

MATHEMATICAL ANALYSIS OF A DELAY-INDUCED THREE-DIMENSIONAL NUTRIENT-PLANKTON SYSTEM WITH HARVESTING

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Abstract Plankton interactions at various trophic levels, play a crucial role in sustaining the aquatic ecosystems. In this paper, a three-component plankton model involving interactions among nutrients, phytoplankton and herbivores zooplankton is considered. Notably we assume that phytoplankton is capable of producing toxic substances and zooplankton population is being harvested at a linear rate. The system is explored for existence of all feasible equilibrium states and their stability analysis has been performed in absence of time delay. The conditions for both local asymptotic stability and the global stability of the system are derived, providing insights into how the populations interact and stabilize over time. For delayed model, the stability analysis has been done and Hopf bifurcation conditions are examined to investigate the system dynamics about interior equilibrium point. The analytical results are justified by numerical simulations. The paper studies the effect of harvesting of zooplankton on the system in presence of toxicity. It is numerically established that increasing harvesting effort moves the system towards stability.

1 Introduction

Researchers widely employ mathematical modelling techniques to gain insights into the dynamics and mechanisms of ecological systems. These models enable the study of the consequences of changes in various atmospheric factors on the system's behaviour. Nutrient-plankton models serve as valuable tools for describing plankton dynamics and exploring biological and physical interactions in aquatic ecosystems, as highlighted by various ecologists[1, 2, 3]. Planktons, crucial components of aquatic ecosystems, play significant roles in functions such as food production, oxygen generation, energy transfer, nutrient cycling, and overall ecosystem productivity. The intricate interplay between these organisms, coupled with nutrient availability and distribution, shapes the dynamics and stability of marine and freshwater ecosystems. Understanding these interactions is vital for predicting and managing ecosystem responses to environmental changes and addressing potential harmful algal blooms. Planktons, which are freely floating unicellular organisms found in oceans and freshwater bodies, may exist individually or in chains. The plant community of plankton is known as phytoplankton, while the animal community is referred to as zooplankton. Phytoplankton, as primary producers, form the foundation of the aquatic food web by converting nutrients into organic matter through photosynthesis [4]. Zooplankton, as primary consumers, exert top-down control on phytoplankton populations by grazing on them, influencing the abundance and composition of phytoplankton communities. The interactions between phytoplankton and zooplankton play a pivotal role in shaping community structure, biomass dynamics, and overall ecosystem functioning. Among

the diverse phytoplankton assemblages some species possess the ability to produce potent toxins. As reported by Wang [5], cyanobacteria, diatoms and dinoflagellates are considered to be main contributors to phytoplankton toxins that are harmful to aquatic life. Some of toxin producing phytoplankton are *pseudo-nitzschia*, *noctiluca scintillans*, *ostreopsis slamensis*, *gambierdiscus toxicus*, *alexandrium centenella*, *schizothrix calcicola*, *oscillatoria nigroviridis*, *karenia brevis* and *anabena cylindrospermopsis* [6, 7]. Zooplankton and herbivorous fish eat phytoplankton and in turn can affect humans either through direct consumption or indirectly through food web. Owing to favourable conditions of nutrient availability, temperature and salinity, phytoplankton sometimes grow excessively to form blooms [8, 9, 10]. This excessive and rapid growth causes deterioration of water quality as dissolved oxygen in water decreases which leads to death of aquatic organisms. The decomposition of dead aquatic organisms, in turn, consumes more oxygen, gradually converting the water body in a dead zone. This may result in destruction of economically important fisheries as reported in [11]. Thus, liberation of toxins by phytoplankton or their excessive growth is injurious not only to aquatic ecosystem but also to humans and economic fisheries. Many studies have been done to understand the complex interplay between nutrients, toxic producing phytoplankton (TPP) and zooplankton [12, 13]. The study in [14] indicated that the intensity of toxic substances released by TPP contributes to stabilizing the system. As per Banerjee and Venturino [10], the increase in magnitude of TPP does not lead zooplankton to extinction. Jang et al. [15] reported that populations undergo cyclic blooms when inhibited by toxins. The effects of allelopathic phytoplankton along with spatial variation in population density in a nutrient plankton model has been investigated by [16, 17, 18].

Research Gap and motivation

Reutilisation of organic matter and nutrients released through excretion and decomposition of plankton biomass is referred to as nutrient recycling. Role of nutrient recycling on pelagic food chain have been undertaken in numerous studies. Nisbet et al. [19], Gard [20], Ruan [21], Ghosh and Sarkar [22], Khare et al. [23], Fan et al. [24], Sharma et al. [25], Jang et al. [15] explored how nutrient recycling affects the behaviour of system. In their investigations, Beretta et al. [26], Ruan [27], Das and Ray [28], Mukhopadhyay and Bhattacharya [29], Rehim et al. [30], Khare et al. [23] and Fan et al. [24] have incorporated the time delay in the nutrient recycling process. Further, Das and Ray, Rehim et al. [28, 30] have also considered time lag in the nutrient recycling process. Sharma et al. [25] have investigated the system behaviour by incorporating recycling of nutrients only from phytoplankton biomass and discrete time variation in the toxin liberation process by phytoplankton species taking into account the time taken by phytoplankton species to release toxins. Fan et al. [24] showed global stability of the system for small input concentration of nutrients. However, in their study of nutrient-plankton system with toxins, Jang et al. [15] concluded that a larger input nutrient concentration will promote species coexistence. Das and Ray explored the length of delay that can preserve the system stability. Fan et al. and Das and Ray [24, 28] have not incorporated toxin liberation terms in their studies.

This paper investigates the dynamics of a system involving nutrients, toxic phytoplankton, and zooplankton, incorporating nutrient recycling. The model assumes linear harvesting of the zooplankton population, with a primary focus on exploring the interactions among nutrients, toxin-releasing phytoplankton, and zooplankton, and their collective influence on the plankton community. While numerous researchers have investigated nutrient planktonic systems, a notable observation is the omission of the harvesting phenomenon in many studies. For instance, in the work referenced as [23], the model is explored in the presence of toxicity but harvesting of the population and the incorporation of a time delay factor have not been considered. On the other hand, [25, 29] examined the model in the absence of toxic substances, and [30] studied the model with a time delay in the presence of toxic substances but did not incorporate harvesting. Rehim et al. have also not taken into consideration the harvesting of species. In our model, we address this gap by considering the harvesting of the predator population.

We initiate our exploration by outlining the theoretical framework and underlying assumptions of our model in section 2. The subsequent sections follow this organizational structure: Section 3 delves into the discussion of the positivity and boundedness of the solution. In Section 4, we thoroughly examine the existence of all positive equilibria and conduct stability analyses for each. Section 5 presents a bifurcation analysis of the non-delayed model, focusing specifically on transcritical and Hopf bifurcations. Moving forward, Section 6 undertakes the stability anal-

ysis of the delayed model around the interior equilibrium point, probing into Hopf bifurcation with τ as the perturbation parameter. Theoretical insights are complemented by numerical simulations in Section 7, which is succeeded by a comprehensive discussion of the study's outcomes in conclusions in section 8.

2 The Model System

Enormous variability can be seen in natural planktonic ecosystems. The phytoplankton growth mainly relies on availability of light and nutrients-mainly nitrogen, phosphorous and trace elements. Nutrients are returned to the system through microbial decomposition of deceased phytoplankton and zooplankton. Besides other biotic factors, like release of toxins, changed rate of predation and harvesting of zooplankton(predator) also exert influence on the system. Let nutrient concentration at any time t be $N(t)$ and concentration of phytoplankton and zooplankton populations be $P(t)$ and $Z(t)$. Mathematically, the model system is described using a system of ODEs as below:

$$\begin{aligned}\frac{dN(t)}{dt} &= N_0 - \alpha N - bNP + \beta_1\mu_1P(t) + \beta_2\mu_2Z(t), \\ \frac{dP(t)}{dt} &= b_1bNP - \mu_1P(t) - \frac{\gamma P(t)Z(t)}{\beta + P(t)}, \\ \frac{dZ(t)}{dt} &= \frac{\gamma_1\gamma P(t)Z(t)}{\beta + P(t)} - \frac{\theta P(t - \tau)Z(t)}{\beta + P(t - \tau)} - (\mu_2 + qE)Z(t).\end{aligned}\tag{2.1}$$

System (2.1) will be studied with the initial conditions:

$$N(0) \geq 0, P(0) \geq 0 \text{ and } Z(0) \geq 0.$$

$$(N(0), P(0), Z(0)) = (\psi_1(\theta), \psi_2(\theta), \psi_3(\theta)) \in C[(-\tau, 0), \mathbb{R}_3^+],$$

$$\psi_1(0) > 0, \psi_2(0) > 0, \psi_3(0) > 0, \theta \in [-\tau, 0], \text{ where}$$

$C[(-\tau, 0), \mathbb{R}_3^+]$ represents the space of continuous functions mapping $(-\tau, 0)$ into $\mathbb{R}_3^+$.

N_0 is the natural availability of various nutrients in the system and α is the rate they get washed out of system. Nutrients are consumed from the system at rate b by phytoplankton and replenished back in the system by putrefaction of dead phytoplankton and zooplankton at a rate β_1 and β_2 respectively. For mathematical simplicity nutrient recycling is assumed to be instantaneous. The rate at which nutrients are converted to phytoplankton biomass is denoted by b_1 ($0 < b_1 < 1$). It is assumed that zooplankton predate phytoplankton as per Holling type II functional response at a rate γ . The half saturation constant is denoted by β and μ_1 is the death rate of phytoplankton due to natural reasons. γ_1 is conversion parameter for zooplankton. θ is the rate at which toxins are released by TPP. μ_2 is the natural death rate of zooplankton. The time gap required for toxic phytoplankton to mature or release toxins is denoted by τ . Further, a linear harvesting is applied to zooplankton at a rate E and q is the catchability coefficient. The values and meaning of various parameters used in the model are given in table 1.

