

MATHEMATICAL MODELLING AND ANALYSIS OF PEST MANAGEMENT IN COCONUT TREES USING BIOCONTROL AGENT AND BOTANICAL INSECTICIDE WITH DISCRETE TIME DELAY

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Abstract In this study, we analyse a system of nonlinear delay differential equations to describe the dynamics of integrated pest management strategies against the Rugose Spiraling Whitefly (RSW) on coconut leaves. Coconut farming is seriously threatened by the RSW, which has an effect on farmer livelihoods and crop productivity. Our model incorporates the interactions between RSW populations (both immature and mature stages), entomopathogenic fungi (*Isaria fumosorosea*) as a biological control agent, and the application of neem oil as a botanical insecticide. A time delay is introduced to capture the delayed impact of insecticide on pest populations. We analyse the stability and existence of equilibrium points in our model by incorporating both the basic reproduction number and time delay factors. Sensitivity analysis is used to explore how variations in key parameters impact the system dynamics. Our findings are supported by numerical simulations carried out with MATLAB dde23 solver. The results from these approaches align well, offering valuable insights into the efficacy of integrated biological control methods. This model provides a detailed framework for enhancing and optimizing pest management strategies in sustainable coconut cultivation.

1 Introduction

The coconut, also known as *Cocos nucifera* (C. nucifera) L., is a significant fruit tree that is found all over the world. It is a source of sustenance for millions of people, particularly in tropical and subtropical countries [1]. One of the greatest challenges to coconut farming is the presence of pests that have a substantial effect on the development and yield of coconut plants. These pests can damage the trees, making it harder for them to absorb water and nutrients that are necessary for long-term growth. The rugose spiraling whitefly, an invasive pest, was first observed on coconut palms in India during august-september 2016. This occurrence was reported in Pollachi taluk, Coimbatore district, Tamil Nadu, by Sundararaj and Selvaraj. Farmers used chemical pesticides to control RSW because they were scared by an unknown bug. The chemicals only worked temporarily, therefore the efforts failed. In addition to being expensive, pesticide use caused environmental damage, the eradication of natural enemies, and health risks for sprayers. Natural insect predators and parasitoids that are safe for the environment, ecologically compatible, and financially successful were investigated for biological pest management. Predators were discovered by observing RSW embryonic feeding in the field [2]. Further, it is possible to control the population of RSWs by the use of several

management measures, such as the application of insecticides, the utilization of biopesticides, and the assistance of natural enemies. Biological control is an effective strategy for managing the Rugose Spirling Whitefly (RSW), utilizing common predators such as *Encarsia guadeloupae*, ladybird beetles, and lacewings, along with entomopathogenic fungi like *Isaria fumosorosea*, which can reduce RSW incidence by 75.1% to 77.8% [3]. According to Mohan et al., there are various approaches to managing RSW effectively. The best tactics to be used are Integrated pest management techniques, awareness campaigns, frequent surveys and monitoring of coconut plants, and the quarantine of foreign products [4]. The nature of population association can be investigated through the use of a mathematical model. Nonlinear differential equations, are widely studied to understand disease dynamics and ecological interactions. Mathematical model using differential equations system for RSWs affecting coconut trees was first proposed and analysed in [5]. Implementing control through natural enemies and awareness campaigns to educate farmers improves their general development as well as that of cultivators has been conducted in [6]. Further, an delay differential equations model has analysed for RSW infestation on coconut tree in [7]. The increasing concern regarding the harmful effects of chemicals has led to the creation of alternatives. Microbial control agents, including viruses, bacteria, fungi, and entomopathogenic nematodes, have been created as alternate approaches for managing a diverse array of pests by inundative and inoculative applications. A mathematical model for the microbial management of pests has been developed and analysed in [8]. Biological control of pests is an environmentally friendly alternative to conventional pest management strategies. A mathematical model that includes the interactions between a plant, a pest, and a virus, and investigates the dynamic behavior of the system for effective pest control [9]. There are limited studies on mathematical models addressing the dynamics of microbial diseases in pest control [10, 11, 12]. A delayed optimal management problem for bio-pesticide and a mathematical model for sustaining plants through promotions for awareness are described in [13]. Sustainable pest management now heavily relies on botanical insecticides made from plants, minerals, or other natural sources. Botanical insecticides are safer for the environment and human health, less harmful to non-target creatures, and typically biodegradable than synthetic chemical pesticides. The seeds of the neem tree are used to make neem oil, one of the most popular natural insecticides. Azadirachtin, one of its active ingredients, prevents numerous insect pests from growing, eating, and reproducing. Its suitability for sustainable pest management is increased by its compatibility with other strategies, such as the application of biological agents [14]. Botanical biopesticides are natural and a better substitute for chemical pesticides and provide an alternative for crop protection worldwide [15]. Numerous studies have explored mathematical modeling approaches to analyze the dynamics of pest control using biocontrol [16, 17, 18, 19]. In this research work, we develop a continuous spray biopesticide model to investigate the impact of continuous biopesticide spraying on both coconut leaves and pest populations. The model uniquely incorporates a delay to capture the lag between the application of neem oil and its impact on pest reproduction and development. Which makes the model to reflect the realistic lag between neem oil application and its effect on RSW populations. This delay is significant because it captures the gradual impact of the pesticide on pest feeding, growth, and reproduction which are often missed in traditional models. By including this feature, the model provides a more accurate and insightful representation of pest control dynamics, improving the understanding of the timing and efficiency of control strategies. While the delay is applied exclusively to neem oil, the model also integrates biological control via entomopathogenic fungi. This dual approach enables an in-depth analysis of how natural and biological agents interact to suppress pest populations. A time delay is incorporated in the model to reflect the real-world lag between neem oil application and its effect on Rugose Spirling Whitefly populations.

The structure of the paper is as follows. In Section 2, the model is developed with the assumption that the spraying rate of neem oil depends on the number of mature pests in the field. To enhance the model's realism, a time delay is introduced to reflect the delayed impact of the control measure on the pest population. By identifying the invariant region, non-negativity and boundedness of the solutions are demonstrated in Section 3. Section 4 uses qualitative theory to study the equilibria, the basic reproduction number, and the stability of the equilibrium. We examine the sensitivity indices of various parameters in Section 5. Numerical simulations are performed to demonstrate our analytical results in Section 6. We conclude with a discussion and conclusion in Section 7.

