

DYNAMICS OF LONG FINITE-AMPLITUDE WAVES IN COMPRESSIBLE ROD COMPOSED OF MOONEY-RIVLIN MATERIAL

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Abstract The manuscript aims to investigate the dynamics of long finite-amplitude waves in compressible rods composed of Mooney-Rivlin materials. Qualitative analysis of nonlinear waves in compressible rods is examined using phase plane analysis in the framework of the evolution equation. The periodic and solitary wave solutions are demonstrated along with the effects of various parameters on these wave characteristics. Additionally, the propagation of two-soliton and the influence of various parameters on the two-solitons are demonstrated. The phase shift and its variation with different values of parameters are also studied. This study will help in understanding the dynamics of nonlinear waves compressible rod composed of Mooney-Rivlin material.

1 Introduction

The investigation of nonlinear waves in elastic rods is crucial for understanding how materials respond under stress and strain. These waves often exhibit phenomena such as solitons, periodic waves, and multi-solitons [1, 2, 3], which are essential for analyzing energy transmission and wave dynamics in nonlinear media. This knowledge is valuable for engineering applications, including impact and shock wave analysis, non-destructive testing, waveguides, and signal transmission technologies. Moreover, nonlinear waves have a significant role in energy harvesting, as elastic rods are capable of efficiently converting mechanical energy into electrical energy. In biomedical engineering, these waves are used in developing advanced medical technologies, such as ultrasound imaging. The study of nonlinear waves also advances mathematical and computational modeling, enhancing the accuracy of wave behavior predictions in complex systems. It further contributes to fundamental physics by clarifying principles of wave dynamics, stability, and interactions.

It is known that waves traveling along a rod experience dispersion due to the rod's transverse dimensions [4, 5, 6, 7]. When the wave's amplitude is sufficiently large, nonlinearity can be as significant as dispersion. Consequently, it is necessary to consider waves influenced by both nonlinearity and dispersion, known as nonlinear dispersive waves. Numerous researchers have studied these waves. Wright [6] used a direct method to demonstrate the existence of solitary and periodic traveling waves in rods. Later, Wright [7] employed intrinsic nonlinear theory to demonstrate the movement of straight, elastic rods, accounting for both inertia and radial shear. Coleman et al. [8] used rod theory to derive solitary and periodic traveling wave solutions in neo-Hookean materials. In the year 1993, Cohen et al. [9] investigated nonlinear axisymmetric waves in a compressible rod which is hyperelastic. They derived a second order soliton solution.

Earlier, in 1987, Soerensen et al. [10] used computational methods to investigate the improved Boussinesq equation, a more accurate model for wave dynamics in uneven elastic rods. The research showcased that a solitary wave may split into multiple waves when transitioning from a wider to a narrower section of the rod. Dai [11] focused on axisymmetric waves in rods made of compressible Mooney-Rivlin material, deriving simplified far-field model equations that accounted for both dispersion and nonlinearity. Later that year, Dai [12] examined nonlinear dissipative waves in hyperelastic rods through an evolution equation, using phase plane analysis to investigate bounded wave patterns.

In 2003, Antman et al. [13] explored the quasilinear evolution equations that govern the planar dynamics in an incompressible rod. Their work established inherent upper and lower stretch limits, preventing viscosity singularities as these stretches approached their extremes. The following year, they [14] further detailed the quasilinear hyperbolic equations governing the planar movements of incompressible, nonlinearly elastic rods. In 2005, Dai et al. [15] focused on radial deformation and boundary conditions with zero lateral traction in phase transitions of thin elastic cylinders, providing analytic solutions for infinite cylinders. Later that year, Cai et al. [16] tackled boundary-value problems related to earlier work on asymptotic model equations, using phase-plane analysis to solve a force-controlled problem. They also derived analytic solutions for displacement-controlled problems, achieving a stress-strain curve closely matching experimental data. In 2004, Dai et al. [17] studied nonlinear waves in elastic rods, deriving the Hamiltonian for the model and concentrating on waves with long but finite amplitude. That same year in a separate study, Dai et al. [18] investigated nonlinear long waves in a compressible Murnaghan material rods, formulating basic governing equations for dispersive nonlinear waves. They also showed solitary wave solutions based on the evolution equations. Bojan et al. [19] in 2009, demonstrated that intrinsically straight planar Kirchhoff elastic rods with nonlinear constitutive relations support compact traveling wave solutions, unlike intrinsically curved rods. In 2012, Luna et al. [20] examined nonlinear waves in heterogeneous elastic rods using second-order homogenization theory and multiple-scale asymptotic expansion, revealing how material heterogeneities affect wave propagation and offering insights into the dynamics of these systems. Finally, in 2014, Murphy [21] studied the stability of thin elastic rods, proposing two stability criteria—one for prescribed axial stretch and torsional twist, and another for prescribed axial force and moment—using Gent’s method. These criteria, which were model-independent, were illustrated through examples of rods under tension and torsion.