3 Positivity and boundedness analysis

In accordance with the biological meaning, the non-negative initial conditions are $N(0), P(0), Z(0) > 0$. The system parameters $\alpha, b, \gamma, \gamma_1, \beta, \beta_1, \beta_2, b_1, \mu_1, \mu_2, \theta, q, E$ are all positive. Note that, we have $\frac{dZ}{dt} < 0$ for $\gamma\gamma_1 < \theta$ or $\gamma\gamma_1 - \theta - \mu_2 - qE < 0$, which is not possible for a biological population. Therefore, in our analysis, we assume that,

$$\gamma\gamma_1 > \theta + \mu_2 + qE\tag{3.1}$$

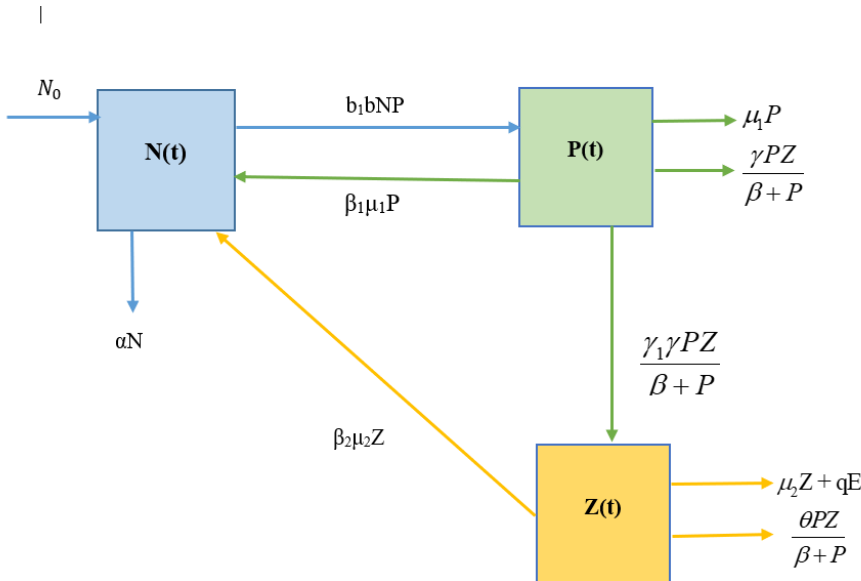
As per the ecological viewpoint, the dynamics of the system are of interest in the positive first octant. It means that all the state variables remain non-negative and do not move towards infinity

Theorem 3.1. *All the solutions of system under consideration are non-negative.*

Proof. The system considered can be written in vector form as

Table 1. Brief description of parameters used for in the model.

Parameter	Description	Values
N_0	Natural availability of nutrients in water	1
α	Wearing out rate of nutrients	0.283
β_1	Nutrient recycle rate from death of phytoplankton	0.31
β_2	Nutrient recycle rate from death of zooplankton	0.43
b	Uptake rate of nutrients by phytoplankton ($0 < b < 1$)	0.75
b_1	Maximal conversion rate of nutrients by phytoplankton ($b_1 < b$)	0.64
μ_1	Natural mortality rate of phytoplankton	0.25
μ_2	Natural mortality rate of zooplankton	0.32
γ	Maximum ingestion rate by zooplankton	1.8
β	Half saturation constant	1
γ_1	Grazing rate of phytoplankton by zooplankton	1.3
θ	Rate of toxin secretion by TPP	0.3092
E	Harvesting effort of zooplankton	0.61
q	Catchability coefficient	0.01

**Figure 1.** Diagram showing interactions among three populations

$$\dot{W}(t) = H(W(t)) \quad (3.2)$$

Where, $W_0(t) = (N(t), P(t), Z(t))^t \in R_3^+$

$$H(W(t)) = \begin{pmatrix} H_1(W(t)) \\ H_2(W(t)) \\ H_3(W(t)) \end{pmatrix} = \begin{pmatrix} N_0 - \alpha N - bNP + \beta_1 \mu_1 P + \beta_2 \mu_2 Z \\ b_1 bNP - \mu_1 P - \frac{\gamma P Z}{\beta + P} \\ (\frac{\gamma_1 \gamma - \theta}{\beta + P})P - (\mu_2 + qE)Z \end{pmatrix}$$

It is clear that $H_i(W(t))_{W_0} \geq 0$, for $i = 1, 2, 3$. In accordance with Nagumo's theorem [32, 33] all the solutions of (3.2) with non-negative initial condition remain positive in the region R_3^+ i.e. $W(t) \in R_3^+$, $\forall t > 0$. $\square$

Theorem 3.2. All the solutions of proposed system with non-negative initial conditions and originating in R_+^3 remain bounded.

Proof. Define a function,

$$U(t) = N(t) + \frac{1}{b_1}P(t) + \frac{\gamma}{b_1(\gamma\gamma_1 - \theta)}Z(t) \quad (3.3)$$

Differentiating with respect to time,

$$\begin{aligned} \frac{dU}{dt} &= \frac{dN}{dt} - \frac{1}{b_1} \frac{dP}{dt} + \frac{\gamma}{b_1(\gamma\gamma_1 - \theta)} \frac{dZ}{dt} \\ &= N_o - \alpha N - \frac{\mu_1}{b_1}(1 - b_1\beta_1)P - \frac{\gamma Z}{b_1(\gamma\gamma_1 - \theta)}(\mu_2 + qE - \frac{b_1}{\gamma}(\gamma\gamma_1 - \theta)\beta_2\mu_2) \\ &\leq N_o - \eta U \\ \text{where } \eta &= \min\left(\alpha, \mu_1(1 - b_1\beta_1), (\mu_2 + qE) - \frac{b_1}{\gamma}(\gamma\gamma_1 - \theta)\beta_2\mu_2\right) \end{aligned}$$

That gives, $\frac{dU}{dt} + \eta U \leq N_o$

By differential inequality theorem, $0 < U < U(0)e^{\eta t} + \frac{N_o}{\eta}$,

Where $U(0) = (N(0), P(0), Z(0))$

As $t \rightarrow \infty, 0 < U < \frac{N_o}{\eta}$

Hence, all the solutions of system (1) initiating in R_+^3 are defined in region

$$X = (N, P, Z) : 0 \leq U(t) \leq \frac{N_o}{\eta}$$

Where $\eta = \min\left(\alpha, \mu_1(1 - b_1\beta_1), (\mu_2 + qE) - \frac{b_1}{\gamma}(\gamma\gamma_1 - \theta)\beta_2\mu_2\right)$

Hence, all the positive solutions of system are uniformly bounded. $\square$

4 Stability analysis of equilibrium points

4.1 Existence of equilibrium points of the model system

The proposed system will have following three feasible non-trivial equilibrium points.

i) Axial equilibrium point: $E_1 = (N_1, 0, 0)$ is plankton free equilibrium.

$$\frac{dN_1}{dt} = 0 \Rightarrow N_o - \alpha N_1 = 0 \Rightarrow N_1 = \frac{N_o}{\alpha}$$

$$\therefore, E_1 = \left(\frac{N_o}{\alpha}, 0, 0\right)$$

ii) Boundary Equilibrium point: $E_2 = (N_2, P_2, 0)$ is zooplankton free equilibrium.

After simplification of equations, $\frac{dN}{dt} = 0, \frac{dP}{dt} = 0$, we have,

$$N_2 = \frac{\mu_1}{bb_1}, P_2 = \frac{N_o bb_1 \alpha \mu_1}{b \mu_1 (1 - b_1 \beta_1)}$$

$$\therefore, E_2 = \left(\frac{\mu_1}{bb_1}, \frac{N_o bb_1 \alpha \mu_1}{b \mu_1 (1 - b_1 \beta_1)}, 0\right)$$

The boundary equilibrium E_2 exists if $N_o bb_1 - \alpha \mu_1 > 0, b_1 \beta_1 < 1$.

i.e., $bb_1 \geq \frac{\alpha \mu_1}{N_o}, b_1 < \frac{1}{\beta_1}$.

Remark1. For the existence of E_2 , growth rate of phytoplankton population should be at least $\frac{\alpha \mu_1}{N_o}$ and at the same time conversion factor from nutrients to phytoplankton should not exceed $\frac{1}{\beta_1}$.

iii) Positive coexisting equilibrium: $E_3 = (N_*, P_*, Z_*)$ is given by solution of equations,

$\frac{dN_*}{dt} = 0, \frac{dP_*}{dt} = 0, \frac{dZ_*}{dt} = 0$. It gives.

$$N_* = \frac{N_o + \beta_1 \mu_1 P_* + \beta_2 \mu_2 Z_*}{\alpha + b P_*}, P_* = \frac{(\mu_2 + qE)\beta}{\gamma\gamma_1 - \theta - \mu_2 - qE}, Z_* = \frac{1}{\gamma}(bb_1 N_* - \mu_1)(\beta + P_*)$$

. The positive interior equilibrium E_* exists if

i) $\gamma\gamma_1 > \theta + \mu_2 + qE$ which is true by equation...

ii) $N_* bb_1 - \mu_1 > 0, N_* > \frac{\mu_1}{bb_1} > \frac{\mu_1}{b}$

Remark2. For existence of interior equilibrium point, zooplankton population should grow at a rate which exceeds their rate of depletion due to natural mortality, harvesting and toxin production by TPP. Also, nutrient concentration in the system at that point should remain more than $\frac{\mu_1}{b}$.

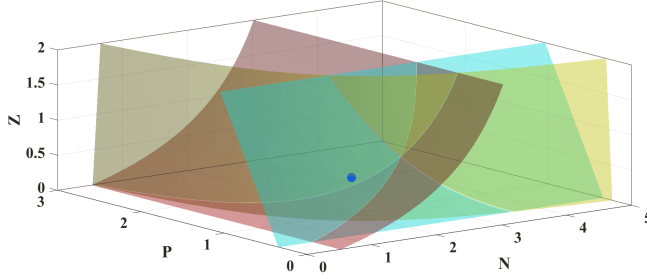


Figure 2. Intersection of nullclines showing existence of coexisting equilibrium

4.2 Stability analysis of equilibrium points for delay free case ($\tau = 0$)

This subsection examines the local asymptotic stability of steady states for the delay-free model, specifically when $\tau = 0$. To accomplish this, the Jacobian matrix is calculated, and the eigenvalues of the matrix are analysed to assess stability.

$$\begin{aligned}\frac{dN(t)}{dt} &= N_0 - \alpha N - bNP + \beta_1\mu_1 P(t) + \beta_2\mu_2 Z(t), \\ \frac{dP(t)}{dt} &= b_1bNP - \mu_1 P(t) - \frac{\gamma P(t)Z(t)}{\beta + P(t)}, \\ \frac{dZ(t)}{dt} &= \frac{\gamma_1\gamma P(t)Z(t)}{\beta + P(t)} - \frac{\theta P(t - \tau)Z(t)}{\beta + P(t - \tau)} - (\mu_2 + qE)Z(t).\end{aligned}\tag{4.1}$$

Theorem 4.1. The plankton free equilibrium point $E_1 = (N_1, 0, 0)$, is stable whenever $bb_1 < \frac{\alpha\mu_1}{N_o}$ holds where $N_1 = \frac{N_o}{\alpha}$.

