

Analysis of the Stability of a Solitary Prey and a Couple of Predators Utilizing a Delay Differential Equation

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Abstract In this study, a prey-predator model with square root functional response and a logistic growth delay for the prey population are taken into account. A new competitor predator introduced into the system with a density-dependent death rate. Calculating the interior equilibrium point and doing a stability analysis of the non-zero equilibrium point. The system's behavior changes from stable to unstable, when the value of delay crosses the critical value, Hopf-Bifurcation is seen. Numerical simulations are done using MATLAB (dde23) to support analytical finding.

1 Introduction

In a predator-prey context, the fundamental and secondary components of predation were presented by C. S. Holling. Because of its significance, the functional response has been studied further in an attempt to understand its characteristics [1]. N. Macdonald investigated whether it is customary to include a temporal delay in the component reflecting the predator's reliance on the prey in deterministic models of prey-predator association [2]. When studying the general differential equations system that characterizes predator-prey dynamics, J. M. Cushing made the assumption that the parameters are cyclic expressions of time [3]. Len Nunnery observed persistent predator-prey interactions. The author discovered that discrete-generation predator-prey models are stabilized by sigmoid functional responses [4]. Andrew Sih found an interesting fact any tactic that lowers the danger of predation can be included in the broad definition of refuge usage [6]. The prevailing theories hold that prey is encountered by predators at random and that the trophic function is solely dependent on prey availability. This strategy is not always acceptable, according to Roger Arditi et al. The trophic function must be considered on the long-time scale of population dynamics, rather than the short behavioral time scale on which the models operate [7]. Yang Kuang et al. studied ratio-dependent prey-predator delayed systems [8]. Chris Cosner and team proposed this unified scientific method for the generation of many forms of functional responses. The argument is based on the concept of mass action, with the key difference that the structure of predator geographic variation and predation possibilities are implicitly taken into consideration [9]. Jill McGrady-Steed et al. discovered that species richness influences population size and stability. Many fundamental theories suggest that complex speciose societies are much less stable than simple ones. Functional groups did not indicate individual population stability. Inter-specific relationships and statistical averaging may both aid in reducing the temporal changes of functional groups generated by aggregating a large number of species [10]. In 20th century several types of prey-predator dynamics studied by different researchers. These dynamics related to density dependent predation by (Rochelle D. Seitz et

al.[11], Garrick T. Skalski et al.[12], John A. Vucetich [14], Israr et al. [16] and Sabbar et al. [13]). Peter A. Abrams et al. investigate the dynamics of each predator as well as the reactions of its mean frequencies to mortality rates enforced on it or its opponents. This model's dynamics investigation reveals previously unknown characteristics such as chaotic oscillations and brief transitory that differ considerably from the ultimate sequences of vibrations [15]. Meng Fan et al. use the Beddington-DeAngelis highly adapted to painstakingly investigate the behaviour of a nondeterministic predator-prey scenario. For the cyclic (almost periodic) situation, a positive (just about) periodic answer and a border (just about) periodic solution are examined, together with persistence, mortality, and global asymptotic stable (generic nonautonomous situation) [17]. When confronted with a group of prey, many attackers become disoriented and less effective in their assaults. J. M. Jeschke et al. studied the influence of predator perplexity on predator dynamics to learn more about the environmental, ethological, and evolutionary implications of this phenomenon. The authors developed the very first functional responder model that takes perplexity into consideration and subjectively and statistically compares it to actual data from multi predator/prey systems [18]. Dongmei Xiao demonstrated physiologically the eradication of the singularity of origin, which was regarded "pathological behaviour" for a ratio-dependent prey-predator system, as well as the avoidance of simultaneous death as a likely result of the predator-prey encounter [19]. Researchers establish a functional reaction that accounts for both prey densities. As a result, depending on how the two prey densities interact, the functional response exhibits types II (manipulating prey densities) and III (switching predator behaviour) studied by Hassan et al. [20]. In a (small) area of the origin, the authors investigated the effects of linear adjustments to intrinsically instability consistent prey-predator scenarios. P. E. Kloeden et al., in particular, proved that based on the parameters chosen, either one universal attractor or a repulsive state arises [21]. Valerio Ajraldi et al. demonstrate certain basic assumptions for traditional two-population system [22]. S. N. Matia et al. discovered approach for herd behaviour as a self-defence mechanism outperforms the strategy of toxic production. The model also exhibits ecologically important dynamics around the origin due to prey herd behaviour [23]. Yun Kang et al. came to the conclusion that species are vulnerable to extinction in the presence of Allee effects and that starting state has a significant bang on both the survival of prey and its matching predator. Researchers also noted that illness could be able to prevent prey from going extinct due to predation and promote cohabitation, while the disease-driven extinction cannot be stopped by a predator [24]. Hongxia Liu et al. used a stable-dependent impulse to examine a prey-predator interaction using a square root response function. Using the result of geometric differential calculus and the successor functional approach, the existence, uniqueness, and attractiveness of first order cyclic results were first examined. The stability first order cycle is next examined using the Poincaré criterion for spontaneous differential equations [25]. Sourav Kumar Sasmal et al. look into the behaviour of a prey-predator scenario hypothesis. This model incorporates collective prey defense against predation via the Monod-Haldane functional response, in addition to a fear effect that slows prey growth rate when a predator is around [26]. Jai Prakash Tripathi et al. described universal attractivity and permanence as basic functions using the continuation theory from coincidence degree theory, which improves on the standard conditions derived by employing limits of the important parameters and the presence of periodic solutions. This method outperforms the Brouwer fixed point theory in the Crowley-Martin basis function [27]. The authors explored the partial differential equation system's characteristic roots, along with the Turing destabilization and Hopf bifurcation. Analysing the normal form of the Turing-Hopf bifurcation [28] yielded the phase diagram. Jawdat Alebraheem suggests a predator-dependent operational and numerical response-based autonomous predator-prey model. The model's co-existence and extinction conditions were validated and identified using Kolmogorov analysis [29]. Few scholars have used mathematical modelling to study the influence of prey growth in a predator-prey paradigm. Predation on adult prey was investigated using delay system of equations. In order to understand the dynamic effects of these sorts of organisms, time lag ought to be part of the mathematical model [30, 31, 32]. Using a mathematical model, this study investigated the behaviour of paired up prey populations in the face of predation [33]. Hattaf et al. in [34, 35] propose some ecological model of a delayed prey-predator model with Hattaf-Yousfi functional response. This research lies in formulating and analyzing a delay differential equation model that incorporates a single prey species interacting with two predator species, coupled through a square-root type functional response. Unlike conventional predator-prey models that