2 Model Formulation

Here, we formulate a nonlinear delay differential equations model to enhance coconut tree growth by controlling pest population using the entomopathogenic fungi and botanical insecticide. We consider five dynamical variables: number of coconut leaves $L(t)$, pest population is divided into immature pest $I(t)$ and mature pest $M(t)$, fungi concentration $F(t)$, and concentration of botanical insecticide (Neem oil) $C(t)$.

We postulate that the coconut leaves adheres to the logistic growth, characterized by an intrinsic growth rate denoted as r and a carrying capacity represented by k , since leaf growth is self-limiting in nature. Both Immature and mature pests attack leaves, leading to a reduction in the growth of leaves at a rate α_1 and α_2 respectively. Let s denote the conversion efficiency. Let fungi be sprayed at a rate ϕ and the infection rate by fungi is denoted by the term $P_1 IF$. We assume that the fungi attacks the immature pest population only and the infected population does not recover or become immune. The fungi treatment is governed by the so-called mass-action incidence. Let θ be the conversion efficiency of the fungi. The immature pest becomes mature at a rate γ . Let d_1 and d_2 are natural mortality rate of mature pests and fungi respectively.

We consider that farmers spray the insecticide proportional to mature pest population at a rate λ . The insecticides naturally deplete at the rate ρ . Due to this consumption of insecticides, pest population declines at the rate δ . The mortality effect is delayed by τ , reflecting the lag between insecticide application and its lethal impact. Based on the above considerations, a schematic diagram is demonstrated in figure 1, and the following set of nonlinear delay differential equations regulates the dynamics of the proposed model:

$$\begin{aligned}
 \frac{dL}{dt} &= rL \left(1 - \frac{L}{k} \right) - \alpha_1 IL - \alpha_2 ML, \\
 \frac{dI}{dt} &= s\alpha_1 IL - P_1 IF - \gamma I, \\
 \frac{dM}{dt} &= \gamma I - d_1 M - \delta C(t - \tau)M, \\
 \frac{dF}{dt} &= \phi + \theta P_1 IF - d_2 F, \\
 \frac{dC}{dt} &= \lambda M - \rho C,
 \end{aligned} \tag{2.1}$$

with the initial conditions as

$$L(\psi) > 0, I(\psi) > 0, M(\psi) > 0, F(\psi) > 0, C(\psi) > 0, \tag{2.2}$$

where $\psi \in [-\tau, 0]$

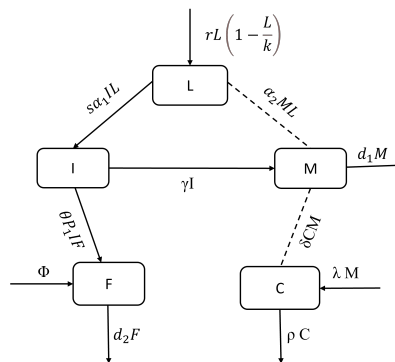


Figure 1: Schematic diagram for model (2.1).

3 Model Analysis

From a biological perspective, positive indicates that the populations are surviving. The methodology put forward by Bodnar [20] and Yang et al. [21] has been used to illustrate this.

Theorem 3.1. *For positive initial data, solutions of system (2.1) are positive for all $t \geq 0$.*

Proof. From (2.1)

$$\frac{dL}{dt} = \left[r \left(1 - \frac{L}{k} \right) - \alpha_1 I - \alpha_2 M \right] dt, \quad (3.1)$$

when on integrating and based on the initial value, we have

$$L(t) = L(0)e^{\int_0^t \left[r \left(1 - \frac{L(\xi)}{k} \right) - \alpha_1 I(\xi) - \alpha_2 M(\xi) \right] d\xi}. \quad (3.2)$$

This implies $L(t) > 0$ for all $t > 0$ since $L(0) > 0$. Hence $L(t)$ is positive for all $t > 0$. The non-negativity of $I(t)$, $M(t)$, $C(t)$ can be proven with the same argument.

For non-negativity of $F(t)$, consider

$$\frac{dF}{dt} = \phi + \theta P_1 I(t) F(t) - d_2 F(t). \quad (3.3)$$

we claim that $F(t) > 0$ for all $t > 0$. Suppose not, there exists $t_1 > 0$ and $v_1 > 0$ such that $F(t) > 0$ for $t < t_1$, $F(t) = 0$ for $t = t_1$ and $F(t) < 0$ for $t \in [t_1, t_1 + v_1)$. Thus

$$\begin{aligned} \frac{dF}{dt} &= \phi + \theta P_1 I(t_1) F(t_1) - d_2 F(t_1), \\ &= \phi > 0, \end{aligned} \quad (3.4)$$

this is a contradiction. Hence $F(t) > 0$ for all $t > 0$. $\square$

Theorem 3.2. *All the solutions of (2.1), that are initiated from $\mathbb{R}_+^5$, will be confined in the region*

$$\Lambda = \left\{ (L, I, M, F, C) \in \mathbb{R}_+^5 : 0 \leq L \leq \Delta, 0 \leq I, M, F \leq \frac{\phi}{\theta}, 0 \leq C \leq \frac{\lambda \phi}{\theta} \right\} \quad (3.5)$$

with $\Delta = \max\{L(0), k\}$.

Proof. From system (2.1) we have,

$$\frac{dL}{dt} = rL \left(1 - \frac{L}{k} \right) - \alpha_1 IL - \alpha_2 ML \leq rL \left(1 - \frac{L}{k} \right), \quad (3.6)$$

this implies, $\limsup_{t \rightarrow \infty} L \leq \Delta := \max\{L(0), k\}$.