Proof. The Jacobian matrix evaluated at E_1 is where $E_1 = (\frac{N_o}{\alpha}, 0, 0)$

$$J_1 = \begin{pmatrix} -\alpha & -b\frac{N_o}{\alpha} + \beta_1\mu_1 & \beta_2\mu_2 \\ 0 & bb_1\frac{N_o}{\alpha} - \mu_1 & 0 \\ 0 & 0 & -\mu_2 - qE \end{pmatrix}$$

It can be seen that eigen values of the characteristic equation of above matrix are $-\lambda_1 = -\alpha$, $\lambda_2 = bb_1\frac{N_o}{\alpha} - \mu_1$, $\lambda_3 = -(\mu_2 + qE)$

Clearly, $\lambda_1, \lambda_3 < 0$.

For λ_2 to be negative, $bb_1 < \frac{\alpha\mu_1}{N_o}$

For the stability of axial equilibrium point, the phytoplankton growth is bounded by factor $\frac{\alpha\mu_1}{N_o}$. $\square$

Theorem 4.2. The boundary equilibrium point $E_2 = (N_2, P_2, 0)$ where, $N_2 = \frac{\mu_1}{bb_1}$, $P_2 = \frac{N_o bb_1 \alpha \mu_1}{b\mu_1(1-b_1\beta_1)}$ is stable whenever $0 < N_o bb_1 \alpha \mu_1 < \frac{\beta b \mu_1 (\mu_2 + qE)(1-b_1\beta_1)}{\gamma\gamma_1 - \theta - \mu_2 - qE}$ holds.

Proof. The Jacobian matrix for the system evaluated at boundary equilibrium point $E_2 = (\frac{\mu_1}{bb_1}, \frac{N_o bb_1 \alpha \mu_1}{b\mu_1(1-b_1\beta_1)}, 0)$ is,

$$J_2 = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & 0 & r_{23} \\ 0 & 0 & r_{33} \end{pmatrix} = \begin{pmatrix} \frac{b_1(\alpha\beta_1\mu_1 - N_o b)}{\mu_1(1-b_1\beta_1)} & -\frac{\mu_1}{\beta_1} + \beta_1\mu_1 & \beta_2\mu_2 \\ \frac{b_1(N_o bb_1 - \alpha\mu_1)}{\mu_1(1-b_1\beta_1)} & 0 & \frac{-\gamma(N_o bb_1 - \alpha\mu_1)}{N_o bb_1 - \alpha\mu_1 + b\beta\mu_1} \\ 0 & 0 & r_{33} \end{pmatrix}$$

where, $r_{33} = -\mu_2 - qE$

The characteristic equation is

$$(\lambda - r_{33}) \left(\lambda^2 + \frac{b_1(bN_o - \alpha\mu_1\beta_1)}{\mu_1(1 - b_1\beta_1)}\lambda + (N_o b b_1 - \alpha\mu_1) \right) = 0 \quad (4.2)$$

The two eigen values λ_1, λ_2 are given by the roots of the equation,

$$\lambda^2 + \frac{b_1(bN_o - \alpha\mu_1\beta_1)}{\mu_1(1 - b_1\beta_1)}\lambda + (N_o b b_1 - \alpha\mu_1) = 0 \text{ and third eigen value is } \lambda = r_{33}$$

From above equation, λ_1, λ_2 will have negative real parts, if

i) $N_o b b_1 - \alpha\mu_1 > 0$, which is true by existence of E_2 ,

ii) $\text{Trace} = \frac{\alpha\mu_1 b_1 \beta_1 - b b_1 N_o}{\mu_1(1 - b_1\beta_1)} < 0$

Which yields $(N_o b b_1 - \alpha\mu_1) + \alpha\mu_1(1 - b_1\beta_1) > 0$. This condition clearly holds for the existence of E_2 .

The eigen value r_{33} is negative when, $N_o b b_1 - \alpha\mu_1 < \frac{\beta b \mu_1 (\mu_2 + qE)(1 - b_1\beta_1)}{\gamma\gamma_1 - \theta - \mu_2 - qE}$

Combining these results, the stability condition of boundary equilibrium is obtained as,

$$0 < N_o b b_1 - \alpha\mu_1 < \frac{\beta b \mu_1 (\mu_2 + qE)(1 - b_1\beta_1)}{\gamma\gamma_1 - \theta - \mu_2 - qE}.$$

□

Theorem 4.3. The positive equilibrium point E_3 is locally asymptotically stable if

$$\frac{b b_1 \beta_2 \mu_2 (\beta + P_*)}{\alpha + b P_*} < \gamma < \frac{(\alpha + b P_*)(\beta + P_*)^2}{P_* Z_*}, \text{ and}$$

$$(\alpha + b P_*)^2 \frac{\gamma Z_*}{(\beta + P_*)^2} - (\alpha + b P_*) \left(\frac{\gamma^2 P_* Z_*^2}{(\beta + P_*)^4} + (b N_* - \beta_1 \mu_1) b b_1 \right)$$

$$+ (b N_* - \beta_1 \mu_1) \frac{b b_1 \gamma Z_*}{(\beta + P_*)^2} < \frac{(\gamma \gamma_1 - \theta) \beta Z_*}{(\beta + P_*)^2} \left(\beta_2 \mu_2 b b_1 - \frac{\gamma^2 P_* Z_*}{(\beta + P_*)^3} \right),$$

hold.

Proof. The variational matrix for the system evaluated at E_3 is

$$J_3 = \begin{pmatrix} -(\alpha + b P_*) & -b N_* + \beta_1 \mu_1 & \beta_2 \mu_2 \\ b_1 b P_* & \frac{\gamma P_* Z_*}{(\beta + P_*)^2} & \frac{\gamma P_*}{(\beta + P_*)} \\ 0 & \frac{(\gamma \gamma_1 - \theta) \beta Z_*}{(\beta + P_*)^2} & 0 \end{pmatrix}$$

The characteristic equation obtained from the variational matrix J_3 is

$$\lambda^3 + B_1 \lambda^2 + B_2 \lambda + B_3 = 0 \quad (4.3)$$

where

$$B_1 = -\text{trace} J_3 = (\alpha + b P_*) - \frac{\gamma P_* Z_*}{(\beta + P_*)^2};$$

$$B_2 = \frac{(\gamma \gamma_1 - \theta) \gamma \beta P_* Z_*}{(\beta + P_*)^3} + (b N_* - \beta_1 \mu_1) b_1 b P_* - \frac{\gamma(\alpha + b P_*) P_* Z_*}{(\beta + P_*)^2}; \quad (4.4)$$

$$B_3 = \text{big} \left(\frac{\gamma(\alpha + b P_*)}{\beta + P_*} - \beta_2 \mu_2 b b_1 \right) \frac{(\mu_2 + qE) \beta Z_*}{\beta + P_*}.$$

Now E_* is locally asymptotically stable if $B_1 > 0$, $B_3 > 0$ and $B_1 B_2 - B_3 > 0$ (Routh Hurwitz criteria) holds. $B_1 > 0$ if $\frac{\gamma P_* Z_*}{(\beta + P_*)^2} < (\alpha + b P_*)$ and $B_3 > 0$ if $\gamma > \frac{b b_1 \beta_2 \mu_2 (\beta + P_*)}{\alpha + b P_*}$. Combining we have,

$$\frac{b b_1 \beta_2 \mu_2 (\beta + P_*)}{\alpha + b P_*} < \gamma < \frac{(\alpha + b P_*)(\beta + P_*)^2}{P_* Z_*} \quad (4.5)$$

$B_1 B_2 - B_3 > 0$ if

$$(\alpha + b P_*)^2 \frac{\gamma Z_*}{(\beta + P_*)^2} - (\alpha + b P_*) \left(\frac{\gamma^2 P_* Z_*^2}{(\beta + P_*)^4} + (b N_* - \beta_1 \mu_1) b b_1 \right) \\ + (b N_* - \beta_1 \mu_1) \frac{b b_1 \gamma Z_*}{(\beta + P_*)^2} < \frac{(\gamma \gamma_1 - \theta) \beta Z_*}{(\beta + P_*)^2} \left(\beta_2 \mu_2 b b_1 - \frac{\gamma^2 P_* Z_*}{(\beta + P_*)^3} \right) \quad (4.6)$$

For the parameters satisfying above conditions, the system will be locally asymptotically stable. If $B_1 B_2 = B_3$ for a set of parameter values, the Hopf bifurcation will be exhibited by the system around E_* and system will no more be stable. $\square$

4.3 Global asymptotic stability of interior equilibrium point

Theorem 4.4. *The coexistence equilibrium point E_* is globally asymptotically stable if following conditions hold.*

- (i) $b_1 b N_* - \mu_1 < \frac{\gamma \beta Z_*}{\beta(\beta + P_*)}$ and
- (ii) $(\mu_2 + qE) + \frac{\gamma \gamma P_*}{\gamma} < \frac{\theta P_*}{\beta + P_*}$

Proof. To derive stability criteria of interior equilibrium point we consider a positive definite function V ,

$$V = \frac{1}{2}(N - N_*)^2 + \frac{A}{2}(P - P_*)^2 + \frac{B}{2}(Z - Z_*)^2 \quad (4.7)$$