use standard Holling-type functional responses (e.g., type I, II, or III) and neglect time delays, this study introduces a nonlinear square-root response function that better captures saturation effects at low prey densities and includes a time delay to represent gestation or handling times. This combined approach provides new insights into the dynamic behavior, stability conditions, and coexistence criteria for multi-predator systems with biologically realistic interactions, which have not been extensively explored in previous literature.

2 Motivation Behind this Paper

The prey-predator model proposed by Valerio Ajraldi et al. with a delay in the logistic growth of the prey population and the introduction of another competitor predator with a density-dependent mortality rate can be suitable for simulating dynamics in ecosystems such as large savannas with herbivores and their predators. This model captures the complex interactions among multiple species in such ecosystems, where prey populations exhibit logistic growth with a time delay, and predators compete for resources while exhibiting density-dependent mortality rates. The dynamics of such a model can be represented by a system of differential equations, which may include terms for the prey population (e.g., herbivores), multiple predator populations (e.g., large predators and competitor predators), and possibly other factors such as environmental influences.

3 Mathematical Model

$$\frac{dR}{dT} = sR \left(1 - \frac{R(T - \tau)}{K_1} \right) - b_1 \sqrt{R} D_1 - b_2 \sqrt{R} D_2, \quad (3.1)$$

$$\frac{dD_1}{dT} = b_1 c_1 \sqrt{R} D_1 - f_1 D_1, \quad (3.2)$$

$$\frac{dD_2}{dT} = b_2 c_2 \sqrt{R} D_2 - f_2 D_2 - g D_2^2. \quad (3.3)$$

With the initial conditions $R(0) > 0$, $D_1(0) > 0$, $D_2(0) > 0$, above system (1)–(3) can be *non-dimensionalized* by introducing new relations:

$$\hat{p} = \frac{R}{K_1}, \quad Z_1 = \frac{D_1 b_1}{s \sqrt{K_1}}, \quad Z_2 = \frac{D_2 b_2}{s \sqrt{K_1}}, \quad t = \frac{sT}{2},$$

and new parameters are used:

$$s_1 = \frac{2c_1 b_1 \sqrt{K_1}}{s}, \quad s_2 = \frac{2c_2 b_2 \sqrt{K_1}}{s}, \quad h_1 = \frac{f_1}{c_1 b_1 \sqrt{K_1}}, \quad h_2 = \frac{f_2}{c_2 b_2 \sqrt{K_1}}, \quad h_3 = \frac{gs}{c_2 b_2^2}.$$

After conversion, the non-dimensional system becomes:

$$\frac{d\hat{p}}{2dt} = \hat{p}(1 - \hat{p}(t - \tau)) - \sqrt{\hat{p}} Z_1 - \sqrt{\hat{p}} Z_2, \quad (3.4)$$

$$\frac{dZ_1}{dt} = s_1 Z_1 (\sqrt{\hat{p}} - h_1), \quad (3.5)$$

$$\frac{dZ_2}{dt} = s_2 Z_2 (\sqrt{\hat{p}} - h_2 - h_3 Z_2). \quad (3.6)$$

The system (4)–(6) has singularity when the Jacobian is calculated, because in (4)–(6) terms are not free from the square root. To understand the system of equations (4)–(6), we assume $\hat{p} = p^2$:

$$\frac{dp}{dT} = p(1 - p^2(t - \tau)) - Z_1 - Z_2, \quad (3.7)$$

$$\frac{dZ_1}{dT} = s_1 Z_1 (p - h_1). \quad (3.8)$$

$$\frac{dz_2}{dT} = s_2 z_2 (p - h_2 - h_3 z_2). \quad (9)$$

With initial conditions $p(0) > 0$, $z_1 > 0$, $z_2 > 0$.

Table 1. Meaning of parameters used in equation (1)–(3)

Symbols	Meaning
R	Prey population at time t
D_1	Predator population at time t
D_2	Population of second predator at time t
f_j	Death rate
c_j	Predation rate
g	Intra-specific competition rate
K_1	Carrying capacity
b_j	Conversion efficiency
τ	Time delay
s	Intrinsic growth rate of prey

The subscript $j = 1$ or 2 .

Equilibrium points of the system

We calculate equilibrium points of (7)–(9):

$$p(1 - p^2) - z_1 - z_2 = 0, \quad (10)$$

$$s_1 z_1 (p - h_1) = 0, \quad (11)$$

$$s_2 z_2 (p - h_2 - h_3 z_2) = 0. \quad (12)$$

After simplification of (10)–(12) system of equations, we calculate five points of equilibrium for the system (7)–(9):

- (i) First, we calculate the trivial equilibrium point, i.e., origin $G_0 = (0, 0, 0)$.
- (ii) Second, we calculate the axial equilibrium point $G_a = (1, 0, 0)$.
- (iii) Third, we calculate boundary equilibrium points

$$G_{B_2} = (h_1, h_1 h_1 (1 - h_1^2), 0), \quad G_{B_2} = (\hat{p}, 0, \tilde{z}_2),$$

where $\tilde{z}_2 = \hat{p}(1 - \hat{p}^2)$ and $\hat{p}$ is the first positive eigen root of the cubic equation

$$h_3 \hat{p}^3 + (1 - h_3) \hat{p} - h_2 = 0. \quad (13)$$

If $h_1 < 1$, then G_{B_1} (the first boundary equilibrium point) exists, and G_{B_2} exists only if $\hat{p} < 1$.

