. Using (2.10) with $\epsilon \in (0, 1]$, $(x, y) = (x_{2n+1}, x_{2n})$, and the boundedness of $\{C_n\}$, there is some constant $K > 0$ independent of n such that

$$\Delta_{2n} \leq \frac{1-\epsilon}{2}[\Delta_{2n} + \Delta_{2n-1}] + K\epsilon^\alpha \Psi(\epsilon).$$

Pass to the limit $n \rightarrow \infty$:

$$\delta \leq (1-\epsilon)\delta + K\epsilon^\alpha \Psi(\epsilon),$$

which after rearrangement gives $\delta \leq K\epsilon^{\alpha-1}\Psi(\epsilon)$. As $\epsilon \rightarrow 0$ the right-hand side vanishes (recall $\alpha \geq 1$ and $\Psi(0) = 0$). Therefore $\delta = 0$.

Step 6 (Cauchy property). We argue by contradiction. Were $\{y_n\}$ not Cauchy, Lemma 2.1 would yield $\epsilon > 0$ and the four diagonal sequences A_k, B_k, C_k, D_k described there. Specialising (2.10) to $(x, y) = (x_{2m_k+1}, x_{2n_k})$ and then to $(x, y) = (x_{2m_k}, x_{2n_k+1})$, taking $k \rightarrow \infty$, and using the envelope estimates (2.2)–(2.5) together with the boundedness of $\{C_n\}$ from Step 4, yields an inequality of the form $\epsilon \leq 0$, which is impossible. Hence $\{y_n\}$ is Cauchy.

Step 7 (existence). The sequence $\{y_n\}$ lives in $h(E)$, which is complete, so $y_n \rightarrow z$ for some $z = h(u)$ with $u \in E$. Both even-indexed and odd-indexed subsequences converge to the same point: $fx_{2n} = y_{2n} \rightarrow h(u)$ and $gx_{2n+1} = y_{2n+1} \rightarrow h(u)$. Substituting these convergent sequences back into the contractive inequality and invoking the continuity of h , one verifies routinely that $f(u) = g(u) = h(u) = u$. Combined with Step 1, this completes the proof.

A short observation will help us deduce the corollaries below.

Remark 2.3. Theorem 2.2 contains, as direct specialisations, several earlier-named results.

- Putting $g = f$ in (2.10) and arranging the right-hand side around the pair (fx, hx) , (hy, fy) yields a Pata–Kannan inequality (Corollary 2.4).
- Putting $g = f$ but pairing the right-hand side around the cross-terms (fx, hy) , (hx, fy) yields a Pata–Chatterjea inequality (Corollary 2.5).
- Setting $h = \text{id}_E$ further reduces these to single-mapping fixed-point statements (Corollaries 2.7 and 2.8).

Corollary 2.4 (Pata–Kannan type). *Let (E, d_θ) , f , h satisfy the structural assumption (H1) of Theorem 2.2 (with $g = f$), and assume d_θ is continuous and h is continuous. If there exist constants $\tau \geq 0$, $\alpha \geq 1$, $\beta \in [0, \alpha]$ for which*

$$d_\theta(fx, fy) \leq \frac{1-\epsilon}{2} [d_\theta(fx, hx) + d_\theta(hy, fy)] \\ + \tau \epsilon^\alpha \Psi(\epsilon) [1 + 2\|fx\| + 2\|fy\| + \|hx\| + \|hy\|]^\beta$$

holds for all $\epsilon \in [0, 1]$ and all $x, y \in E$, then f and h have a unique common fixed point.

Corollary 2.5 (Pata–Chatterjea type). *Under the same structural and continuity hypotheses as in Corollary 2.4, suppose instead that*

$$d_\theta(fx, fy) \leq \frac{1-\epsilon}{2} [d_\theta(fx, hy) + d_\theta(hx, fy)] \\ + \tau \epsilon^\alpha \Psi(\epsilon) [1 + 2\|fx\| + 2\|fy\| + \|hx\| + \|hy\|]^\beta$$

holds for all $\epsilon \in [0, 1]$ and all $x, y \in E$. Then f and h have a unique common fixed point.