Let $W = I + M + \frac{1}{\theta} F$

$$\frac{dW(t)}{dt} = s\alpha_1 IL - P_1 IF - \gamma I + \gamma I - \delta C(t - \tau)M + \frac{\phi}{\theta} + P_1 IF - \frac{d_2}{\theta} F, \quad (3.7)$$

Choose a constant $\zeta > 0$ such that

$$\frac{dW(t)}{dt} + \zeta W \leq \frac{\phi}{\theta} + I(s\alpha_1 k - \zeta) - (\delta C - \zeta)M - (d_2 - \zeta) \frac{F}{\theta}, \quad (3.8)$$

Define $\zeta = \min\{s\alpha_1 k, \delta C, d_2\}$ such that

$$\frac{dW(t)}{dt} \leq \frac{\phi}{\theta}, \quad (3.9)$$

$$0 \leq W(t) \leq \frac{\phi}{\zeta \theta} + \left(W(0) - \frac{\phi}{\zeta \theta} \right) e^{-\zeta t}. \quad (3.10)$$

As $t \rightarrow \infty$, $W(t) \rightarrow \frac{\phi}{\xi\theta}$.

$$\frac{dC}{dt} = \lambda M - \rho C \quad (3.11)$$

$$\frac{dC}{dt} + \rho C \leq \frac{\lambda\phi}{\theta} \quad (3.12)$$

Then $\sup_{t \rightarrow \infty} C = \frac{\lambda\phi}{\rho\theta}$. $\square$

4 Equilibria and stability

4.1 Existence of equilibria

We consider the following equilibrium points for the model (2.1):

- (i) The trivial equilibrium $E_1(0, 0, 0, 0, 0)$
- (ii) The pest-free equilibrium $E_2(k, 0, 0, \frac{\phi}{d_2}, 0)$
- (iii) The co-existence equilibrium $E_3(L^*, I^*, M^*, F^*, C^*)$

where,

$$C^* = \frac{\lambda M^*}{\rho}, \quad I^* = \frac{d_1 \rho M^* + \delta \lambda M^{*2}}{\gamma \rho}, \quad F^* = \frac{\phi \gamma \rho}{d_2 \gamma \rho - \theta P_1 (d_1 \rho M^* - \delta \lambda M^{*2})},$$

$$L^* = \frac{\gamma^2 \rho d_2 - \gamma \theta P_1 (d_1 \rho M^* + \delta \lambda M^{*2}) + P_1 \theta \gamma \rho}{s \alpha_1 (\gamma \rho d_2 - \theta P_1 (\rho d_1 M^* + \delta \lambda M^{*2}))}$$

M^* is the positive root of the equation,

$$\begin{aligned} & \delta^2 k \lambda^2 P_1 s \theta \alpha_1^2 M^{*4} + (2d_1 \delta k \lambda P_1 \rho s \theta \alpha_1^2 + \delta \gamma k \lambda P_1 \rho s \theta \alpha_1 \alpha_2) M^{*3} + (d_1^2 k P_1 \\ & \rho^2 s \theta \alpha_1^2 + d_1 \gamma k P_1 \rho^2 s \theta \alpha_1 \alpha_2 - \delta \gamma k \lambda P_1 \rho s \theta \alpha_1 - d_2 \delta \gamma k \lambda \rho s \alpha_1^2 + \delta \gamma^2 \lambda P_1 r \\ & \rho \theta) M^{*2} + (-d_1 \gamma k P_1 \rho^2 s \theta \alpha_1 - d_1 d_2 \gamma k \rho^2 s \alpha_1^2 - d_2 \gamma^2 k \rho^2 s \alpha_1 \alpha_2 + d_1 \gamma^2 P_1 \rho^2 \\ & \theta) M^* - d_2 \gamma^2 k \rho^2 s \alpha_1 + \phi \gamma^2 P_1 \rho^2 + d_2 \gamma^3 \rho^2. \end{aligned} \quad (4.1)$$

On checking the sign of the coefficients of (4.1), leading coefficient is positive. We can conclude that there exists atleast one positive root when $\phi \gamma^2 P_1 \rho^2 + d_2 \gamma^3 \rho^2 - d_2 \gamma^2 k \rho^2 s \alpha_1 < 0$.

4.2 Stability analysis of the system without delay

We first study the system without delay. The Jacobian of (2.1) with no delay, $\tau = 0$, at any equilibrium point E is given by

$$J(E) = \begin{bmatrix} r - \frac{2rL}{k} - \alpha_1 I - \alpha_2 M & -\alpha_1 L & -\alpha_2 L & 0 & 0 \\ s \alpha_1 I & s \alpha_1 L - P_1 F - \gamma & 0 & -P_1 I & 0 \\ 0 & \gamma & -d_1 - \delta C & 0 & -\delta M \\ 0 & \theta P_1 F & 0 & \theta P_1 I - d_2 & 0 \\ 0 & 0 & \lambda & 0 & -\rho \end{bmatrix}. \quad (4.2)$$

The jacobian matrix of system (2.1) evaluated at the trivial equilibrium (0,0,0,0,0) is

$$J|_{(0,0,0,0,0)} = \begin{bmatrix} r & 0 & 0 & 0 & 0 \\ 0 & -\gamma & 0 & 0 & 0 \\ 0 & \gamma & -d_1 & 0 & 0 \\ 0 & 0 & 0 & -d_2 & 0 \\ 0 & 0 & \lambda & 0 & -\rho \end{bmatrix}. \quad (4.3)$$

The eigenvalues in this case are $r, -\gamma, -d_1, -d_2, -\rho$. The origin $E_1(0, 0, 0, 0, 0)$ is unstable, in view of the fact that one of its eigenvalues ($r > 0$) is positive.

The system's basic reproduction number of the is provided by $R_0 = \frac{\alpha_1 d_2 k s}{d_2 \gamma + P_1 \phi}$.

The next-generation matrix method is used to obtain the expression for reproduction number. Let $X = (I, M)^T$.

We write

$$X' = F(X) - V(X), \quad (4.4)$$

Where

$$F(X) = \begin{bmatrix} s\alpha_1 IL \\ 0 \end{bmatrix} \quad \text{and} \quad V(X) = \begin{bmatrix} -P_1 IF - \gamma I \\ \gamma I - d_1 M - \delta C(t - \tau)M \end{bmatrix}. \quad (4.5)$$

The jacobian matrix of $F(X)$ and $V(X)$ denoted by $DF(E_2)$ and $DV(E_2)$, at the pest-free equilibrium point E_2 respectively, are given as follows

$$DF(E_2) = \begin{bmatrix} s\alpha_1 k & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad DV(E_2) = \begin{bmatrix} -\gamma - \frac{P_1 \phi}{d_2} & 0 \\ 0 & -d_1 \end{bmatrix}. \quad (4.6)$$