Using Lyapunov's method, differentiating with respect to t ,

$$\begin{aligned} \frac{dV}{dt} &= \frac{\partial V}{\partial N} \frac{dN}{dt} + \frac{\partial V}{\partial P} \frac{dP}{dt} + \frac{\partial V}{\partial Z} \frac{dZ}{dt} \\ &= (N - N_*)(N_0 - \alpha N - bNP + \beta_1 \mu_1 P + \beta_2 \mu_2 Z - N_0) \\ &\quad + \alpha N_* + bN_* P_* - \beta_1 \mu_1 P_* - \beta_2 \mu_2 Z_*) + \\ &\quad A(P - P_*) \left(-b_1 b N_* P_* + \mu_1 P_* + \frac{\gamma P_* Z_*}{\beta + P_*} + b_1 b NP - \mu_1 P - \frac{\gamma PZ}{\beta + P} \right) + \\ &\quad B(Z - Z_*) \left(\frac{\gamma_1 \gamma PZ}{\beta + P} - \frac{\theta PZ}{\beta + P} - (\mu_2 + qE)Z - \frac{\gamma_1 \gamma P_* Z_*}{\beta + P_*} + \frac{\theta P_* Z_*}{\beta + P_*} + (\mu_2 + qE)Z_* \right). \end{aligned} \quad (4.8)$$

After some algebraic calculations, we get

$$\begin{aligned} \frac{dV}{dt} &\leq (N - N_*)^2 (\alpha + bP) + (P - P_*)^2 A (bb_1 N_* - \mu_1) - \frac{\gamma \beta Z_*}{\beta(\beta + P_*)} \\ &\quad + (Z - Z_*)^2 B \left((\mu_2 + qE) + \frac{\gamma \gamma_1 P}{\beta} - \frac{\theta P_*}{\beta + P_*} \right) \end{aligned} \quad (4.9)$$

Assuming $A = \frac{bN_* + \beta_1 \mu_1}{bb_1 P}$ and $B = \frac{\gamma(bN_* + \beta_1 \mu_1)(P_* - \beta)}{bb_1 P \beta (\gamma \gamma_1 Z_* - \theta Z)}$.

Therefore, $\frac{dV}{dt} < 0$ for

$$(bb_1 N_* - \mu_1) < \frac{\gamma \beta Z_*}{\beta(\beta + P_*)} \text{ and } (\mu_2 + qE) + \frac{\gamma \gamma_1 P_*}{\beta} < \frac{\theta P_*}{\beta + P_*}. \quad (4.10)$$

Hence the theorem. $\square$

5 Bifurcation Analysis

In dynamical systems, bifurcation analysis is used to study the qualitative behaviour of the system as one or more parameters are varied to explore how system behaves and stabilises with time. In this section, we explore the conditions for occurrence of transcritical and Hopf bifurcations in the system.

5.1 Transcritical bifurcation

A transcritical bifurcation occurs when an equilibrium point exchanges its stability conditions with some other steady point. Let $X = (N, P, Z)$, so the system can be expressed as $X' = F(X)$, $F = (f, g, h)^T$. Some general expressions used have been evaluated as follows:

$$DF = \begin{pmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ 0 & F_{32} & F_{33} \end{pmatrix}$$

Where, $F_{11} = -\alpha - bP$, $F_{12} = -bN + \beta_1\mu_1$, $F_{13} = \beta_2\mu_2$, $F_{21} = bb_1P$,
 $F_{22} = bb_1N - \mu_1 - \frac{\beta\gamma Z}{(\beta+P)^2}$, $F_{23} = \frac{-\gamma P}{\beta+P}$, $F_{32} = \frac{\beta(\gamma\gamma_1-\theta)Z}{(\beta+P)^2}$, $F_{33} = \frac{(\gamma\gamma_1-\theta)P}{\beta+P} - (\mu_2 + qE)$.

Theorem 5.1. *The system does not go through bifurcation about the plankton free steady state $E_1 = (N_1, 0, 0)$.*

Proof. Eigen values of jacobian matrix J_1 are given by $-\alpha$, $bb_1\frac{N_o}{\alpha} - \mu_1$ and $-(\mu_2 + qE)$.

If $-(\mu_2 + qE) = 0$, then $E = \frac{-\mu_2}{q}$, which is not possible in view of positivity assumption of parameters.

Take $bb_1\frac{N_o}{\alpha} - \mu_1 = 0$. Then, eigenvectors of J_1 and J_1^T associated with zero eigenvalue are $v = (t, 1, 0)^T$ and $w = (0, 1, 0)^T$, where $t = \frac{1}{\alpha}(-\frac{bN_o}{\alpha}) + \beta_1\mu_1$. It is verified that:

$$(i)w^T[F_\alpha(E_1, \alpha = \alpha^*)] = 0, (ii)w^T[DF_\alpha(E_1, \alpha = \alpha^*)v] = 0. \quad (5.1)$$

Hence, steady state E_1 does not experience transcritical bifurcation or saddle node bifurcation. $\square$

Theorem 5.2. *The system undergoes transcritical bifurcation around the zooplankton free equilibrium point $E_2 = (N_2, P_2, 0)$.*

Proof. For $E = E_* = \frac{1}{q}(\frac{(\gamma\gamma_1-\theta)(N_o bb_1 - \alpha\mu_1)}{N_o bb_1 - \alpha\mu_1 + b\beta\mu_1(1-b_1\beta_1)} - \mu_2)$, the jacobian matrix J_2 has a zero-eigenvalue. Corresponding eigen vectors of J_2 and J_2^T are $v = (t_1, t_2, 1)^T$ and $w = (0, 0, 1)^T$, where $t_1 = \frac{-r_{23}}{r_{21}}$, $t_2 = \frac{1}{r_{12}}(\frac{r_{11}r_{23}}{r_{21}} - r_{13})$. Taking E as parameter, we examine the transversality conditions for the occurrence of transcritical bifurcation [35]. We first compute,

$$F_E = \begin{pmatrix} 0 \\ 0 \\ -qZ \end{pmatrix}, \quad DF_E = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -q \end{pmatrix} \quad \text{It is observed that,}$$

$$1.w^T[F_E(E_2, E = E_*)] = 0, 2.w^T[DF_E(E_2, E = E_*)v] = (0, 0, 1)(0, 0, -q)^T = -q \neq 0, \quad (5.2)$$

$$3.D^2F(E_1, E = E_{TC})(v, v) = F_{NN}(E_2, E = E_*)t_1^2 + 2F_{NP}(E_2, E = E_*)t_1t_2 \\ + F_{PP}(E_2, E = E_*)t_2^2 + 2F_{PZ}(E_2, E = E_*)t_2 \\ + F_{ZZ}(E_2, E = E_*)1^2 + 2F_{NZ}(E_2, E = E_*)t_1 \quad (5.3)$$

$$D^2F(E_1, E = E_{TC})(v, v) = 2t_2 \begin{pmatrix} -bt_1 \\ bb_1t_1 - \frac{\beta\gamma}{(\beta+P_2)^2} \\ \frac{\beta(\gamma\gamma_1-\theta)}{(\beta+P_2)^2} \end{pmatrix}$$

$$w^T D^2F(E_1, E = E_{TC})(v, v) = \frac{2\beta(\gamma\gamma_1 - \theta)t_2}{(\beta + P_2)^2} \quad (5.4)$$

Therefore, using Sotomayor's theorem [34, 35], it is concluded that system undergoes transcritical bifurcation about equilibrium point E_2 .

Remark: The occurrence of transcritical bifurcation at zooplankton free steady state E_2 is also verified from the figure (6) for the considered set of parameters. $\square$

5.2 Hopf bifurcation

Theorem 5.3. Hopf bifurcation is experienced by the system around positive steady point E_3 as the bifurcation parameter E crosses its critical value E_H , relative to following conditions:

- (i) $B_i(E_H) > 0$, $i = 1, 2, 3$ (ii) $B_1B_2 - B_3 = 0$ at $E = E_H$
 (iii) The transversality condition holds good, $(B_1B'_2 + B'_1B_2 - B'_3)_{E=E_H} \neq 0$.

Proof. The characteristic equation for coexisting steady state is given by equation (4.3). The Hopf bifurcation occurs at critical value $E = E_H$, where $B_i > 0$, $i = 1, 2, 3$ and $B_1B_2 - B_3 = 0$. So the characteristic equation can be written as

$$(\lambda^2 + B_2)(\lambda + B_1) = 0. \quad (5.5)$$

The equation has a real root $\lambda_2 = -B_1$ and a pair of conjugate imaginary roots, $\lambda_1 = \iota\sqrt{B_2}$, $\bar{\lambda}_1 = -\iota\sqrt{B_2}$. Writing in general form, $\pm\iota\sqrt{B_2} = a(E) + ib(E)$ and $c = -B_1$. The characteristic equation can be written as

$$(\lambda - \lambda_1)(\lambda - \bar{\lambda}_1)(\lambda - c) = 0 \quad (5.6)$$

which gives,

$$\lambda^3 - (c + 2a)\lambda^2 + (2ac + |\lambda_1|^2)\lambda - c|\lambda_1|^2 = 0 \quad (5.7)$$

Comparing coefficients of (4.3) and above equation, and simplifying, we get

$$B_1B_2 + 2aB_2 + 2(B_1^2 + 4a^2B_1) - B_3 = 0 \quad (5.8)$$

Differentiating the equation with respect to E at $E = E_H$ gives

$$B_1(E_H)B'_2(E_H) + B'_1(E_H)B_2(E_H) + 2a'(E_H)B_2(E_H) + 2B_1^2(E_H)a'(E_H) - B'_3(E_H) = 0, \quad (5.9)$$

$$a'(E_H) = \left[\frac{dRe\lambda_i(E)}{dE} \right]_{E=E_H} = \frac{B_1(E_H)B'_2(E_H) + B'_1(E_H)B_2(E_H) - B'_3(E_H)}{2(B_2(E_H) + B_1^2(E_H))}, \quad (5.10)$$

$$\left[\frac{dRe\lambda_i(E)}{dE} \right]_{E=E_H} \neq 0, \text{ provided } (B_1B'_2 + B'_1B_2 - B'_3)_{E=E_H} \neq 0. \quad (5.11)$$

All the conditions for existence of Hopf bifurcation are satisfied at $E = E_H$. Hence the theorem. $\square$

6 Qualitative analysis of system with time delay ($\tau \neq 0$)

This subsection examines the local stability of coexisting steady state for the model taking into account the time delay τ ($\tau \neq 0$). Further, the conditions for existence of Hopf bifurcations are examined.