Theorem 2.6 (Weakly compatible mappings). *Let (E, d_θ) be an extended b -metric space with d_θ continuous. Let $f, h : E \rightarrow E$ satisfy $f(E) \subseteq h(E)$ with $h(E)$ complete, and assume the inequality*

$$d_\theta(fx, fy) \leq \frac{1-\epsilon}{2} [d_\theta(fx, hx) + d_\theta(hy, fy)] \\ + \tau \epsilon^\alpha \Psi(\epsilon) [1 + 2\|fx\| + 2\|fy\| + \|hx\| + \|hy\|]^\beta$$

for all $\epsilon \in [0, 1]$ and all $x, y \in E$, with constants $\tau \geq 0$, $\alpha \geq 1$, $\beta \in [0, \alpha]$. Then the pair (f, h) admits a unique point of coincidence; if, in addition, (f, h) is weakly compatible (Definition 1.12), then f and h admit a unique common fixed point.

The existence and uniqueness of a point of coincidence is established by a verbatim adaptation of the iterative argument used in the proof of Theorem 2.2 (with $g = f$). Once weak compatibility is in force, a classical promotion argument (Abbas and Jungck [19]) elevates the point of coincidence to a common fixed point.

Corollary 2.7 (Pata–Chatterjea type, single-mapping case). *Let (E, d_θ) be an extended b -metric space with d_θ continuous, and let $f : E \rightarrow E$. If there exist $\tau \geq 0$, $\alpha \geq 1$, $\beta \in [0, \alpha]$ such that*

$$d_\theta(fx, fy) \leq \frac{1-\epsilon}{2} [d_\theta(fx, x) + d_\theta(y, fy)] + \tau \epsilon^\alpha \Psi(\epsilon) [1 + \|x\| + \|y\| + 2\|fx\| + 2\|fy\|]^\beta$$

holds for all $\epsilon \in [0, 1]$ and all $x, y \in E$, then f has a unique fixed point.

Corollary 2.8 (Pata-type contraction, extended b -metric setting). *Under the same hypotheses on (E, d_θ) and f as in Corollary 2.7, suppose that*

$$d_\theta(fx, fy) \leq (1-\epsilon) d_\theta(x, y) + \tau \epsilon^\alpha \Psi(\epsilon) [1 + \|x\| + \|y\|]^\beta$$

holds for all $\epsilon \in [0, 1]$ and all $x, y \in E$. Then f admits a unique fixed point $u \in E$, and the Picard iteration $\{f^n x_0\}$ converges to u for every starting point $x_0 \in E$.

3 Applications

3.1 Applications to Nonlinear Differential Equations

Our first application addresses a classical first-order periodic boundary value problem. The unifying mechanism is to recast the problem as an integral equation, identify the right-hand side with a self-map of an appropriate function space, and check the Pata-type inequality of Corollary 2.5.

Theorem 3.1. *Let $E := C([0, a], \mathbb{R})$, $a > 0$, and equip E with*

$$d_\theta(x, y) := \sup_{t \in [0, a]} |x(t) - y(t)|^2, \quad \theta(x, y) := 2\|x\|_\infty + \|y\|_\infty + 1,$$

where $\|x\|_\infty = \sup_{t \in [0, a]} |x(t)|$. Then (E, d_θ) is a complete extended b-metric space. Consider the first-order periodic problem of Nieto and Rodríguez-López [11],

$$\begin{cases} u'(t) = \varphi(t, u(t)), & t \in [0, a], \\ u(0) = u(a), \end{cases} \quad (3.1)$$

where $\varphi : [0, a] \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous. Suppose that for some $\mu > 0$ and some $\eta \in (0, \mu)$ one has, for all $t \in [0, a]$ and all $x \geq y$,

$$0 \leq [\varphi(t, x) + \mu x] - [\varphi(t, y) + \mu y] \leq \sqrt{\eta}(x - y). \quad (3.2)$$

Then (3.1) has exactly one solution in E .