- (iv) Fourth, the interior equilibrium point $G^* = (p^*, z_1^*, z_2^*)$, where

$$p^* = h_1, \quad z_1^* = h_1(1 - h_1^2) - \frac{h_1 - h_2}{h_3}, \quad z_2^* = \frac{h_1 - h_2}{h_3}.$$

4 Dynamical Behaviour when $\tau = 0$

To study the nature of the system of equation (7)–(9) about the five feasible equilibrium points, the variational matrix is

$$W = \begin{bmatrix} 1 - 3p^2 & -1 & -1 \\ s_1 z_1 & s_1(p - h_1) & 0 \\ s_2 z_2 & 0 & s_2(p - h_2 - 2h_3 z_2) \end{bmatrix}$$

Characteristic polynomial of W corresponding to eigenvalue λ is given by

$$(1 - \lambda)(s_1 h_1 + \lambda)(s_2 h_2 + \lambda) = 0. \quad (14)$$

Because equation (14) contains one positive root definitely, G_0 is unstable.

At the equilibrium point G_a , the characteristic polynomial of W is

$$(2 + \lambda)(s_1(1 - h_1) - \lambda)(s_2(1 - h_2) - \lambda) = 0.$$

Eigenvalues of W at G_a are $(s_1(1 - h_1), -2, s_2(1 - h_2))$, so the equilibrium point G_a of the system (7)–(9) is stable if $h_1 > 1$ and $h_2 > 1$.

One interesting result is obtained: if the equilibrium point G_a is stable, then the remaining equilibrium points G_{B_1} , G_{B_2} , and G^* do not exist, and if equilibrium point G_{B_1} , G_{B_2} , and G^* exist, then the equilibrium point G_a is not stable.

The characteristic equation of W at G_{B_1} is

$$(s_2(h_1 - h_2) - \lambda)(\lambda^2 + (3h_1^2 - 1)\lambda + s_1 h_1) = 0. \quad (15)$$

Only if $h_2 > h_1$ and $h_1 > \frac{1}{\sqrt{3}}$ then equilibrium point G_{B_1} of the system (7)–(9) is locally asymptotically stable. During calculation, we notice that when G_{B_1} (equilibrium point) is stable, then G^* (equilibrium point) does not exist and vice versa.

For G_{B_2} , the characteristic polynomial of W is

$$(s_1(\hat{p} - h_1) - \lambda)(\lambda^2 + (3\hat{p}^2 + s_2 h_3 \tilde{z}_2 - 1)\lambda + s_2 \tilde{z}_2) = 0. \quad (16)$$

If $\hat{p} > \tilde{p}$ and $\hat{p} > \frac{1}{\sqrt{3}}$, the point G_{B_2} is locally asymptotic stable.

At point G^* , the characteristic polynomial of the system (7)–(9) is given by

$$\lambda^3 + J_1 \lambda^2 - J_2 \lambda + J_3 = 0, \quad (17)$$

where

$$J_1 = s_2 h_3 z_2^* + 3h_1^2 - 1, \quad J_2 = (3h_1^2 - 1)s_2 h_3 z_2^* + s_1 z_1^*, \quad J_3 = s_1 h_3 z_1^* s_2 z_2^*.$$

By the Routh–Hurwitz criterion, the equilibrium point G^* in system (7)–(9) is asymptotically locally stable if $h_1 > \frac{1}{\sqrt{3}}$.

5 Study Behavior when $\tau > 0$

The characteristic equation of (7)–(9) around any equilibrium point is

$$\begin{vmatrix} 1 - p^2 - 2p^2 e^{-\lambda\tau} - \lambda & -1 & -1 \\ s_1 z_1 & s_1(p - h_1) - \lambda & 0 \\ s_2 z_2 & 0 & s_2(p - h_2 - 2h_3 z_2) - \lambda \end{vmatrix} = 0.$$

After simplification

$$\lambda^3 + L_1 \lambda^2 + L_2 \lambda + L_3 + e^{-\lambda\tau}(Q_1 \lambda^2 + Q_2 \lambda + Q_3) = 0. \quad (18)$$

Where

$$\begin{aligned} L_2 = & -s_1h_1 - s_2h_2 + s_1h_1s_2h_2 + s_1p + s_2p - s_1s_2h_1p \\ & - s_1s_2h_2p + s_1s_2p^2 + s_1h_1p^2 + s_2h_2p^2 - s_1p^3 - s_2p^3 \\ & + s_1z_1 + s_2z_2 - 2s_2z_2h_3 + 2s_1s_2h_2z_2h_3 \\ & - 2s_1s_2pz_2h_3 + 2s_2z_2h_2h_3p^2. \end{aligned}$$

$$Q_2 = 2s_1h_1p^2 + 2s_2h_2p^2 - 2s_1p^3 - 2s_2p^3 + 4s_2^2z_2h_3p^2,$$

$$\begin{aligned} L_3 = & -s_1h_1s_2h_2 + s_1h_1s_2p + s_1ps_2h_2 - s_1s_2p^2 \\ & + s_1h_1p^3 - s_1h_1s_2p^3 - s_2h_2p^3 \\ & + s_1s_2h_2p^3 + s_1h_1s_2h_2p^2 - s_1h_1p^2 \\ & - s_2h_2p^2 + s_1p^3 + s_2p^3 - s_1h_1s_2h_2p^3 - s_1h_1s_2p^2 \\ & + 2s_1s_2h_2p^2h_3z_2 - s_1s_2h_2p^2 \\ & + 2s_1s_2ph_3z_2 - 2s_2h_2ph_3z_2 + 4s_1s_2p^2h_3^2z_2^2, \end{aligned}$$

$$Q_3 = 2s_1h_1s_2h_2p^2 - 2s_1h_1s_2p^3 - 2s_1h_2s_2p^3 + 4s_1s_2p^2h_1h_2z_2 - 4s_1s_2p^3h_2h_3z_2.$$

If the eigenvalues of the characteristic polynomial (18) do not have positive real part, then the equilibrium point is stable. It is not an easy task when all latent roots of (18) have no positive real part.