The spectral radius of $DF(E_2)(DV(E_2))^{-1}$ gives the reproduction number. Thus,

$$R_0 = \frac{\alpha_1 d_2 k s}{d_2 \gamma + P_1 \phi}. \quad (4.7)$$

Theorem 4.1. *The system (2.1) around the pest-free equilibrium $E_2(k, 0, 0, \frac{\phi}{d_2}, 0)$ is locally asymptotically stable(LAS), provided $R_0 < 1$.*

Proof. We linearize the system around $E_2(k, 0, 0, \frac{\phi}{d_2}, 0)$. Then, the jacobian matrix at this equilibrium is:

$$\begin{bmatrix} -r & -\alpha_1 k & -\alpha_2 k & 0 & 0 \\ 0 & s\alpha_1 k - \frac{P_1 \phi}{d_2} - \gamma & 0 & 0 & 0 \\ 0 & \gamma & -d_1 & 0 & 0 \\ 0 & \frac{\theta P_1 \phi}{d_2} & 0 & -d_2 & 0 \\ 0 & 0 & \lambda & 0 & -\rho \end{bmatrix}. \quad (4.8)$$

At the pest-free equilibrium E_2 , the characteristic equation is provided by

$$(\lambda + r)(\lambda - s\alpha_1 k + \frac{P_1 \phi}{d_2} + \gamma)(\lambda + d_1)(\lambda + d_2)(\lambda + \rho) = 0. \quad (4.9)$$

Thus, the eigenvalues are $-r, s\alpha_1 k - \frac{P_1 \phi}{d_2} - \gamma, -d_1, -d_2, -\rho$.

Therefore, E_2 is locally stable if $s\alpha_1 k - \frac{P_1 \phi}{d_2} - \gamma < 0$, from which $sd_2 \alpha_1 k < P_1 \phi + d_2 \gamma$, that is, $\frac{\alpha_1 d_2 k s}{d_2 \gamma + P_1 \phi} < 1$.

As expected, we determine that $R_0 < 1$. □

Theorem 4.2. *The system (2.1) at the endemic equilibrium $E_3(L^*, I^*, M^*, F^*, C^*)$ is locally asymptotically stable if $q_i > 0$ ($i = 1, 2, \dots, 5$), $q_1 q_2 q_3 > q_3^2 + q_1^2 q_4$ and $(q_1 q_4 - q_5)(q_1 q_2 q_3 - q_3^2 - q_1^2 q_4) > q_5(q_1 q_2 - q_3)^2 + q_1 q_5^2$.*

Proof. The model's Jacobian matrix at E_3 is provided by

$$J(E_3) = \begin{bmatrix} r - \frac{2rL^*}{k} - \alpha_1 I^* - \alpha_2 M^* & -\alpha_1 L^* & -\alpha_2 L^* & 0 & 0 \\ s\alpha_1 I^* & s\alpha_1 L^* - P_1 F^* - \gamma & 0 & -P_1 I^* & 0 \\ 0 & \gamma & -d_1 - \delta C^* & 0 & -\delta M^* \\ 0 & \theta P_1 F^* & 0 & \theta P_1 I^* - d_2 & 0 \\ 0 & 0 & \lambda & 0 & -\rho \end{bmatrix}. \quad (4.10)$$

The characteristic equation is obtained as $\lambda^5 + q_1\lambda^4 + q_2\lambda^3 + q_3\lambda^2 + q_4\lambda + q_5 = 0$. There will be negative real components in the characteristic equation's roots, when $q_i > 0$ ($i = 1, 2, \dots, 5$), $q_1q_2q_3 > q_3^2 + q_1^2q_4$ and $(q_1q_4 - q_5)(q_1q_2q_3 - q_3^2 - q_1^2q_4) > q_5(q_1q_2 - q_3)^2 + q_1q_5^2$. Hence the system (2.1) around E_3 is LAS. $\square$

4.3 Stability analysis of the system with delay

We examine the model system (2.1) with delay in this section. The stability requirements for the equilibrium E_3 are also derived.

The following is obtained by linearizing the system around the equilibrium point:

$$\frac{dX}{dt} = FX(t) + GX(t - \tau), \quad (4.11)$$

with F,G the 5×5 matrices, given as below

$$F = [f_{ij}] = \begin{bmatrix} r - \frac{2rL^*}{k} - \alpha_1 I^* - \alpha_2 M^* & -\alpha_1 L^* & -\alpha_2 L^* & 0 & 0 \\ s\alpha_1 I^* & s\alpha_1 L^* - P_1 F^* - \gamma & 0 & -P_1 I^* & 0 \\ 0 & \gamma & -d_1 - \delta C^* & 0 & -\delta M^* \\ 0 & \theta P_1 F^* & 0 & \theta P_1 I^* - d_2 & 0 \\ 0 & 0 & \lambda & 0 & -\rho \end{bmatrix}, \quad (4.12)$$

$$G = [g_{ij}] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\delta M^* \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \quad (4.13)$$

The characteristic equation of the delay system (2.1) is given by

$$H(\lambda) = |\epsilon I - F - e^{-\epsilon\tau} G| = 0, \quad (4.14)$$

This gives,

$$\epsilon^5 + a_1\epsilon^4 + a_2\epsilon^3 + a_3\epsilon^2 + a_4\epsilon + a_5 + e^{-\epsilon\tau}(a_6\epsilon^3 + a_7\epsilon^2 + a_8\epsilon + a_9) = 0, \quad (4.15)$$