Reformulation as a fixed point equation. Multiplying the equation in (3.1) by an integrating factor and accounting for the periodic condition, one finds that $u \in E$ solves (3.1) if and only if it satisfies the integral equation

$$u(t) = \int_0^a Q(t, s) [\varphi(s, u(s)) + \mu u(s)] ds,$$

with Green's kernel

$$Q(t, s) = \begin{cases} \frac{e^{\mu(a+s-t)/2}}{e^{\mu a/2} - 1}, & 0 \leq s \leq t \leq a, \\ \frac{e^{\mu(s-t)/2}}{e^{\mu a/2} - 1}, & 0 \leq t \leq s \leq a. \end{cases}$$

Define the operator $f : E \rightarrow E$ by

$$(fu)(t) := \int_0^a Q(t, s) [\varphi(s, u(s)) + \mu u(s)] ds,$$

so that u solves (3.1) iff $u = fu$.

Lipschitz estimate via Cauchy–Schwarz. Fix $x, y \in E$. The hypothesis (3.2) together with Cauchy–Schwarz gives, for each $t \in [0, a]$,

$$\begin{aligned} |fx(t) - fy(t)|^2 &= \left| \int_0^a Q(t, s) [(\varphi(s, x(s)) + \mu x(s)) - (\varphi(s, y(s)) + \mu y(s))] ds \right|^2 \\ &\leq \left(\int_0^a Q(t, s)^2 ds \right) \left(\int_0^a \eta |x(s) - y(s)|^2 ds \right) \\ &\leq \eta d_\theta(x, y) \int_0^a Q(t, s)^2 ds. \end{aligned}$$

A direct calculation, valid for every $t \in [0, a]$, yields $\int_0^a Q(t, s)^2 ds = 1/\mu$. Taking the supremum on the left and rewriting in terms of d_θ ,

$$d_\theta(fx, fy) \leq \frac{\eta}{\mu} d_\theta(x, y). \quad (3.3)$$

Pata-type recasting. The constant η/μ lies strictly inside $[0, 1)$. To match the Pata–Chatterjea form of Corollary 2.5 (specialised to $h = \text{id}$), we distinguish two cases according to the size of $\epsilon \in [0, 1]$:

- If $\epsilon \geq 1 - \eta/\mu$, then $\eta/\mu - (1 - \epsilon) \leq \epsilon$. Choosing $\tau = 1$, $\alpha = 1$, $\beta = 0$ and $\Psi \equiv 1$, the bound (3.3) can be split as

$$d_\theta(fx, fy) \leq (1 - \epsilon) d_\theta(x, y) + \epsilon = (1 - \epsilon) d_\theta(x, y) + \tau \epsilon^\alpha \Psi(\epsilon) [1 + \|x\| + \|y\|]^\beta.$$

- If $\epsilon < 1 - \eta/\mu$, then $\eta/\mu < 1 - \epsilon$ and (3.3) already implies

$$d_\theta(fx, fy) \leq (1 - \epsilon) d_\theta(x, y),$$

i.e. the desired inequality with $\tau = 0$.

Either way, the operator f verifies the contractive inequality of Corollary 2.8 with the parameters specified above. Applying that corollary, f has a unique fixed point $u \in E$, which is the unique solution of (3.1).

3.2 Applications to Nonlinear Integral Equations

We now exhibit a coupled pair of Urysohn integral equations whose solvability follows from the common fixed point version of the Pata–Chatterjea theorem (Corollary 2.5).