When $\tau = 0$, equation (18) transforms to

$$\lambda^3 + (L_1 + Q_1)\lambda^2 + (L_2 + Q_2)\lambda + L_3 + Q_3 = 0. \quad (19)$$

By Routh–Hurwitz criterion and the following condition

$$\begin{aligned} (L_1 + Q_1) & > 0, \\ (L_3 + Q_3) & > 0, \\ (L_1 + Q_1)(L_2 + Q_2) & > (L_3 + Q_3). \end{aligned} \quad (19a)$$

Then equation (19) has all the roots negative real parts.

Let us suppose that if $\lambda = 0$ is a solution of (18), then $L_3 + Q_3 = 0$, a contradiction to fact in (19a). Hence, $\lambda = 0$ is not an eigenvalue of equation (18).

Suppose that for some arbitrary $\tau \geq 0$, $i\omega$ with a condition $\omega > 0$ is a solution of (18), so

$$-i\omega^3 - L_1\omega^2 + iL_2\omega + L_3 + (\cos \omega\tau - i \sin \omega\tau)(-Q_1\omega + iQ_2\omega + Q_3) = 0. \quad (20)$$

Separate real and imaginary parts,

$$L_3 - L_1\omega^2 + (Q_3 - Q_1\omega^2) \cos \omega\tau + Q_2\omega \sin \omega\tau = 0, \quad (21)$$

$$L_2\omega - \omega^3 + Q_2\omega \cos \omega\tau - (Q_3 - Q_1\omega^2) \sin \omega\tau = 0. \quad (22)$$

Which gives

$$\omega^6 + l\omega^4 + m\omega^3 + n = 0, \quad (23)$$

where

$$l = L_1^2 - Q_1^2 - 2L_2, \quad m = L_3^2 - Q_3^2 + 2Q_1Q_3 - 2L_1L_3, \quad n = L_3^2 - Q_3^2.$$

Let $\alpha = \omega^2$, then equation (23) becomes

$$\alpha^3 + l\alpha^2 + m\alpha + n = 0. \quad (24)$$

Where $o(\alpha) = \alpha^3 + l\alpha^2 + m\alpha + n$.

Lemma 3.3 For equation (24), we have results:

(i) If $n < 0$, the equation (24) has one positive root definitely.

- (ii) If $n \geq 0$, and $(l^2 - 3m) \leq 0$, then equation (24) has all roots non-negative.
 (iii) If $n \geq 0$, and $(l^2 - 3m) > 0$, then positive real root for equation (24) iff

$$\alpha_i^* = \frac{-l \pm \sqrt{l^2 - 3m}}{3} > 0 \quad \text{and} \quad \omega(\alpha_i^*) \leq 0.$$

Assume equation (24) has a minimum one positive real root $\alpha_0 = \sqrt{\alpha_0}$. From equation (21)–(22), we obtain

$$\cos \omega \tau = \frac{-(Q_2 \omega^2 (L_2 - \omega^2) + (L_3 - L_1 \omega^2)(Q_3 - Q_1 \omega^2))}{(Q_3 - Q_1 \omega^2)^2 + (Q_2 \omega)^2}, \quad (25)$$

$$\tau_i = \frac{1}{\omega_0} \arccos \left(\frac{-(Q_2 \omega_0^2 (L_2 - \omega_0^2) + (L_3 - L_1 \omega_0^2)(Q_3 - Q_1 \omega_0^2))}{(Q_3 - Q_1 \omega_0^2)^2 + (Q_2 \omega_0)^2} \right) + \frac{2j\pi}{\omega_0}, \quad (26)$$

where $j = 0, 1, 2, 3, \dots$

Lemma 3.4 Suppose $\alpha_0 = (3\alpha_0^2 + l\alpha_0 + m)$ and condition in (19a) are satisfied. For $(i = 0, 1, 2, 3, 4, \dots)$, let $\lambda(\tau) = \beta(\tau) + i\omega(\tau)$ be the latent value of equation (18) satisfying $\beta(\tau_i) = 0$, $\omega(\tau_i) = \omega_0$, where

$$\tau_i = \frac{1}{\omega_0} \arccos \left(\frac{-(Q_2 \omega_0^2 (L_2 - \omega_0^2) + (L_3 - L_1 \omega_0^2)(Q_3 - Q_1 \omega_0^2))}{(Q_3 - Q_1 \omega_0^2)^2 + (Q_2 \omega_0)^2} \right) + \frac{2i\pi}{\omega_0}.$$

Then $\pm i\omega_0$ are latent values. If condition of transversality (26a) exists

$$\beta'(\tau_i) = \left. \frac{\Re \lambda(\tau)}{d\tau} \right|_{\lambda=i\omega_0} \neq 0, \quad (26a)$$

then Hopf-bifurcation is observed in system (7)–(9) at some random point of equilibrium and $\tau = \tau_i$.

Proof. Suppose $\lambda = \lambda(\tau)$ be the solution of equation (18). Put $\lambda(\tau)$ in equation (18) and differentiating equation (18) w.r.t. τ , we observed that

$$[(3\lambda^2 + 2L_1\lambda + L_2) + ((Q_2\lambda + Q_2 + Q_3)(-\tau) + (2\lambda Q_1 + Q_2))e^{-\lambda\tau}] \frac{d\lambda}{d\tau} = \lambda(2Q_1\lambda + Q_2 + Q_3)e^{-\lambda\tau}.$$

Thus,

$$\frac{d\lambda}{d\tau} = \frac{-(3\lambda^2 + 2L_1\lambda + L_2)e^{\lambda\tau} + (2\lambda Q_1 + Q_2)}{\lambda(2Q_1\lambda + Q_2 + Q_3)e^{\lambda\tau} - \tau/\lambda}.$$

From equation (21)–(23), we have

$$\begin{aligned} \beta'(\tau_0) &= \Re \left(\frac{(3\lambda^2 + 2L_1\lambda + L_2)e^{\lambda\tau}}{\lambda(2Q_1\lambda + Q_2 + Q_3)} + \Re \left(\frac{2\lambda Q_1 + Q_2}{\lambda(2Q_1\lambda + Q_2 + Q_3)} \right) \right) \\ &= \frac{1}{\lambda} \left[3\omega_0^2 + 2L_1\omega_0^2 - (L_1^2 - 2L_2)\omega_0^4 + (L_3^2 - Q_3^2 + 2Q_1Q_3 - 2L_1L_3)\omega_0^2 \right]. \end{aligned}$$

Where $A = (Q_3 - Q_1\omega_0^2)^2 + (Q_2\omega_0)^2$. Note that when $A > 0$ and $\omega_0 > 0$, we concluded that

$$\text{sign}[\beta'(\tau_0)] = \text{sign}[\sigma'(\alpha_0)].$$

This proves the lemma.