where,

$$\begin{aligned}
 a_1 &= -(f_{11} + s\alpha_1 L^* - P_1 F^* - \gamma - d_1 - \delta C^* + \theta P_1 I^* - d_2 - \rho), \\
 a_2 &= f_{11}(-d_1 - \delta C^* - \theta P_1 I^* + d_2 + \rho - s\alpha_1 L^* + P_1 F^* + \gamma) - s\alpha_1^2 I^* L^* \\
 &\quad - \theta P_1^2 I^* F^* - (s\alpha_1 L^* - P_1 F^* - \gamma)(-d_1 - \delta C^* + \theta P_1 I^* - d_2 - \rho) \\
 &\quad - \rho(\theta P_1 I^* - d_2) - (d_1 + \delta C^*)(\theta P_1 I^* - d_2 - \rho), \\
 a_3 &= f_{11}(s\alpha_1 L^* - P_1 F^* - \gamma)(-d_1 - \delta C^* + \theta P_1 I^* - d_2 - \rho) \\
 &\quad - f_{11}(d_1 + \delta C^*)(\theta P_1 I^* - d_2 - \rho) - s\alpha_1^2 I^* L^*(s\alpha_1 L^* - P_1 F^* - \gamma) \\
 &\quad (-d_1 - \delta C^* + \theta P_1 I^* - d_2 - \rho) - s\gamma\alpha_1\alpha_2 L^* I^* - (s\alpha_1 L^* - P_1 F^* - \gamma) \\
 &\quad (d_1 + \delta C^*)(\theta P_1 I^* - d_2 - \rho) - \theta P_1^2 I^* F^*(f_{11} - d_1 - \delta C^* - \rho) \\
 &\quad - \rho(\theta P_1 I^* - d_2)(f_{11} + s\alpha_1 L^* - P_1 F^* - \gamma - d_1 - \delta C^*), \\
 a_4 &= f_{11}[(s\alpha_1 L^* - P_1 F^* - \gamma)(-d_1 - \delta C^*)(\theta P_1 I^* - d_2 - \rho) - \rho(\theta P_1 I^* - d_2) \\
 &\quad (s\alpha_1 L^* - P_1 F^* - \gamma - d_1 - \delta C^*) + \theta P_1^2 I^* F^*(-d_1 - \delta C^* - \rho)] \\
 &\quad + (\theta P_1 I^* - d_2 - \rho)[s\alpha_1 I^*(\gamma(-\alpha_2 L^*) + \alpha_1 L^*(d_1 + \delta C^*))] \\
 &\quad + \rho(d_1 + \delta C^*)[(s\alpha_1 L^* + P_1 F^* + \gamma)(\theta P_1 I^* - d_2) - \theta P_1^2 I^* F^*] \\
 a_5 &= s\alpha_1 \rho I^*(d_2 - \theta P_1 I^*)[\alpha_1 L^*(d_1 + \delta C^*) - \gamma\alpha_2 L^*] + f_{11}\rho(d_1 + \delta C^*) \\
 &\quad [(s\alpha_1 L^* - P_1 F^* - \gamma)(-d_1 - \delta C^*) + \theta P_1^2 I^* F^*], \\
 a_6 &= -\lambda \delta M^*, \\
 a_7 &= -\lambda \delta M^*(f_{11} + s\alpha_1 L^* - P_1 F^* - \gamma + \theta P_1 I^* - d_2), \\
 a_8 &= -f_{11}\lambda \delta M^*(s\alpha_1 L^* - P_1 F^* - \gamma + \theta P_1 I^* - d_2) + \lambda \delta M^*(s\alpha_1^2 I^* L^* \\
 &\quad + \theta P_1^2 I^* F^* + (s\alpha_1 L^* - P_1 F^* - \gamma)(\theta P_1 I^* - d_2)), \\
 a_9 &= -f_{11}\lambda \delta M^*(\theta P_1^2 I^* F^* + (s\alpha_1 L^* - P_1 F^* - \gamma)(\theta P_1 I^* - d_2)) \\
 &\quad + \lambda \delta M^* s\alpha_1^2 I^* L^*(\theta P_1 I^* - d_2).
 \end{aligned}$$

Evaluating the sign of roots in the presence of τ is extremely complex. E_3 is LAS if all roots of the associated characteristic equation possess negative real parts. It is considered unstable if a root with a positive real component exists. Stability switches occur when entirely imaginary roots are present. The transcendental Eq. (4.15) possesses an infinite number of complex roots. Consequently, we initiate our analysis by establishing the delay at zero and thereafter deriving requirements for stability when $\tau > 0$.

Case I: When $\tau = 0$ If there is no delay, the characteristic Eq. (4.15) can be written up as follows:

$$\varepsilon^5 + a_1 \varepsilon^4 + (a_2 + a_6) \varepsilon^3 + (a_3 + a_7) \varepsilon^2 + (a_4 + a_8) \varepsilon + (a_5 + a_9) = 0, \quad (4.16)$$

and the stability is the same as obtained for system without delay.

Case II: When $\tau > 0$ Eq. (4.15) is a transcendental equation in ε possessing infinitely many roots. The coexistence equilibrium E_3 is locally stable if the roots of the related characteristic equation possess negative real parts, and unstable if they possess positive real parts. Let $\varepsilon = iw(\tau)$ be a root of Equation (4.15). Subsequently,

$$iw^5 + a_1 w^4 - a_2 iw^3 - a_3 w^2 + a_4 w + a_5 + (-a_6 w^3 - a_7 w^2 + a_8 iw + a_9) e^{\tau(iw)} = 0. \quad (4.17)$$

The resulting equations are obtained by sorting the real and imaginary parts:

$$w^5 - a_2 w^3 + a_4 w = (a_6 w^3 - a_8 w) \cos w\tau - (a_7 w^2 - a_9) \sin w\tau. \quad (4.18)$$

$$a_1 w^4 - a_3 w^2 + a_5 = (a_7 w^2 - a_9) \cos w\tau + (a_6 w^3 - a_8 w) \sin w\tau. \quad (4.19)$$

When the real and imaginary parts are squared and added, we obtain

$$w^{10} + (a_1^2 - 2a_2)w^8 + (a_2^2 - 2a_1a_3 + 2a_4 - a_6^2)w^6 + (a_3^2 + 2a_1a_5 - 2a_2a_4 + 2a_6a_8 - a_7^2)w^4 + (a_4^2 - 2a_3a_5 + 2a_7a_9 + a_8^2)w^2 + (a_5^2 - a_9^2) = 0. \quad (4.20)$$

Let $\theta = w^2$. Then, Eq. (4.20) becomes

$$H(\theta) = \theta^5 + S_1\theta^4 + S_2\theta^3 + \theta^2S_3 + \theta S_4 + S_5 = 0. \quad (4.21)$$

where,

$$\begin{aligned} S_1 &= a_1^2 - 2a_2, \\ S_2 &= a_2^2 - 2a_1a_3 + 2a_4 - a_6^2, \\ S_3 &= a_3^2 + 2a_1a_5 - 2a_2a_4 + 2a_6a_8 - a_7^2, \\ S_4 &= a_4^2 - 2a_3a_5 + 2a_7a_9 + a_8^2, \\ S_5 &= a_5^2 - a_9^2. \end{aligned}$$

Eq. (4.21) will not have a positive real root if the coefficients in $H(\theta)$ meet the requirements of the Routh-Hurwitz criterion. As a result, we might not obtain a positive value for ω that satisfies the transcendental equations (4.18) and (4.19). In this instance, the outcome could be expressed as the following theorem.