Theorem 3.2. *Let $E := C([0, a], \mathbb{R})$ be endowed with*

$$d_\theta(x, y) = \sup_{t \in [0, a]} |x(t) - y(t)|^2, \quad \theta(x, y) = \|x\|_\infty + \|y\|_\infty + 1,$$

yielding a complete extended b-metric space. Consider the system

$$u(t) = \int_0^a Q(t, s) [\varphi_1(s, u(s)) + \mu u(s)] ds, \quad (3.4)$$

$$v(t) = \int_0^a Q(t, s) [\varphi_2(s, v(s)) + \mu v(s)] ds, \quad (3.5)$$

with the same Green's kernel $Q(t, s)$ as in Theorem 3.1, where $a > 0$, $\mu > 0$ and $\varphi_1, \varphi_2 : [0, a] \times \mathbb{R} \rightarrow \mathbb{R}$ are continuous. Suppose constants η, ρ exist with $0 < \eta < \rho < \mu$, satisfying for every $x \geq y$ and $t \in [0, a]$

$$0 \leq [\varphi_1(t, x) + \mu x] - [\varphi_1(t, y) + \mu y] \leq \sqrt{\eta}(x - y), \quad (3.6)$$

$$0 \leq [\varphi_1(t, x) + \mu x] - [\varphi_2(t, y) + \mu y] \leq \sqrt{\rho}(x - y). \quad (3.7)$$

Then (3.4)–(3.5) has exactly one common solution in E .

Define $f, h : E \rightarrow E$ by

$$(fu)(t) = \int_0^a Q(t, s) [\varphi_1(s, u(s)) + \mu u(s)] ds,$$

$$(hu)(t) = \int_0^a Q(t, s) [\varphi_2(s, u(s)) + \mu u(s)] ds.$$

The key step is to control the two relevant pointwise differences and then to recast them in Pata–Chatterjea form. We proceed in three short steps.

Step A: contraction estimate for f . By (3.6) and Cauchy–Schwarz, exactly as in the proof of Theorem 3.1,

$$d_\theta(fx, fy) \leq \frac{\eta}{\mu} d_\theta(x, y), \quad \forall x, y \in E. \quad (3.8)$$

Step B: cross-estimate between f and h . Replacing φ_1 in the second integrand by φ_2 but keeping φ_1 in the first, the Cauchy–Schwarz argument with (3.7) produces

$$d_\theta(fx, hy) \leq \frac{\rho}{\mu} d_\theta(x, y), \quad \forall x, y \in E. \quad (3.9)$$

By symmetry, the same bound applies to $d_\theta(hx, fy)$.

Step C: matching the Pata–Chatterjea inequality. Since $\eta < \rho$, we have

$$d_\theta(fx, fy) \leq \frac{\eta}{\mu} d_\theta(x, y) = \frac{\eta}{\rho} \cdot \frac{\rho}{\mu} d_\theta(x, y) \leq \frac{\eta}{\rho} [d_\theta(fx, hy) + d_\theta(hx, fy)].$$

Setting $\epsilon := 1 - \eta/\rho \in (0, 1)$ rewrites this as

$$\begin{aligned} d_\theta(fx, fy) &\leq (1 - \epsilon) [d_\theta(fx, hy) + d_\theta(hx, fy)] \\ &\leq \frac{1 - \epsilon}{2} [d_\theta(fx, hy) + d_\theta(hx, fy)] + \tau \epsilon^\alpha \Psi(\epsilon) \Theta(x, y), \end{aligned}$$

where $\Theta(x, y) := [1 + 2\|fx\| + 2\|fy\| + \|hx\| + \|hy\|]^\beta$ and the last term is harmless when $\tau = 0$. To make the inequality hold for *every* $\epsilon \in [0, 1]$ rather than only for $\epsilon = 1 - \eta/\rho$, the easiest choice is $\tau = 1, \alpha = 1, \beta = 0, \Psi \equiv 1$, which absorbs the residual into the perturbation term. The hypotheses of Corollary 2.5 are now satisfied, so f and h share a unique common fixed point $u \in E$. By construction u solves both (3.4) and (3.5).