At equilibrium point G^* , the equation (18) becomes

$$\lambda^3 + R_1\lambda^2 + R_2\lambda + R_3 + e^{-\lambda\tau}(R_3\lambda^2 + R_4\lambda + R_5) = 0, \quad (27)$$

and l, m, n becomes $R_1^2 - R_2 - 2R_2$, $R_2^2 - R_1^2 + 2R_3R_5$, and $-R_5^2$ respectively, and τ_i becomes

$$\tau_i = \frac{1}{\omega_0} \arccos \left(\frac{-(R_4\omega_0^2(R_2 - \omega_0^2) - R_3\omega_0(R_5 - R_3\omega_0^2))}{(R_5 - R_3\omega_0^2)^2 + (R_4\omega_0)^2} \right). \quad (28)$$

Where

$$R_1 = -s_1 - s_2 + s_1 h_1 + s_2 h_2, \quad R_2 = s_1 s_2 - s_1 s_2 h_1 - s_1 s_2 h_2 + s_1 s_2 h_1 h_2, \quad R_3 = 2, \\ R_4 = -2s_1 - 2s_2 + 2s_1 h_1 + 2s_2 h_2, \quad R_5 = -2s_1 s_2(-1 + h_1)(1 - h_2).$$

Theorem 3.5. Suppose that axioms in (19a) and (26a) are satisfied. Then

- (i) The system (18) latent points G_A are asymptotically stable for every $\tau \in [0, \tau_A)$.
- (ii) If it holds, in that case system (18) passes through a Hopf-bifurcation at the equilibrium position G_A when $\tau = \tau_A$. In the same way, at the equilibrium point G_{B_1} , the equation (15) changes to

$$\lambda^3 + S_1 \lambda^2 + S_2 \lambda + S_3 + e^{-\lambda \tau} (S_3 \lambda^2 + S_4 \lambda + S_5) = 0. \quad (5.1)$$

$$\lambda^3 + S_1 \lambda^2 + S_2 \lambda + S_3 + e^{-\lambda \tau} (S_3 \lambda^2 + S_4 \lambda + S_5) = 0, \quad (29)$$

where l, m, n becomes

$$l = S_1^2 - S_2, \quad m = S_2^2 - S_5^2 - 2S_1 S_3, \quad n = S_3^2,$$

and τ_i becomes

$$\tau_{B_1} = \frac{1}{\omega_0} \arccos \left(\frac{-(S_3 \omega_0^2 (S_2 - \omega_0^2) - S_3 \omega_0 (S_3 - S_1 \omega_0^2))}{(S_3 - S_1 \omega_0^2)^2 + (S_5 \omega_0)^2} \right). \quad (30)$$

Where,

$$S_1 = -1 - s_1 h_1 + h_1^2 + s_2, \quad S_2 = h_1 s_1 + h_1 s_2 - s_1 h_1^2 - s_2 h_1^3, \\ S_3 = h_1 s_1 h_1 - s_1 h_1^2 h_2, \quad S_4 = 2h_1^2, \quad S_5 = -2s_2 h_2 + 2s_1 h_1 + 2s_2 h_1^2 h_2.$$

Theorem 3.6. Let axioms in (19a) and (26a) are fulfilled and

- (i) If $n \geq 0$ and $(l^2 - 3m) \leq 0$, all solutions of equation (31) has no real part with positive sign for all $\tau \geq 0$, and so G_{B_2} , the point of equilibrium of mathematical equation (7)–(9) is stable asymptotic for every $\tau \geq 0$.
- (ii) If either $n < 0$ or $n \geq 0$ and $(l^2 - 3m) > 0$, $\alpha_i^* > 0$ and $\omega(\alpha_i^*) \leq 0$ satisfy, then G_{B_2} , the equilibrium points of the system (7)–(9) is stable asymptotic for every $\tau \in [0, \tau_{B_2})$.
- (iii) If all axioms in (b) and $\sigma'(\alpha_0) \neq 0$ hold, then the system (7)–(9) at G_{B_2} occurred a Hopf-Bifurcation when $\tau = \tau_{B_2}$. Similarly at the equilibrium point G^* , the equation (18) becomes

$$\lambda^3 + W_1 \lambda^2 + W_2 \lambda + W_3 + e^{-\lambda \tau} (W_4 \lambda^2 + W_5 \lambda + W_6) = 0, \quad (31)$$

l, m, n changes to $l = W_1^2 - W_2 - 2W_2$, $m = W_2^2 - W_5^2 + 2W_4 W_6 - 2W_1 W_3$, $n = W_3^2 - W_6^2$ respectively, and τ_i becomes

$$\tau_{B_2} = \frac{1}{\omega_0} \arccos \left(\frac{-(W_5 \omega_0^2 (W_2 - \omega_0^2) + (W_4 - W_1 \omega_0^2)(W_6 - W_4 \omega_0^2))}{(W_6 - W_4 \omega_0^2)^2 + (W_5 \omega_0)^2} \right). \quad (32)$$