Theorem 4.3. *The endemic equilibrium E_3 of model system (2.1) is asymptotically stable for all delays $\tau > 0$ if the coefficients in $H(\theta)$ meet the requirements of the Routh-Hurwitz criterion, as long as it remains stable without delay.*

In contrast, assuming that the Routh-Hurwitz criterion is not met by the values of S_i 's, ($i = 1, 2, 3, 4, 5$), in (4.21). In this instance, $S_5 < 0$ is a straightforward condition for the possibility of a positive root of equation (4.21), which provides

$$a_5^2 - a_9^2 < 0. \quad (4.22)$$

Eq. (4.21) has a positive root m_0 if condition (4.23) is true, and Eq. (4.19) has two entirely imaginary roots $\pm im_0$. Transcendental equations (4.18) and (4.19) allow us to derive

$$\tan w\tau = \frac{D_1}{D_2}, \quad (4.23)$$

$$\begin{aligned} D_1 &= (a_1w_0^4 - a_3w_0^2 + a_5)(a_6w_0^3 - a_8w_0) + (w_0^5 - a_2w_0^3 + a_4w_0)(a_9 - a_7w_0^2), \\ D_2 &= (a_1w_0^4 - a_3w_0^2 + a_5)(a_7w_0^2 - a_9) - (w_0^5 - a_2w_0^3 + a_4w_0)(a_8w_0 - a_6w_0^3). \end{aligned}$$

Now τ_k corresponding to this positive value of w_0 is given as follows:

$$\tau_k = \frac{k\pi}{w_0} + \left(\frac{1}{w_0} \right) \tan^{-1} \left(\frac{D_1}{D_2} \right), \quad (4.24)$$

where $k = 0, 1, 2, 3, \dots$

The endemic equilibrium of model system (2) is stable for $\tau < \tau_0$.

5 Sensitivity Analysis

The effect of different parameter values on the functional value of the reproduction number R_0 is shown in this section. Finding the essential parameter is necessary since it could be a crucial criterion for managing diseases. The dynamics of the disease are significantly influenced by each of the model's parameters. Sensitivity analysis is taken into consideration for every reproduction number parameter. The following are the algebraic representations of R_0 for the sensitivity index

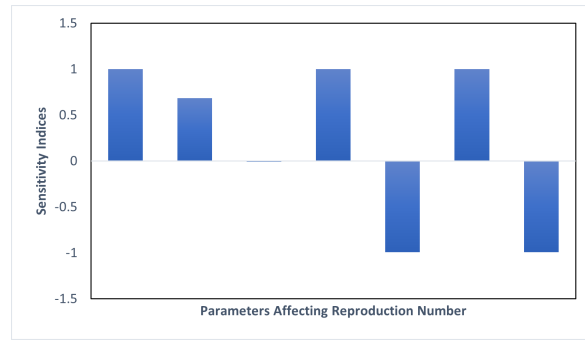


Figure 2: Sensitivity analysis: plot of sensitivity indices of different parameters affecting reproduction number R_0 .

to the parameters $\alpha_1, d_2, k, s, \gamma, P_1, \phi$:

$$\begin{aligned} \frac{\partial R_0}{\partial \alpha_1} &= \frac{d_2 k s}{d_2 \gamma + P_1 \phi}, & \frac{\partial R_0}{\partial d_2} &= \frac{\alpha_1 k s P_1 \phi (1 - d_2)}{(d_2 \gamma + P_1 \phi)^2}, & \frac{\partial R_0}{\partial \gamma} &= \frac{-\alpha_1 (d_2)^2 k s}{(d_2 \gamma + P_1 \phi)^2}, \\ \frac{\partial R_0}{\partial k} &= \frac{\alpha_1 d_2 s}{d_2 \gamma + P_1 \phi}, & \frac{\partial R_0}{\partial s} &= \frac{\alpha_1 d_2 k}{d_2 \gamma + P_1 \phi}, & \frac{\partial R_0}{\partial P_1} &= \frac{-\alpha_1 d_2 k s \phi}{(d_2 \gamma + P_1 \phi)^2}, \\ \frac{\partial R_0}{\partial \phi} &= \frac{-\alpha_1 d_2 k s P_1}{(d_2 \gamma + P_1 \phi)^2}. \end{aligned}$$

The elasticity is estimated with the proportional response to the proportional perturbation.

$$\begin{aligned} E_{\alpha_1} &= \frac{\alpha_1}{R_0} \frac{\partial R_0}{\partial \alpha_1} = 1, & E_{d_2} &= \frac{d_2}{R_0} \frac{\partial R_0}{\partial d_2} = 0.6857142857, & E_k &= \frac{\sigma}{R_0} \frac{\partial R_0}{\partial k} = 1, \\ E_\gamma &= \frac{\gamma}{R_0} \frac{\partial R_0}{\partial \gamma} = \frac{-\alpha_1 (d_2)^2 k s}{(d_2 \gamma + P_1 \phi)^2} < 0, & E_{P_1} &= \frac{P_1}{R_0} \frac{\partial R_0}{\partial P_1} = \frac{-\alpha_1 d_2 k s \phi}{(d_2 \gamma + P_1 \phi)^2} < 0, \\ E_s &= \frac{\alpha_1}{R_0} \frac{\partial R_0}{\partial s} = 1, & E_\phi &= \frac{\phi}{R_0} \frac{\partial R_0}{\partial \phi} = \frac{-\alpha_1 d_2 k s P_1}{(d_2 \gamma + P_1 \phi)^2} < 0. \end{aligned}$$

It is observed from figure 2, that $E_{\alpha_1}, E_{d_2}, E_k$ and E_s are positive. This suggest that an increase in the parameters α_1, d_2, k and s leads to an increase in the value of the basic reproduction number R_0 . The basic reproduction number can be significantly influenced by even the slightest alteration in these parameters. Careful calculation is necessary for a highly sensitive parameter, as even a minor deviation can result in significant quantitative modifications to the system.