In 2016, Lembo [22] expanded Kirchhoff and Clebsch’s nonlinear theory to nonlocal material rods, deriving equations of motion and proving energy balance. His work focused on pure bending deformations, comparing the behavior of classical and nonlocal materials in cantilevers under end forces. That same year, Lei Zhang et al. [23] introduced a wavelet method for analyzing large deflection bending and post-buckling in nonlinearly elastic rods, improving the accuracy of traditional finite element and shooting methods. In 2021, Motaghian et al. [24] examined the effects of cylindrical eigenstrains on traveling waves in neo-Hookean rods, proposing a general framework for traveling wave solutions based on specific eigenstrain parameters. Later that year, Celik et al. [25] studied solitary waves in nonlinear circular elastic rods using a fourth-order nonlinear derivative wave equation. By employing Lie symmetry analysis, they identified ideal systems and traveling wave solutions, which were confirmed through numerical simulations and supported by derived local conservation laws. In 2022, Othman [26] investigated wave dispersion in auxetic rods, analyzing the effects of Poisson’s ratio (ν) on wave velocity while considering radial inertia and shear stiffness. He solved the Pochhammer-Chree equation for ν values ranging from -1 to 0.5 , demonstrating precise control of wave velocity across different frequencies and wavenumbers. In 2023, Silambarasan et al. [27] derived a dispersive nonlinear wave equation for elastic rods using Love’s relation and the Navier-Bernoulli hypothesis, studying solitons and validating multiple solutions. In 2024, Gaibotti et al. [28] addressed bifurcation in a circular Euler-Bernoulli rod under various loads and clarified discrepancies in previous studies by examining differences in external constraints. Their work, the first to experimentally realize centrally-directed loads on a circular rod, aligned with theoretical predictions and showed potential applications in flexible robotics using inextensible cables. Also in 2024, Andrea Nobili [29] refined the Love hypothesis for nonlinear wave propagation in slender

hyperelastic rods through slow-time perturbation, offering excellent numerical agreement with existing solutions while maintaining accuracy in the unimodal Lagrangian framework. Recently, in 2024, T. Tang et al. [30] introduced an Enriched Finite Element Method for modeling discontinuous wave propagation in impacting rods. By enriching linear displacement solutions with harmonic functions, this method reduced dispersion error and provided guidance on mesh size and time step. Experimental results from rod-wall collision tests closely matched numerical simulations, demonstrating minimal strain amplitude decay and precise stress wave behavior.

Phase plane analysis, a method introduced by mathematicians like Henri Poincare in the late 19th century, is both a graphical and analytical tool for studying second-order systems. This technique generates equilibrium points and motion trajectories in the state space of a second-order dynamic system (a two-dimensional phase plane) for different initial conditions, allowing for the qualitative analysis of these trajectories [31, 32].

In 1993, Cohen [33] studied nonlinear axisymmetric waves in a compressible rod made of a compressible Mooney-Rivlin material. Their objective was to obtain governing equations that incorporate both dispersion and nonlinearity. For waves with long but finite amplitude, they derived an evolution equation. But in their study they did not study the qualitative analysis of the dynamical system corresponding to the derived evolution equations. In this paper, our aim is to conduct a qualitative analysis of nonlinear waves in compressible Mooney-Rivlin elastic rods using phase plane analysis. We also present the corresponding periodic wave and solitary wave solutions, along with the impact of parameters like material constants (κ and β), characteristic axial displacement (h), characteristic wavelength (l), radius of the rod (a) and speed of traveling wave (V) on these solutions. Further we study the two-soliton phenomenon and the phase shifts along with the impact of different parameters on these solutions.