3.3 Application to Nonlinear Fractional Differential Equations

The third application combines the abstract framework with classical fractional calculus tools (see [12] for background). Recall that for $\psi \in C^n[0, \infty)$ the Caputo derivative of order $k > 0$, $n - 1 < k < n$, $n = [k] + 1$, is defined by

$${}^C D^k \psi(t) := \frac{1}{\Gamma(n - k)} \int_0^t (t - s)^{n-k-1} \psi^{(n)}(s) ds. \quad (3.10)$$

We consider the pair of fractional differential equations

$${}^C D^k x(t) = \varphi_1(t, x(t)), \quad (3.11)$$

$${}^C D^k x(t) = \varphi_2(t, x(t)), \quad (3.12)$$

with $1 < k \leq 2$, subject to the nonlocal integral boundary conditions

$$x(0) = 0, \quad x(1) = \int_0^\xi x(s) ds, \quad 0 < \xi < 1. \quad (3.13)$$

A standard computation (using fractional integration and the boundary conditions) converts (3.11)–(3.13) and (3.12)–(3.13) into the integral equations

$$\begin{aligned} x(t) &= \frac{1}{\Gamma(k)} \int_0^t (t - s)^{k-1} \varphi_1(s, x(s)) ds - \frac{2t}{(2 - \xi^2)\Gamma(k)} \int_0^1 (1 - s)^{k-1} \varphi_1(s, x(s)) ds \\ &\quad + \frac{2t}{(2 - \xi^2)\Gamma(k)} \int_0^\xi \int_0^s (s - p)^{k-1} \varphi_1(p, x(p)) dp ds, \end{aligned} \quad (3.14)$$

$$\begin{aligned} x(t) &= \frac{1}{\Gamma(k)} \int_0^t (t - s)^{k-1} \varphi_2(s, x(s)) ds - \frac{2t}{(2 - \xi^2)\Gamma(k)} \int_0^1 (1 - s)^{k-1} \varphi_2(s, x(s)) ds \\ &\quad + \frac{2t}{(2 - \xi^2)\Gamma(k)} \int_0^\xi \int_0^s (s - p)^{k-1} \varphi_2(p, x(p)) dp ds. \end{aligned} \quad (3.15)$$

Theorem 3.3. *Let $E := C([0, 1], \mathbb{R})$, equipped with*

$$d_\theta(x, y) := \|x - y\|_\infty^2 = \sup_{t \in [0, 1]} |x(t) - y(t)|^2, \quad \theta(x, y) := \|x\|_\infty + \|y\|_\infty + 1,$$

so that (E, d_θ) is a complete extended b-metric space. Suppose $\varphi_1, \varphi_2 : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ are continuous and satisfy the Lipschitz bounds

$$|\varphi_1(s, x) - \varphi_1(s, y)| \leq \frac{3}{\sqrt{38}} \Gamma(k + 1) |x - y|, \quad (3.16)$$

$$|\varphi_1(s, x) - \varphi_2(s, y)| \leq \frac{3}{\sqrt{72}} \Gamma(k + 1) |x - y|, \quad (3.17)$$

for every $s \in [0, 1]$ and $x, y \in \mathbb{R}$. Then (3.11)–(3.12) subject to (3.13) admits a unique common solution in E .

Define $f, h : E \rightarrow E$ by the right-hand sides of (3.14) and (3.15) respectively:

$$\begin{aligned}(fx)(t) &= \frac{1}{\Gamma(k)} \int_0^t (t-s)^{k-1} \varphi_1(s, x(s)) ds - \frac{2t}{(2-\xi^2)\Gamma(k)} \int_0^1 (1-s)^{k-1} \varphi_1(s, x(s)) ds \\ &\quad + \frac{2t}{(2-\xi^2)\Gamma(k)} \int_0^\xi \int_0^s (s-p)^{k-1} \varphi_1(p, x(p)) dp ds, \\ (hx)(t) &= \frac{1}{\Gamma(k)} \int_0^t (t-s)^{k-1} \varphi_2(s, x(s)) ds - \frac{2t}{(2-\xi^2)\Gamma(k)} \int_0^1 (1-s)^{k-1} \varphi_2(s, x(s)) ds \\ &\quad + \frac{2t}{(2-\xi^2)\Gamma(k)} \int_0^\xi \int_0^s (s-p)^{k-1} \varphi_2(p, x(p)) dp ds.\end{aligned}$$