6.1 Stability analysis of interior equilibrium point

To check stability of model with delay at interior equilibrium point, we introduce variables $U = N - N_*$, $V = P - P_*$, $W = Z - Z_*$, so that model is transformed to

$$\begin{aligned} \frac{dU}{dt} &= a_{11}U(t) + a_{12}V(t) + a_{13}W(t), \\ \frac{dV}{dt} &= a_{21}U(t) + a_{22}V(t) + a_{23}W(t), \\ \frac{dW}{dt} &= a_{32}V(t) + b_{32}V(t). \end{aligned} \quad (6.1)$$

$$A = [a_{ij}]_{3 \times 3} = \begin{pmatrix} -(\alpha + \beta P_*) & (\mu_1\beta_1 - bN_*) & \mu_2\beta_2 \\ bb_1P_* & \frac{\gamma P_* Z_*}{(\beta + P_*)^2} & \frac{-\gamma P_*}{\beta + P_*} \\ 0 & \frac{\gamma\gamma_1\beta Z_*}{(\beta + P_*)^2} & 0 \end{pmatrix},$$

$$B = [b_{ij}]_{3 \times 3}, b_{32} = \frac{\theta\beta Z_*}{(\beta + P_*)^2} \text{ and } b_{ij} = 0 \text{ for } i = 1, 2; j = 1, 3.$$

The associated characteristic equation is $|A + Be^{-\lambda\tau} - \lambda I| = 0$,

Which yields transcendental characteristic equation,

$$\lambda^3 + m_1\lambda^2 + m_2\lambda + m_3 + e^{-\lambda\tau}(m_4\lambda + m_5) = 0 \quad (6.2)$$

Where $m_1 = (\alpha + \beta P_*) - \frac{\gamma P_* Z_*}{(\beta + P_*)^2}$, $m_2 = \frac{\gamma^2 \gamma_1 \beta P_* Z_*}{(\beta + P_*)^3} - (\alpha + \beta P_*) \frac{\gamma P_* Z_*}{(\beta + P_*)^2} - bb_1 P_* (\mu_1 \beta_1 - b N_*)$,
 $m_3 = \frac{\gamma \gamma_1 \beta Z_*}{(\beta + P_*)^2} [(\alpha + \beta P_*) \frac{\gamma P_*}{\beta + P_*} - \mu_2 \beta_2 bb_1 P_*]$, $m_4 = \frac{\theta \alpha \beta P_* Z_*}{(\beta + P_*)^3}$, $m_5 = \frac{\theta \beta Z_*}{(\beta + P_*)^2} [(\alpha + \beta P_*) \frac{\gamma P_*}{\beta + P_*} - \mu_2 \beta_2 bb_1 P_*]$.

For Hopf bifurcation around co-existing equilibrium, put $\lambda = iq, q > 0$ for $\tau \neq 0$, in above transcendental equation

$$(iq)^3 + m_1(iq)^2 + m_2(iq) + m_3 + e^{-iq\tau}(m_4(iq) + m_5) = 0 \quad (6.3)$$

It gives,

$$\begin{aligned} m_5 \cos(q\tau) + m_4 q \sin(q\tau) &= m_1 q^2 - m_3, \\ m_4 q \cos(q\tau) + m_5 q \sin(q\tau) &= q^3 - m_2 q, \end{aligned} \quad (6.4)$$

which is simplified to obtain

$$(q)^6 + (m_1^2 - 2m_2)(q)^4 + (m_2^2 - 2m_1 m_3 - m_4^2)(q)^2 + m_3^2 - m_5^2 = 0, \quad (6.5)$$

Taking $(p = q^2)$,

$$\phi(p) = (p)^3 + r_1(p)^2 + r_2(p) + r_3 = 0, (p = q^2) \quad (6.6)$$

Where $r_1 = m_1^2 - 2m_2$, $r_2 = m_2^2 - 2m_1 m_3 - m_4^2$, $r_3 = m_3^2 - m_5^2$.

Note that $r_3 > 0$ as $\gamma \gamma_1 > \theta$. So, product of roots of $\phi(p) = -r_3 < 0$.

Hence one or all three roots of cubic $\phi(p)$ have negative real parts.

Proposition. There exists positive root of equation (6.6) if $\phi'(p)$ has a positive root p_1 and value of $\phi(p)$ at p_1 is negative.

Proof. Roots of $\phi'(p)$ are $p_i = \frac{1}{3}(-r_1 \pm \sqrt{d})$, where $d = r_1^2 - 3r_2, i = 1, 2$.

For $d < 0$, $\phi'(p)$ will have a pair of complex conjugate roots or one repeated root. Now, one of the roots of $\phi(p)$ is < 0 and $\phi(0) = r_3 > 0$.

So, $\phi(p)$ is monotonically increasing function which will have no positive root if $d \leq 0$ or if $d > 0$ and $p_1 = \frac{1}{3}(-r_1 + \sqrt{d}) < 0$.

Also, for $d > 0$ and $p_1 > 0$ and $\phi(p_1) > 0$, $\phi(p)$ will have no positive root.

For: $\phi''(p_2) = 6p_2 + 2r_1 = -2\sqrt{d} < 0$, which shows p_2 is local maximum of $\phi(p)$.

So, $\phi(p_1) < \phi(p_2)$ and $\phi(0) > 0$ which means no positive root for $\phi(p) > 0$ in this case.

Therefore, $\phi(p_1) > 0$ will have positive roots for $\phi'(p_1) = 0|_{p_1 > 0}$ and $\phi(p_1) < 0$.

Theorem 6.1. The coexistence equilibrium E_3 of considered system is asymptotically stable for all $\tau \geq 0$ for $3r_2 \geq r_1^2$.

Proof. As seen in preposition, $\phi(p)$ has negative real part for all the roots for $d < 0$ i.e. for $3r_2 \geq r_1^2$.

So, E_* is locally asymptotically stable for $3r_2 \geq r_1^2$. $\square$

6.2 Hopf bifurcation

Theorem 6.2. (i) Hopf bifurcation is observed to occur in the system about E_* when $\tau = \tau^*$, for $3r_2 \geq r_1^2$ and $\phi(p_1) < 0$ where $\phi'(p_1) = 0$ and $p_1 > 0$.

(ii) The transversality condition requires that $\frac{d \operatorname{Re} \lambda(\tau^*)}{d\tau} \big|_{\tau=\tau^*} \neq 0$, if $q_o^2 = p_*$ and $\phi'(p_*) \neq 0$.

Proof. (i) Let p_0 be a positive root of $\phi(p)$. Accordingly, q_o^2 is a positive root of equation (6.5). Solving equations (6.4) for τ , corresponding to the value of q_o^2 , we have

$$\cos(q_o\tau) = \frac{m_4 q_o^4 + (m_1 m_5 - m_2 m_4) q_o^2 - m_3 m_5}{m_5^2 + m_4^2 q_o^2}, \quad (6.7)$$

$$\tau_n = \frac{1}{q_o} \cos^{-1} \left(\frac{m_4 q_o^4 + (m_1 m_5 - m_2 m_4) q_o^2 - m_3 m_5}{m_5^2 + m_4^2 q_o^2} \right) + \frac{2n\pi}{q_o}, \sin(q_o \tau) > 0, n = 1, 2, 3 \dots \quad (6.8)$$

For $\tau = \tau_n$, characteristic equation (6.2) has a pair of roots $\pm i q_o$ lying on imaginary axis.

Take $\tau^* = \min(\tau_n)_{n>0}$. $\therefore$, Hopf bifurcation occurs at E_* for $\tau = \tau^*$. E_* is asymptotically stable for $0 \leq \tau \leq \tau^*$ and becomes unstable for $\tau > \tau^*$.

(ii) Let the characteristic equation (6.2) has root, $\lambda(\tau) = p(\tau) + i q(\tau)$ about $\tau = \tau^*$ such that $p(\tau) = 0$

Write $q(\tau^*)^2 = p(\tau^*) = p_*$, $q(\tau^*) = q_*$, Differentiating (6.2) with time and simplifying, we get,

$$\frac{d\lambda}{dt} = \frac{\lambda(m_4 \lambda + m_5) e^{-\lambda \tau}}{(3\lambda^2 + 2m_1 \lambda + m_2) + m_4 e^{-\lambda \tau} - (m_4 \lambda + m_5) \tau e^{-\lambda \tau}} \quad (6.9)$$

$$\begin{aligned} \frac{d \operatorname{Re} \lambda(\tau^*)}{d\tau} \Big|_{\tau=\tau^*} &= \operatorname{Re} \left(\frac{(3\lambda^2 + 2m_1 \lambda + m_2) e^{\lambda \tau} + m_4}{\lambda(m_4 \lambda + m_5)} - \frac{\tau}{\lambda} \right) \\ &= \frac{\phi'(p_*)}{(m_4^2 q_*^2 + m_5^2)} \neq 0 \text{ if } \phi'(p_*) \neq 0. \end{aligned} \quad (6.10)$$

□

7 Numerical Simulations

Case I Initially, we consider the case where $\tau = 0$. Numerical simulations demonstrate that the positive equilibrium point $E^* = (1.22, 0.84, 0.34)$ is stable in the absence of time delay. The results are presented in figure (3). Taking $\theta = 0.6092$, $q = 0.13$, $E = 3.61$ and other parameters

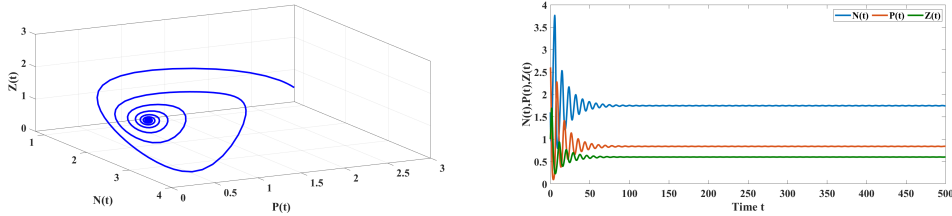


Figure 3. Local stability of interior equilibrium point in absence of time delay. $E^* = (1.22, 0.84, 0.34)$ is a stable focus and time evolution is smooth and non-oscillatory.

as listed in table 1, $E^* = (1.22, 0.84, 0.34)$. By numerical calculations, $B_1 = 0.7162 > 0$, $B_3 = 0.1208 > 0$, $B_2 = 0.3419$, $B_1 B_2 - B_3 = 0.12407 > 0$. This shows numerically that Routh Hurwitz Criteria holds good and local asymptotic stability at E^* . LAS about is also depicted in figure (3).