Where

$$W_1 = -1 + s_1 h_1 - s_1 \hat{p} + \hat{p}^2 + s_2 h_2,$$

$$W_2 = -s_1 h_1 + s_1 \hat{p} + s_1 h_1 \hat{p}^2 - s_1 \hat{p}^3 + s_2 \hat{p} - s_1 s_2 h_2 + s_1 h_1 s_2 h_2 \hat{p} - s_1 s_2 h_2 \hat{p}^2 + s_2 h_3 \hat{p}^2,$$

$$W_3 = s_1 s_2 h_2 - s_1 s_2 h_2 \hat{p}^2,$$

$$W_4 = 2\hat{p}^2,$$

$$W_5 = 2s_1h_1\hat{p}^2 - 2s_1\hat{p}^3 + 2s_2h_2\hat{p}^2,$$

$$W_6 = 2s_1s_2h_1h_2\hat{p}^2 - 2s_1s_2\hat{p}^2.$$

Theorem 3.7. Let (19a) and (26a) are satisfied and

If $n \geq 0$ and $(l^2 - 3m) \leq 0$, all solution of equation (31) has no positive real parts for all $\tau \geq 0$, and hence G_{B_2} , the point of equilibrium of system (7)–(9) is stable asymptotic for every $\tau \geq 0$.

If either $n < 0$ or $n \geq 0$ and $(l^2 - 3m) > 0$, $\alpha_i^* > 0$ and $\omega(\alpha_i^*) \leq 0$ satisfy, then G_{B_2} , the equilibrium points of the system (7)–(9) is stable asymptotic for every $\tau \in [0, \tau_{B_2})$.

If all axioms in (b) and $\sigma'(\alpha_0) \neq 0$ hold, then the system (7)–(9) at G_{B_2} occurred a Hopf-Bifurcation, when $\tau = \tau_{B_2}$. Similarly at the equilibrium point G^* , the equation (18) becomes

$$\lambda^3 + P_1\lambda^2 + P_2\lambda + P_3 + e^{-\lambda\tau}(P_4\lambda^2 + P_5\lambda) = 0, \quad (33)$$

l, m, n becomes $l = P_1^2 - P_2 - 2P_2$, $m = P_2^2 - P_5^2 - 2P_1P_3$, $n = P_3^2$ respectively, and τ_i becomes

$$\tau_0 = \frac{1}{\omega_0} \arccos \left(\frac{-(P_5\omega_0^2(P_2 - \omega_0^2) - P_4\omega_0^3(P_3 - P_1\omega_0^2))}{(P_4\omega_0^2)^2 + (P_5\omega_0)^2} \right). \quad (34)$$

$$\begin{aligned} P_1 &= -1 + h_1^2 + s_2h_3z_2^*, \\ P_2 &= s_1z_1^* + s_2z_2^* - s_2h_3z_2^* + s_2h_3h_1^2z_2^*, \\ P_3 &= s_1s_2h_3z_1^*z_2^*, \\ P_4 &= 2h_1^2, \\ P_5 &= 2s_2h_3h_1^2z_2^*. \end{aligned} \quad (5.2)$$

Theorem 3.8 Let (19a) and (26a) are satisfied and if

If $(l^2 - 3m) \leq 0$, for every solution of equation (33) have no positive real parts for every $\tau \geq 0$, and hence G^* point of equilibrium for (7)–(9) is stable asymptotic for every $\tau \geq 0$.

If $(l^2 - 3m) > 0$, $\alpha_i^* > 0$ and $\omega(\alpha_i^*) \leq 0$ satisfied, then G^* point of equilibrium of the system (7)–(9) is stable asymptotic for every $\tau \in [0, \tau_0)$.

If all axioms stated in (b) and $\sigma'(\alpha_0) \neq 0$ satisfied, then the system (7)–(9) at G^* equilibrium point shows a Hopf-Bifurcation when $\tau = \tau_0$.

6 Numerical Simulation

In this section, we solve the ecological problem of prey-predator as stated by (7)–(9) to properly understand the dynamical behavior. We will also show the validity of our analytical work with the help of large-scale simulation. The numerical simulation is performed with the help of MATLAB software. We chose the value of parameters in a feasible way to perform the visualization of the analytical results. More specifically, the values are as follows:

$$\begin{aligned} s &= 2.5, \quad K_1 = 10, \quad b_1 = 0.15, \quad b_2 = 0.1, \quad c_1 = 0.002, \quad c_2 = 0.01, \\ f_1 &= 0.001, \quad f_2 = 0.001, \quad g = 0.01, \quad \tau = 0.6 < \tau_0 = 0.7. \end{aligned} \quad (35)$$

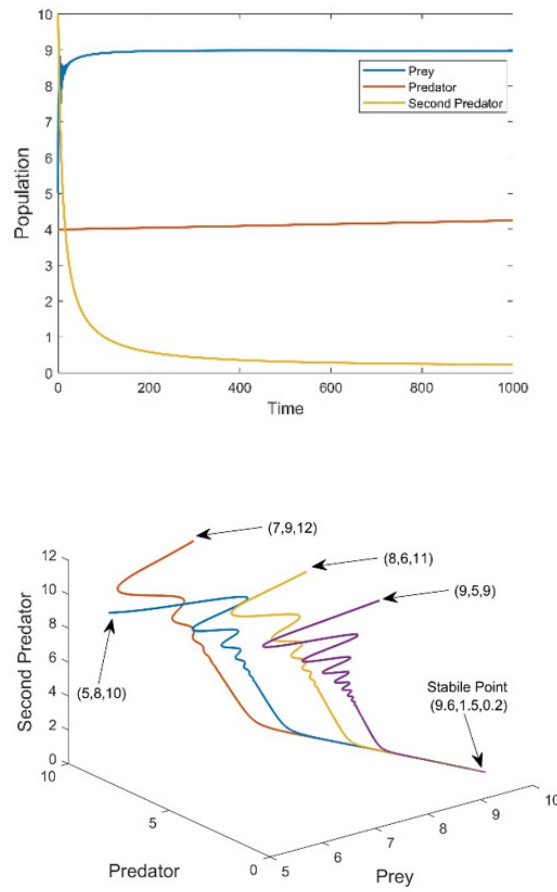


Figure 1. The global stability behavior of the interior equilibrium point with the data in equation (5).