6 Numerical Simulation

We have used the MATLAB dde23 solver to compute the numerical simulation. The system's initial values are assumed to be $L(0) = 7, I(0) = 50, M(0) = 10, F(0) = 10, C(0) = 0$. Experimental data and values from the literature [3, 6] are used to derive the values that correspond to the parameters, and these are summarized in Table 1. The carrying capacity of the leaf per tree is considered approximately as 60-70. The fungal effect on the pest is calculated based on the data given in [22]. We assume that the effective time delay for neem oil to start controlling rugose spiraling whitefly on coconut leaves is typically around 1 to 3 days. For analysis, we assume that the effectiveness of neem oil in controlling the rugose spiraling whitefly population is delayed by approximately 3 days ($\tau = 3$), due to factors such as ingestion time and physiological response.

Table 1: Parameter values used for analysis

Symbol	Meaning	Unit	Value taken for analysis
r	Intrinsic growth rate of coconut leaves	day^{-1}	0.027 - 0.057
k	Carrying capacity of leaf		60 - 70
α_1	Predation rate by immature pests	$pest^{-1} day^{-1}$	0.00001 - 0.0005
α_2	Predation rate by mature pests	$pest^{-1} day^{-1}$	0.0002 - 0.0005
s	Conversion efficiency of whitefly	day^{-1}	0.01 - 0.04
P_1	Infection rate by fungi	day^{-1}	0.001 - 0.003
γ	Conversion to mature pests	day^{-1}	0.02 - 0.03
d_1	Mortality rate of mature whitefly	day^{-1}	0.005 - 0.02
d_2	Mortality rate of fungi	day^{-1}	0.04 - 0.06
δ	Depletion rate of pest due to insecticide	day^{-1}	0.001
θ	Conversion efficiency of fungi	day^{-1}	0.33 - 0.6
ϕ	Amount of fungi sprayed in the field		2
λ	Rate of spraying insecticide	$pest^{-1}$	0.03
ρ	Natural depletion rate of insecticide	day^{-1}	0.173

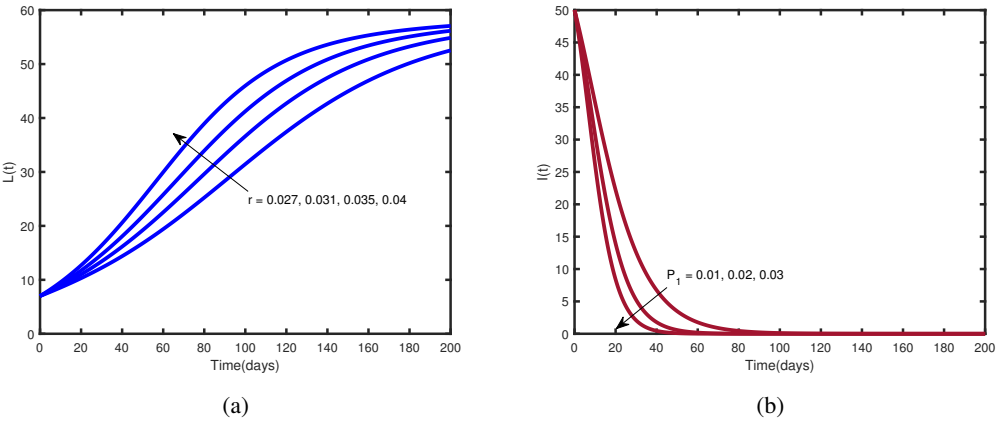


Figure 3: (a) Profile of coconut leaves $L(t)$ versus time (days) for system (2.1) for different values of growth rate of leaves r . (b) Profile of immature pests $I(t)$ versus time (days) for system (2.1) when compared to the various values of the infection rate P_1 and the other parameter values that are presented in Table 1.

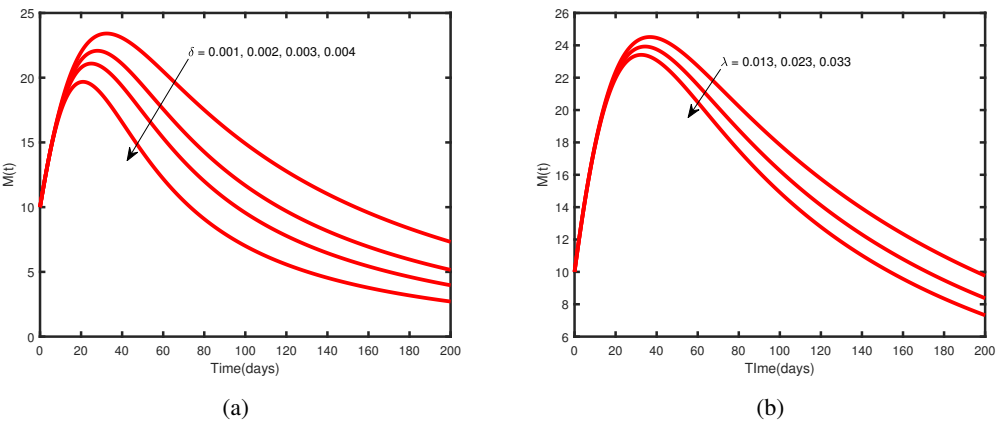


Figure 4: Profile of mature pests $M(t)$ versus time for system (2.1), (a) Variation due to effect of insecticide δ . (b) Variation due to rate of spraying insecticide λ with other parameter values as given in Table 1.