7.1 Impact of harvesting on the system

The system's dynamics exhibit significant behavioral changes as the parameter E varies. A decrease in E leads to the emergence of stable limit cycles, marking a loss of stability through a Hopf bifurcation. Initially, at $E = 3.1$, the system is asymptotically stable around positive equilibrium, as illustrated in figure 4(a). However, as E decreases, the system undergoes a Hopf bifurcation, leading to the emergence of stable limit cycles. At $E = 2.9$ these limit cycles appear (figure 4(b)), and as E is further reduced to 2.1, the amplitude of oscillations increases significantly, as shown in figure 4(c). The bifurcation diagram in figure (5) further illustrates the occurrence of Hopf bifurcation with respect to E , capturing the system's transition from stable equilibrium to sustained periodic oscillations.

The harvesting may help in controlling the system dynamics leading to sustainability in the system. However, for the system permanence, harvesting should be optimal keeping in view the available resources. Linear harvesting of predator population has been considered for the

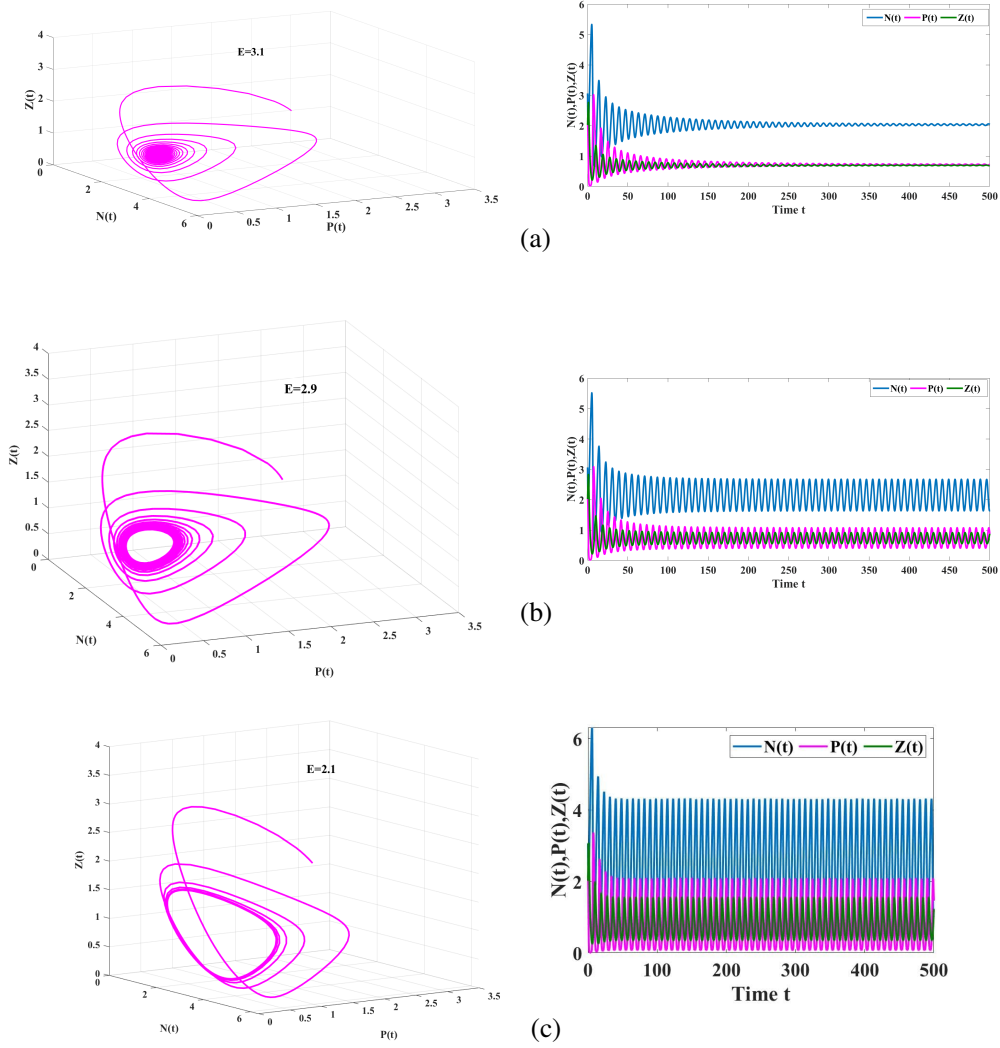


Figure 4. (a) Stability at positive equilibrium for $E=3.1$ (b) Limit cycles emerge for decrease in value of E , as indicated by phase portraits and time evolution series at $E=2.9$ (c) Oscillatory behaviour increases at $E=2.1$

proposed system. Figure (6) shows one parameter bifurcation of the system w.r.t. harvesting parameter. The bifurcation diagram has been split into different regions. The yellow zone shows the positive equilibrium, indicating that system destabilises at low harvesting levels. For the harvesting level between 0 to 2.99, the positive equilibrium remains unstable and undergoes a Hopf bifurcation at Hopf point $E_H = 2.99$. After this, the system stabilizes, allowing the coexistence of species. As harvesting increases further, the populations can no longer coexist. A transcritical bifurcation occurs at $E_t = 7.63$, leading to the stabilization of a zooplankton-free equilibrium $E_2(1.74, 0.835, 0)$ represented by the green zone. As a result, harvesting above the critical value (E_t) drives the zooplankton species to extinction, while for $0 \leq E \leq 7.63$, coexistence of the species is maintained.

7.2 Impact of toxin liberation parameter θ

The toxin liberation parameter, θ , significantly impacts the system's dynamics. By holding other parameters constant and varying θ , the system transitions from stability to instability, indicated by a Hopf bifurcation. In figure 7(a), the system converges to a stable focus at $\theta = 0.5$. As θ decreases to 0.43, the system exhibits limit cycles, as shown in Figure 7(b), with oscillation

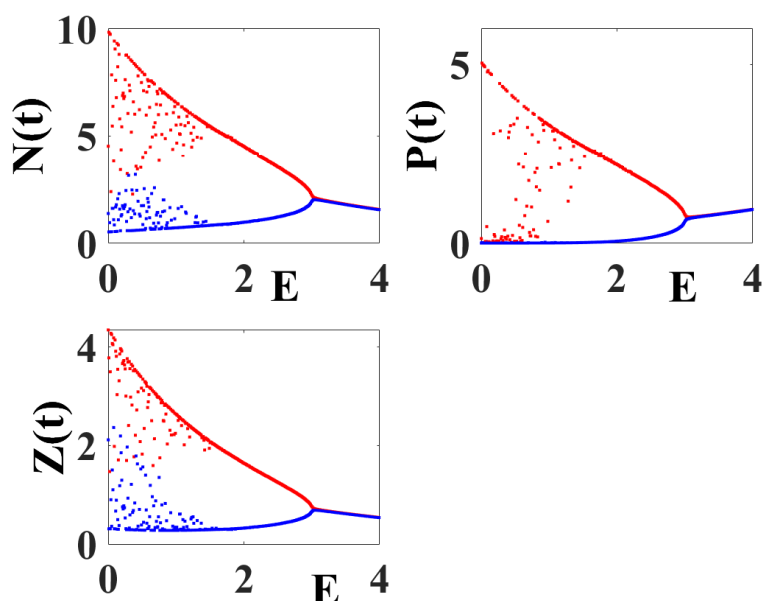


Figure 5. Bifurcation diagram of system relative to harvesting effort E

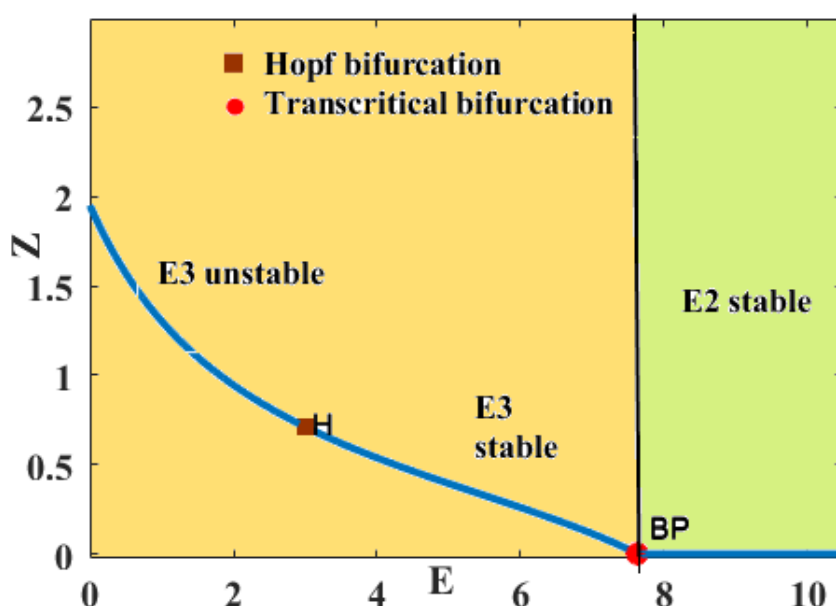


Figure 6. One parameter bifurcation of the system relative to harvesting effort E

amplitude increasing as θ is further reduced to 0.28 (7(c)).