The graphical result for this case is presented in Figure (1). Here, the dynamical behavior approaches to globally asymptotically stable of the interior equilibrium point $G^* = (9.6, 1.5, 0.2)$.

Now, we have checked the death rate effect of all predator and second predator respectively on the dynamical behavior of our model. The solution of this model is shown in Figures (2) and (3), using the parameter values in equation (35) with some different initial points. Figure (2), the model shows a global stable of the equilibrium point $G_{B_2} = (10, 0, 0.2)$ when $f_1 = 0.01$ with keeping other parameters in equation (35) and G^* becomes unstable. In Figure (3), the time series and 3D plots of our model show the global stable of $G_{B_1} = (9.3, 3.3, 0)$ when $f_2 = 0.01$ with keeping other parameters in equation (35) and G^* becomes unstable.

To observe the effect of time delay on the dynamical behavior of our model near each of the stable points, we have plotted the solution of the model considering the parameter values in equation (35) with $\tau = 0.8$, and shown the results in Figures (4),(5) and (6). In Figure (4), the interior equilibrium point G^* becomes unstable when $\tau = 0.8$. Figure(5) shows the delay time effect on the stability of G_{B_2} equilibrium point when $f_1 = 0.01$ and $\tau = 0.8$. Finally, in Figure(6), the bifurcation occurs when $f_2 = 0.01$ with $\tau = 0.8$ and the G_{B_1} equilibrium point becomes unstable. On the other hand, when using the parameter values in equation (35) with $\tau = 0.7$ with two different initial points, we get the bistable of the interior equilibrium point G^* but one of them stable and the other is periodic, see Figure (7).

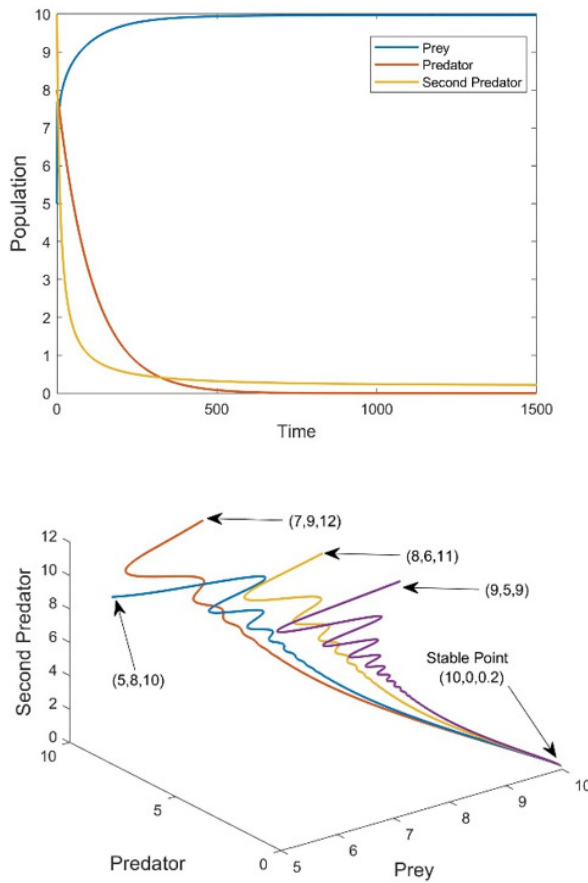


Figure 2. The global stability behavior of the $G_{B2} = (10, 0, 0.2)$ equilibrium point with the data in equation (5) with $f_1 = 0.01$.

7 Conclusion

Limit cycles occur spontaneously in the Arajdi et al. [22] prey-predator model under acceptable circumstances involving model parameters. In this study, we looked at a one-prey, two-predator system with a square root functional response and a delay in the prey's logistic growth rate. The inclusion of another rival predator makes the system more stable than in [22], and limit cycles do not arise spontaneously within the structure without delay. The stability of the equilibrium points leads to the conclusion that the prey population will survive and the predator population will be at risk of extinction if the ratio of their death rates to the square root of the prey population's carrying capacity is higher than the product of respective conversion efficiencies and predation rates. The prey population (R) and one of the predator populations (D_1) survive, and the predator population (D_2) that exhibits a density-dependent mortality rate goes to extinction if the ratio of the death rate of the predator (D_1) to the product of its conversion efficiency and its predation rate is less than the ratio of death rate of the predator (D_2) to the product of its conversion efficiency and its predation rate. This implies that the square root of one-third of carrying capacity of the prey population. After considering the effect of time lag in the system, limit cycle appears for all the equilibrium points when time delay crosses some critical value. The future work of this model are can studied under some effects such as fraction order, fear in prey, harvesting and vigilance.

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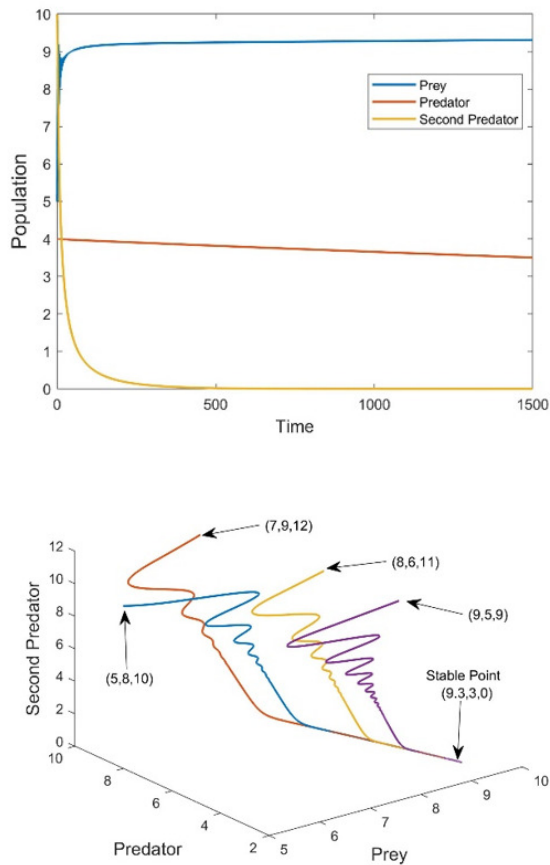


Figure 3. The global stability behavior of the $G_{B1} = (9.3, 3, 0)$ equilibrium point with the data in equation (5) with $f_2 = 0.01$.