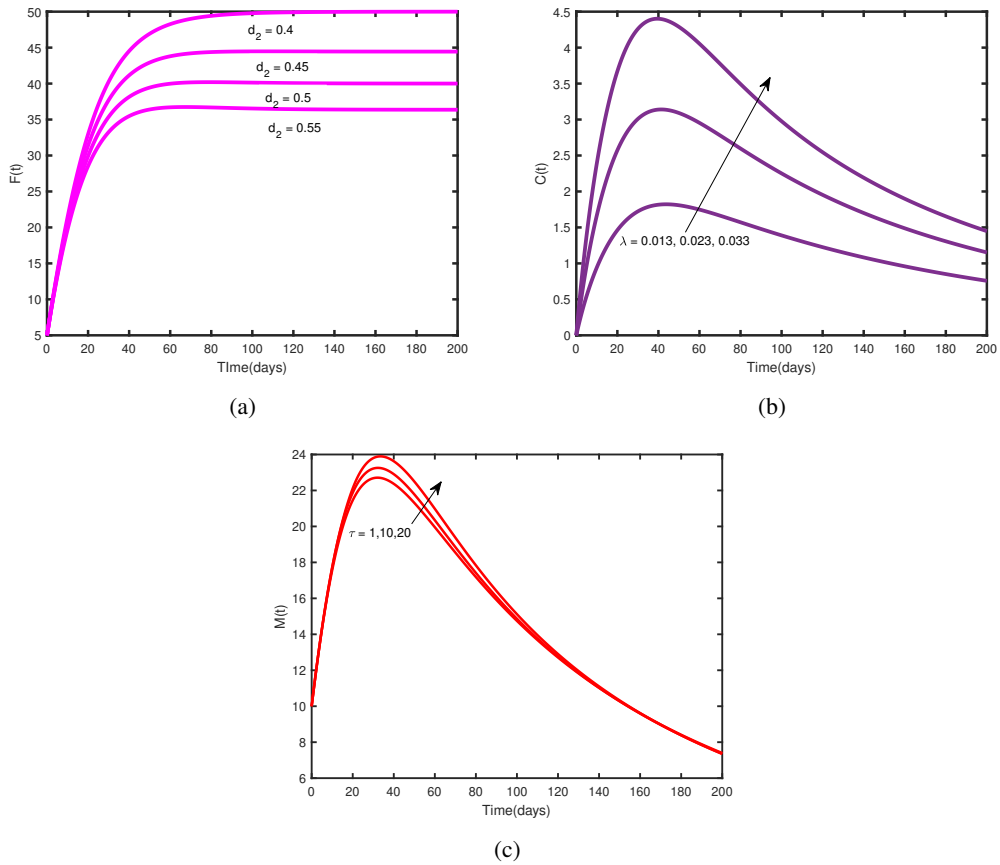


Figure 5: (a) Profile of fungi population $F(t)$ versus time for system (2.1) variation due to mortality rate d_2 . (b) Profile of concentration of insecticide $C(t)$ versus time for system (2.1) rate of spraying insecticide λ . (c) Profile of mature pests $M(t)$ versus time for various values of τ .

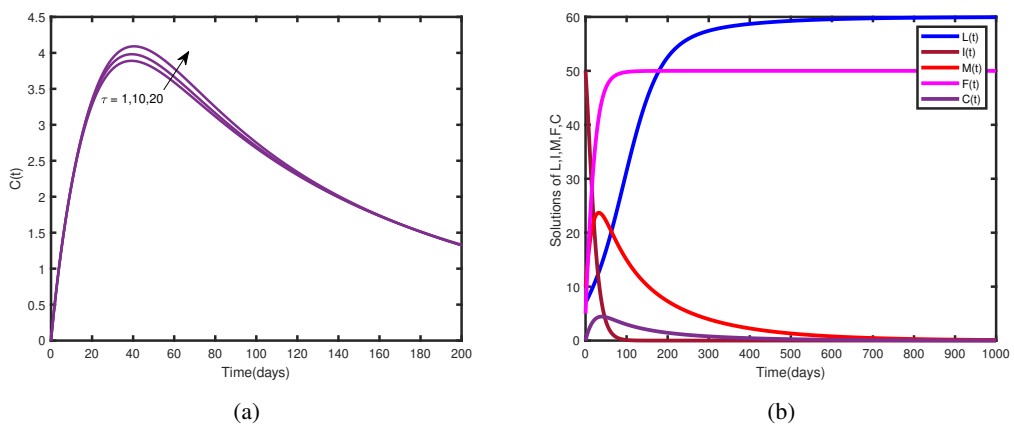


Figure 6: (a) Profile of fungi population $F(t)$ versus time for various values of τ . (b) Profile of the populations with respect to time

From figure 3a, it can be observed that the growth of coconut leaves increases in accordance with the value of r . From figure 3b, it can be inferred that, when there is an increase in the value of P_1 , the number of immature whiteflies decreases. As we can see from Figure 4a, the population of adult whiteflies drops as the values of δ grow. Similarly, we can see from figure 4b, the population of mature whiteflies reduces with time as the values of λ increase. As seen in figure 5a, it is evident that the population of fungus reduces as the values of d_2 increases. It can be observed in figure 5b that the concentration of the botanical insecticide increases in proportion to the value of λ with increasing values. From figure 5c, it is observed that as the time delay in the action of the bioinsecticide increases, the mature pest population also increases. This is because the delayed response allows pests to continue reproducing and surviving before the bioinsecticide effect becomes active. Thus, timely action of the bioinsecticide is crucial, as any delay in its effect allows the mature pest population to grow and weakens overall control efficiency. From figure 6a, it can be inferred that a longer delay in the bioinsecticide action implies slower utilization by the pest population. Hence, the bioinsecticide accumulates in the host tissue before exhibiting its effect, leading to a higher observed concentration over time. A representation of the variation of all the populations with regard to time is depicted in figure 6b.

7 Conclusion

This study explores the dynamics of biocontrol strategies to control RSW in coconut trees, incorporating both entomopathogenic fungi and neem oil. By developing a delay differential equations model, we capture the delayed impacts of these biological and botanical control agents on pest dynamics, accounting for the natural lag in their effectiveness after application. Our study shows that the control strategies that are specific to the stages of the RSW can keep them under control on coconut leaves. Spraying entomopathogenic fungus mostly kills the young pests, while using neem oil kills the adult pests. The model reflects the realistic timing and dynamics of pest suppression by using these dual control tactics coupled with time delay. This gives useful information for creating more effective and targeted pest management method for coconut farming. The analysis includes the determination of existence and local stability criteria for equilibria and sensitive parameters based on the basic reproduction number and time delay. The numerical simulations is conducted in MATLAB to validate our theoretical findings and demonstrate the potential impact of various parameters on pest population dynamics. Through this approach, we contribute to a better understanding of biocontrol strategies in coconut trees, supporting the broader goal of sustainable and productive coconut farming. Future work could extend the model to include delays in other processes, such as pest maturation or fungal infection, and use it to design and test optimal spraying schedules for neem oil and fungi to effectively suppress pest population while minimizing costs and environmental impact.

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