Figure (8) presents a bifurcation diagram illustrating the dynamics of the system's species as a function of the parameter θ , which represents toxin liberation. To further investigate how the system's behaviour changes with θ , another bifurcation diagram in figure (9) emphasizes the role of the toxin liberation parameter in influencing species coexistence and the overall system stability. The distinct regions in this diagram indicate where steady states become stable or unstable based on the value of θ . Specifically, the green zone in the diagram denotes stable coexistence among all populations, with a Hopf bifurcation occurring at $\theta_h = 0.43$. Below this threshold, the system exhibits oscillatory behaviour. However, for values of θ between the

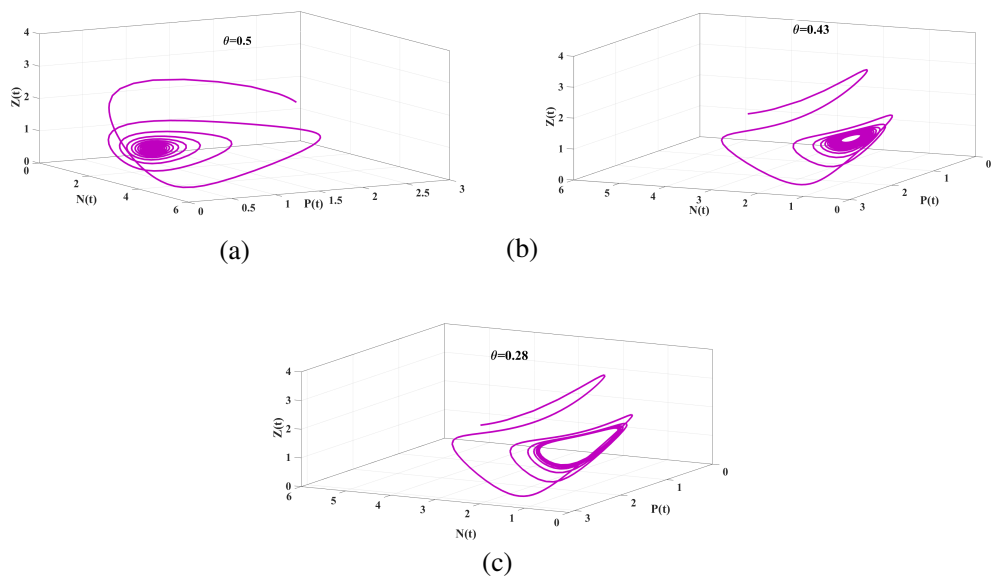


Figure 7. (a) Coexisting equilibrium is a stable focus at $\theta = 0.5$ (b) Occurrence of limit cycles for decrease in value of θ , as indicated by phase portraits at $\theta = 0.43$ (c) Oscillatory behaviour increases at $\theta = 0.28$

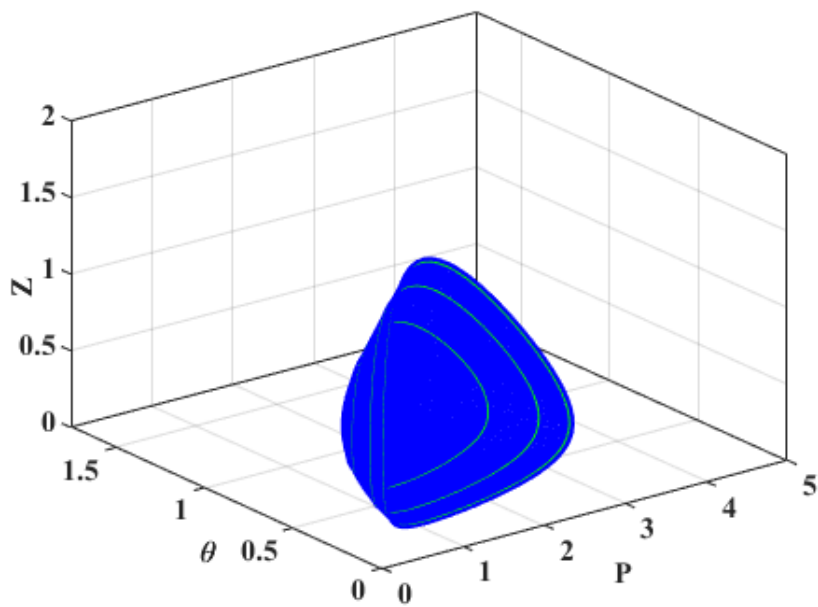


Figure 8. Bifurcation diagram of system relative to toxin liberation parameter θ .

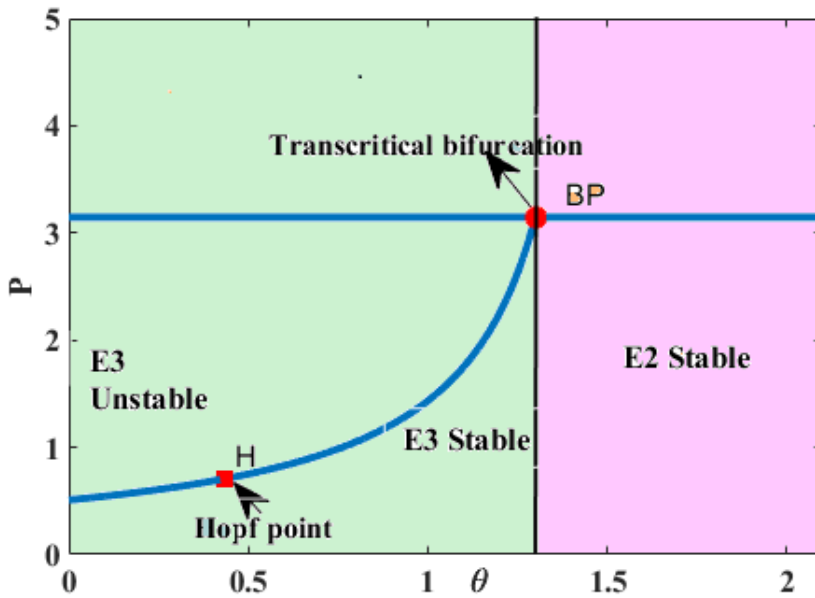


Figure 9. One parameter bifurcation of the system relative to toxin liberation parameter θ .

Hopf threshold θ_h and 1.29, species coexist. At the transcritical bifurcation point $\theta_t = 1.29$, the zooplankton-free equilibrium $E_2(0.520, 3.146, 0)$ becomes stable, marking a transition as zooplankton start to decline within the system. The pink zone in the diagram represents the stability region for E_2 , where zooplankton are absent. Beyond θ_t , zooplankton go extinct, leaving only nutrients and phytoplankton populations in the system.

Figure (10) further illustrates the effects of increased toxin liberation on population dynamics: as toxin levels rise, phytoplankton populations increase markedly, while nutrient and predator (zooplankton) populations drop sharply, ultimately disappearing from the system.

7.3 The system with delay, $\tau \neq 0$

Case II We now explore the system with delay.

The time gap τ represents a time delay in the system which can correspond to various biological or ecological processes. In many toxic phytoplankton species (e.g., *Alexandrium*, *Karenia*), there's a lag between population growth and the onset of toxin production. This lag, denoted by τ , can be due to various reasons like maturation period required for cells to develop toxin-production, a threshold in population density or environmental triggers (e.g., nutrient levels, light) that must be reached before toxins are produced, physiological delay in the biochemical pathways leading to toxin synthesis.

Theoretical analysis reveals that the Nutrient Plankton system can transition through various states by varying the delay parameter τ , leading to the occurrence of Hopf bifurcation. This theoretical insight is further substantiated through simulations in this section, utilizing parameter values listed in table (1). An incremental increase in the critical parameter τ induces a notable shift in the system's stability. The initial values of populations are taken as $[0.0620.0830.021]$. The system exhibits asymptotic stability for τ values within the interval $[0, 4.1)$ (see figure 11). However, at $\tau = 4.1$, the occurrence of periodic orbits is observed, marking the loss of stability through a Hopf bifurcation. Time evolution and phase portrait of stable limit cycles for $\tau = 4.5$ is shown in figure 11(c). A further surge in τ values, with other parameters held constant, results in the emergence of chaos within the system 11(d).

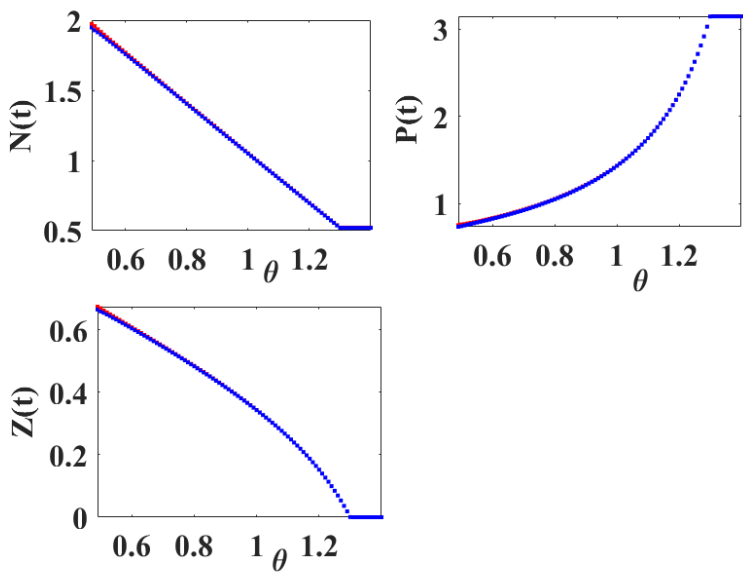


Figure 10. Illustration of population density variations in the system with changes in toxin release.

7.4 Impact of harvesting of zooplankton on planktonic system

In this subsection, we shall observe the impact of harvesting parameter E on planktonic system, keeping other parametric values same as listed in table1.The system moves from chaos and instability towards stability.The simulation results for E , leading the system towards stability are shown by bifurcation diagrams for all three populations in figure (12). For $E \in (0, 35)$ system is oscillatory and chaotic, for $E > 35$, stability is exhibited.

Dynamics of system has been investigated in the presence and absence of toxins. The densities of all three populations for different harvesting effort for $\theta = 0$ (i.e.absence of toxins) and $\theta = 0.3092$ is given in the following tables.

E	$N(t)$	$P(t)$	$Z(t)$
0.6	2.642	0.1912	0.7310
2	2.603	0.1996	0.7192
5	2.520	0.2174	0.6994
10	2.390	0.2497	0.6632
20	2.161	0.3134	0.5894

Table 2. Effect of harvesting on the ecosystem for $\theta = 0, \tau = 3.5$

E	$N(t)$	$P(t)$	$Z(t)$
0.6	2.531	0.2159	0.6489
2	2.458	0.2261	0.6396
5	2.368	0.2492	0.6199
10	2.227	0.2894	0.5878
20	1.967	0.3767	0.5262

Table 3. Effect of harvesting on the ecosystem for $\theta = 0.3092, \tau = 3.5$

A comparison of table (2) and table (3) reveals that as harvesting effort increases, zooplankton population density is higher when $\theta = 0$ than when $\theta = 0.3092$, indicating that toxin-producing phytoplankton have a detrimental effect on zooplankton density. Conversely, phytoplankton density is greater at $\theta = 0.3092$ than at $\theta = 0$, suggesting that the presence of toxins promotes the growth of toxin-producing phytoplankton (TPP). A similar trend is evident in Figure 10, which illustrates variations in the biomass of both phytoplankton and zooplankton for different values of θ . Escalation in value of θ inhibits the coexistence of two species and leads to predator free equilibrium. Biologically, this may be due to a reduction in zooplankton growth or predation rates when toxic substances are present, which leads to an increase in the phytoplankton population. This can be seen as an anti-predator strategy by the phytoplankton.