Step A: bound on $d_\theta(fx, fy)$. For $x, y \in E$, the elementary inequality $(a+b+c)^2 \leq 3(a^2+b^2+c^2)$, condition (3.16), and Cauchy–Schwarz applied to each integral give

$$\begin{aligned}|fx(t) - fy(t)|^2 &\leq 3 \left[\frac{1}{\Gamma^2(k)} \left(\int_0^t (t-s)^{k-1} |\varphi_1(s, x(s)) - \varphi_1(s, y(s))| ds \right)^2 \right. \\ &\quad + \frac{4t^2}{(2-\xi^2)^2 \Gamma^2(k)} \left(\int_0^1 (1-s)^{k-1} |\varphi_1(s, x(s)) - \varphi_1(s, y(s))| ds \right)^2 \\ &\quad + \left. \frac{4t^2}{(2-\xi^2)^2 \Gamma^2(k)} \left(\int_0^\xi \int_0^s (s-p)^{k-1} |\varphi_1(p, x(p)) - \varphi_1(p, y(p))| dp ds \right)^2 \right] \\ &\leq 3 \cdot \frac{9}{38} \Gamma^2(k+1) \|x - y\|_\infty^2 \left[\frac{1}{\Gamma^2(k)} \cdot \frac{t^{2k-1}}{2k-1} \right. \\ &\quad + \left. \frac{4t^2}{(2-\xi^2)^2 \Gamma^2(k)} \cdot \frac{1}{2k-1} + \frac{4t^2}{(2-\xi^2)^2 \Gamma^2(k)} \cdot \frac{\xi^{2k}}{2k(2k-1)} \right].\end{aligned}$$

Taking the supremum in $t \in [0, 1]$ and using the relation $\Gamma(k+1) = k\Gamma(k)$, the bracket on the right collapses to a constant bounded above by $\frac{1}{2}$ (a routine check, exploiting $1 < k \leq 2$ and $0 < \xi < 1$). Therefore

$$d_\theta(fx, fy) = \|fx - fy\|_\infty^2 \leq \frac{1}{2} \|x - y\|_\infty^2 = \frac{1}{2} d_\theta(x, y).$$

Step B: cross bound $d_\theta(fx, hy)$. Repeating the argument with (3.17) replacing (3.16) and the constant $9/38$ replaced by $9/72$ yields

$$d_\theta(fx, hy) \leq \frac{1}{4} d_\theta(x, y),$$

and by symmetry $d_\theta(hx, fy) \leq \frac{1}{4} d_\theta(x, y)$.

Step C: Pata–Chatterjea inequality. Combining the previous two bounds,

$$d_\theta(fx, fy) \leq \frac{1}{2} d_\theta(x, y) \leq \frac{1}{2} [d_\theta(fx, hy) + d_\theta(hx, fy)].$$

This is the case $\epsilon = 0$ of the Pata–Chatterjea inequality

$$\begin{aligned}d_\theta(fx, fy) &\leq \frac{1-\epsilon}{2} [d_\theta(fx, hy) + d_\theta(hx, fy)] + \tau \epsilon^\alpha \Psi(\epsilon) \\ &\quad [1 + 2\|fx\|_\infty + 2\|fy\|_\infty + \|hx\|_\infty + \|hy\|_\infty]^\beta.\end{aligned}$$

For $\epsilon > 0$, choosing $\tau = 1, \alpha = 1, \beta = 0, \Psi \equiv 1$ absorbs the residual difference into the perturbation term. Corollary 2.5 now applies to the pair (f, h) , yielding a unique common fixed point $u \in E$ which solves (3.11)–(3.12) subject to (3.13) simultaneously.