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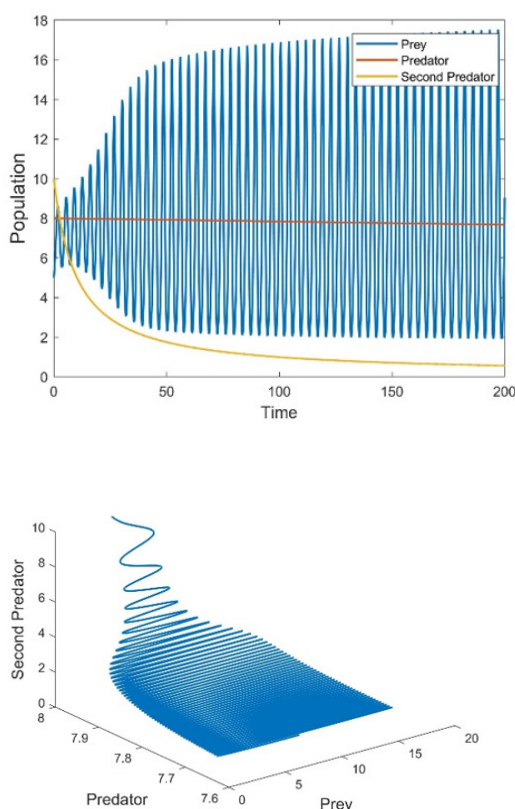


Figure 4. When $\tau = 0.8 > \tau_0$, keeping the other parameter values, the bifurcation periodic solution from the interior equilibrium point G^* occurs..

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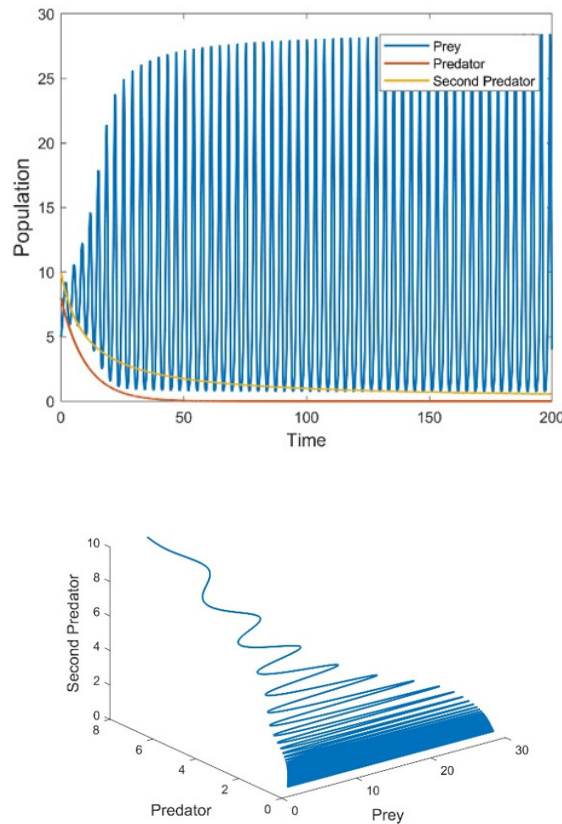


Figure 5. When $\tau = 0.8 > \tau_0$ and $f_2 = 0.01$, keeping the other parameter values, the bifurcation periodic solution from G_{B2} occurs.

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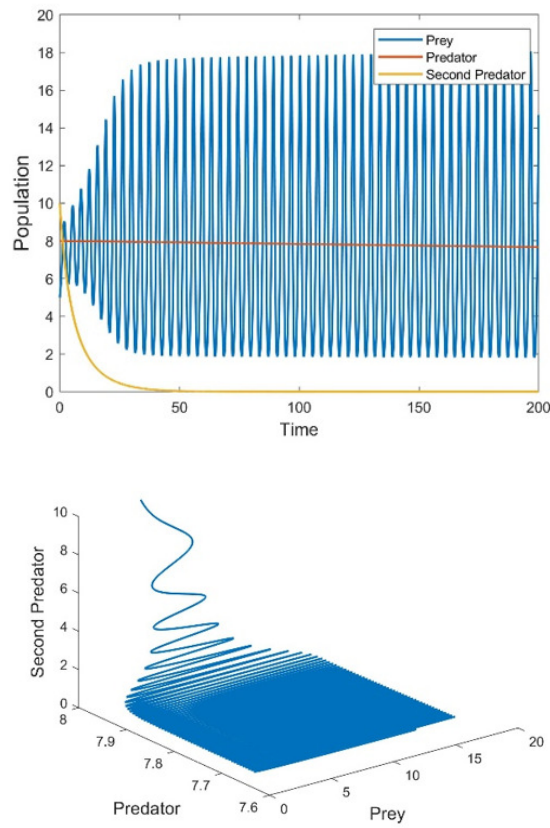


Figure 6. When $\tau = 0.8 > \tau_0$ and $f_2 = 0.01$, keeping the other parameter values, the bifurcation periodic solution from G_{B1} occurs.

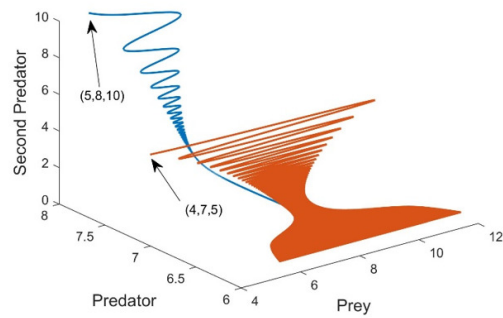


Figure 7. The bistable behavior of the interior equilibrium point G^* when $\tau = 0.7$.

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