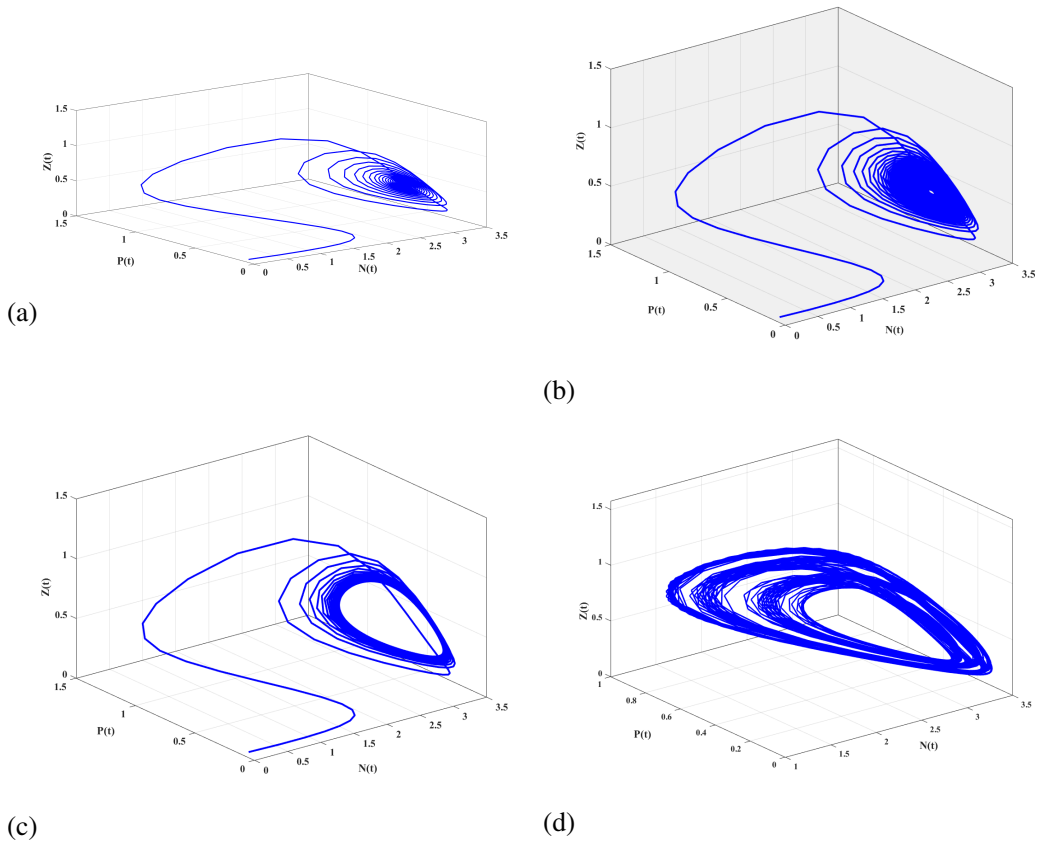


Figure 11. a) Interior Equilibrium point E_3 corresponding to $\tau = 3.5$ is stable for the parametric values listed in table 1. (b) The phase portrait and time series show appearance of limit cycles at $\tau = 4$. (c) Periodic oscillations at $\tau = 4.5 > 4.10 = \tau^*$ (d) Instability in the system owing to multiple periodic oscillations at $\tau = 50$.

8 Conclusion

In the proposed nutrient-plankton model, we have assumed linear harvesting of zooplankton population. Predation of phytoplankton by zooplankton and dissemination of toxins in zooplankton are addressed using Holling type II response function. It is assumed that nutrients are recycled from deceased planktonic biomass by microbial decomposition. The system is analysed both in the presence and absence of time delay. Biologically feasible equilibrium points are identified, and local stability analysis is conducted for each equilibrium point. Global stability conditions around the positive steady state are established using a Lyapunov function. To enhance understanding, the system is further investigated for bifurcations. Transcritical bifurcations are observed with respect to the parameter E at the critical value $E_T = 7.63$ (figure 6). Additionally, figure (9) illustrates transcritical bifurcations occurring with respect to the toxin liberation parameter at $\theta_T = 1.29$, where the stability of the coexisting equilibrium point shifts to a zooplankton-free equilibrium point. A significant shift in the system's stability conditions is observed through Hopf bifurcations with respect to E at the critical value $E_H = 2.99$. Tables 2 and 3 reveal that the effect of harvesting is more pronounced in reducing population densities when toxins are present compared to when they are absent. It is observed through theoretical as well as numerical simulation that system shows varied dynamics with time gap τ as perturbation parameter. An incremental shift in the critical parameter, τ , has a notable impact on the system's stability. A critical value of delay is obtained. Within the range of $[0, 4.1)$, the system remains in a state of asymptotic stability. Figure 11(a) provides a visual representation of equilibrium stability for $\tau = 3.5$, depicting the absence of periodic orbits. However, as τ surpasses the

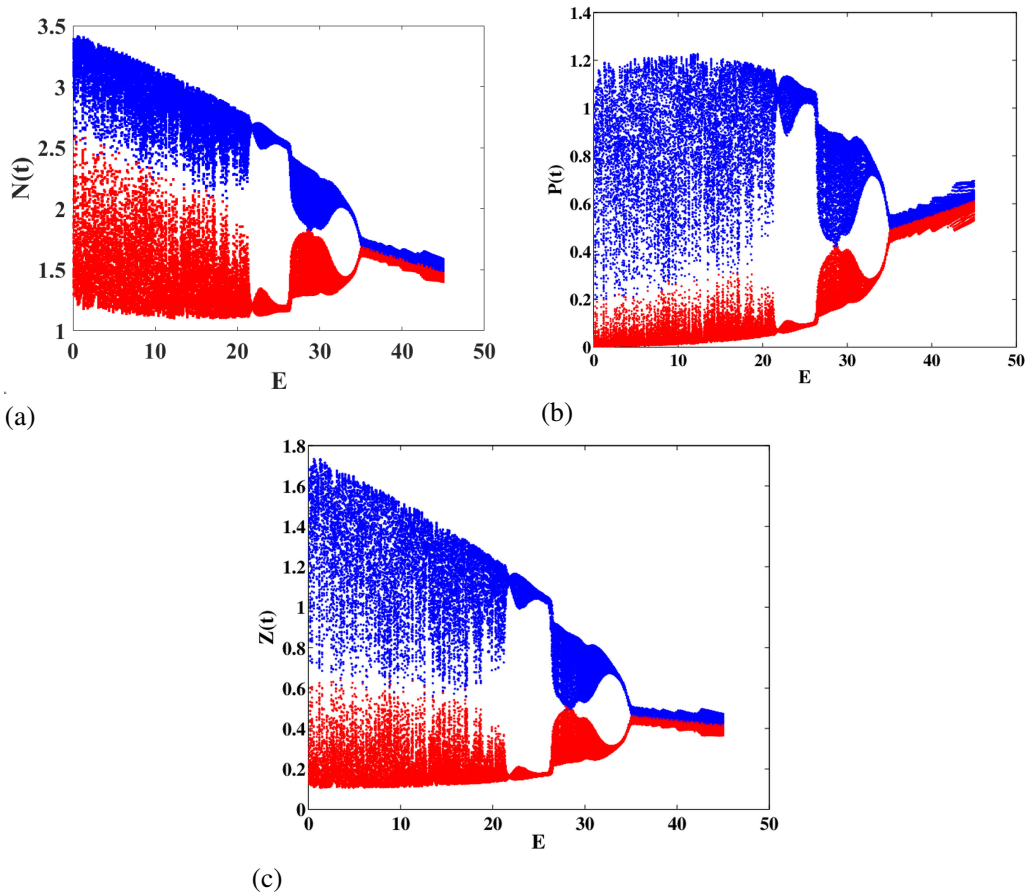


Figure 12. Bifurcation diagrams of three populations relative to parameter E

threshold of 4.1, a significant shift in the system's stability occurs. This transition is characterized by the emergence of periodic orbits in figure 11(b) and the subsequent loss of stability at the equilibrium point due to Hopf bifurcation. Figure 11(c) offers phase diagrams that illustrate the behaviour of the system when $\tau = 4.5$, marking the shift from stability to instability beyond $\tau^* = 4.1$. As τ continues to rise, while keeping other parameters constant, the system's dynamics enter a state of instability. Numerical simulations confirm that the system exhibits chaotic behaviour when τ reaches 50, as shown in figure 11(d). This transition to chaos highlights the intricate and unpredictable nature of the system as τ continues to increase. It is also observed that escalation in value of θ , the rate of toxin secretion, inhibits the coexistence of two species and leads to predator free equilibrium. The system's performance exhibits sensitivity to changes in the harvesting parameter, E , while keeping all other parameters constant. The harvesting parameter E serves as a control or bifurcation parameter that not only determines the dynamic regime of a population system, but also offers an ecological lever for management. By carefully adjusting E , the system can transition from chaotic or unstable behaviors to stable states, meaning that sustainable harvesting can be set where all species persist in equilibrium. In practical terms, this enables to maintain population viability and ecosystem health by operating within appropriate ranges of E , ensuring that species can continue to exist without collapsing or exhibiting undesirable fluctuations.

The discussion highlights the manuscript's relevance concerning with ecosystem resilience, conservation and fisheries management. The findings can be generalized or adapted to other ecosystems that exhibit delayed responses and are subject to resource extraction, making the work of interest to ecologists and environmental managers.

Enhancing the model's realism and interest can be achieved by introducing a delay term in nutrient recycling, as this process is not instantaneously realized in practice. This aspect could

be explored further as an avenue for future research.

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