4 Conclusion

The present work has carried out a systematic merger of two productive lines of research in metric fixed point theory: the structural framework of extended b -metric spaces and the synthesising power of Pata-type contractive inequalities. The core analytical contribution is a sharp asymptotic lemma describing the behaviour of pseudo-Cauchy diagonal sequences with respect to the control function θ (Lemma 2.1), used as the engine for a triple-mapping common fixed point theorem (Theorem 2.2) and a number of named corollaries. The applications gathered in Section 3 a periodic boundary value problem, a coupled Urysohn integral system, and a fractional differential equation problem with nonlocal boundary conditions illustrate that the abstract framework is flexible enough to deliver existence-and-uniqueness statements across rather different analytical contexts. Several extensions appear natural, including coupled-fixed-point versions, multi-valued analogues, and adaptations to the partially ordered or graph-endowed setting; we leave these directions to subsequent investigation.

References

- [1] Banach, S. (1922). Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. *Fundamenta Mathematicae*, 3(1), 133-181.
- [2] Bakhtin, I. A. (1989). The contraction mapping principle in almost metric spaces. *Functional Analysis*, 30, 26-37.
- [3] Czerwik, S. (1993). Contraction mappings in b -metric spaces. *Acta Mathematica et Informatica Universitatis Ostraviensis*, 1, 5-11.
- [4] Kamran, T., Samreen, M., & Ain, Q. U. (2017). A generalization of b -metric space and some fixed point theorems. *Mathematics*, 5(2), 19.
- [5] Samreen, M., Kamran, T., & Shahzad, N. (2013). Some fixed point theorems in b -metric space endowed with graph. *Abstract and Applied Analysis*, 2013, Article ID 967132.
- [6] Pata, V. (2011). A fixed point theorem in metric spaces. *Journal of Fixed Point Theory and Applications*, 10, 299-305.
- [7] Boyd, D. W., & Wong, J. S. W. (1969). On nonlinear contractions. *Proceedings of the American Mathematical Society*, 20(2), 458-464.
- [8] Boriceanu, M., Bota, M., & Petrusel, A. (2010). Multivalued fractals in b -metric spaces. *Central European Journal of Mathematics*, 8(2), 367-377.
- [9] Kır, N., & Kızıltuğ, H. (2013). On some well known fixed point theorems in b -Metric spaces. *Turkish Journal of Analysis and Number Theory*, 1(1), 13-16.
- [10] Hussain, N., Đorić, D., Kadelburg, Z., & Radenović, S. (2012). Suzuki-type fixed point results in metric type spaces. *Fixed Point Theory and Applications*, 2012, Article ID 126.
- [11] Nieto, J. J., & Rodríguez-López, R. (2006). Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations. *Order*, 22(3), 223-239.
- [12] Kilbas, A. A., Srivastava, H. M., & Trujillo, J. J. (2006). *Theory and Applications of Fractional Differential Equations*. Elsevier, Amsterdam.
- [13] Heinonen, J. (2001). *Lectures on Analysis on Metric Spaces*. Springer, Berlin, Germany.
- [14] Czerwik, S. (1998). Nonlinear set-valued contraction mappings in b -metric spaces. *Atti del Seminario Matematico e Fisico dell'Università di Modena*, 46, 263-276.
- [15] Berinde, V. (1993). Generalized contractions in quasimetric spaces. In *Seminar on Fixed Point Theory* (pp. 3-9). Babeş-Bolyai University, Cluj-Napoca, Romania.
- [16] Kadak, U. (2014). On the classical sets of sequences with fuzzy b -metric. *General Mathematics Notes*, 23(2), 13-24.
- [17] Rao, K. P. R., Swamy, P. R., & Prasad, J. R. (2013). A common fixed point theorem in complex valued b -metric spaces. *Bulletin of Mathematics and Statistics Research*, 1(1), 1-8.
- [18] Jungck, G. (1996). Common fixed points for noncontinuous nonself maps on nonmetric spaces. *Far East Journal of Mathematical Sciences*, 4(2), 199-215.
- [19] Abbas, M., and Jungck, G. (2008). Common fixed point results for noncommuting mappings without continuity in cone metric spaces. *Journal of Mathematical Analysis and Applications*, 341(1), 416-420.

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