The manuscript is structured as follows:

Section 2 outlines the fundamental governing equations and the evolution equation. In Section 3, we present the phase portrait of the dynamical system corresponding to the evolution equation for a rod composed of compressible Mooney-Rivlin material. Sections 4 and 5 focus on the solitary and periodic wave solutions, respectively, and further examine the influence of various parameters on these solutions. Section 6 discusses the two-soliton solution and the associated phase shift, along with the effects of different parameters. Finally, Section 7 provides the conclusion.

2 The basic equations and the nonlinear evolution equation

For elastic rods made of compressible, isotropic homogeneous materials with radius a , let z , r , and θ denote the cylindrical coordinates of a material point, whose coordinates in the undisturbed reference system are Z , R , and Θ , with T representing time.

Cohen et al. [33] obtained the following evolution equation:

$$u_\tau + Auu_\xi + Bu_{\xi\xi\xi} = 0, \quad (2.1)$$

where

$$A = \frac{\eta}{2(3\eta + 1)}\sigma_1, \quad (2.2)$$

$$B = \frac{\nu}{\epsilon} \frac{\eta(4\eta + 1)s}{8(2\eta + 1)(3\eta + 1)}, \quad (2.3)$$

and

$$\sigma_1 = 24(\eta - \kappa + 1) + \frac{10}{\eta} + \frac{2}{\eta^2} + \frac{4}{2\eta + 1},$$

$$\eta = \kappa + \frac{1}{2} - \beta,$$

$$\epsilon = \frac{h}{l},$$

$$\nu = \frac{a^2}{l^2}.$$

Here, $\kappa > 0$ and $-\frac{1}{2} < \beta < \frac{1}{2}$ are constant material coefficients, while h denotes the characteristic axial displacement and l represents the characteristic wavelength.

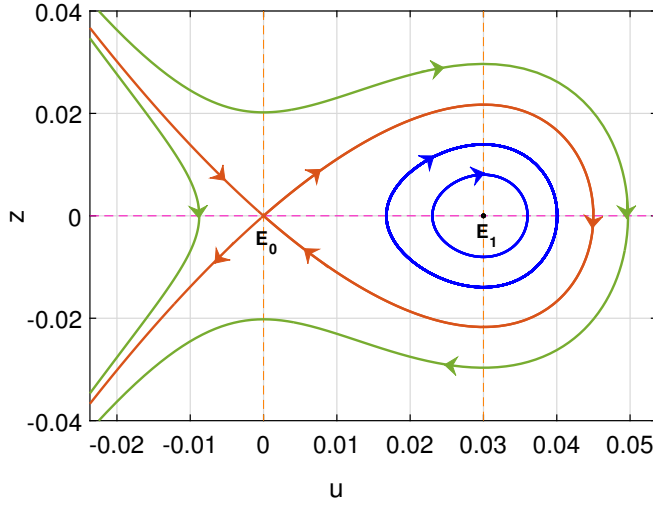


Figure 1. Phase portrait of the dynamical system (3.2) with $\kappa = 0.2$, $\beta = 0.4$, $h = 0.1$, $l = 0.2$, $a = 0.5 = 1$ and $V = 0.1$.

3 Phase plane analysis

Phase plane analysis is a method that uses both graphical and analytical techniques to study dynamical systems. It brings to light the main features of a dynamical system. It has been noted that any orbit in the phase plane, whether homoclinic, heteroclinic, or periodic, corresponds to the solution of a traveling wave [32].

We take the following transformation into consideration [34, 35]:

$$\eta = \xi - V\tau, \quad (3.1)$$

where V represents the speed of the traveling wave. Using the transformation (3.1) in equation (2.1), we obtain the following dynamical system:

$$\begin{cases} \frac{du}{d\eta} = z, \\ \frac{dz}{d\eta} = Cu - Du^2. \end{cases} \quad (3.2)$$

Here $C = \frac{V}{B}$ and $D = \frac{A}{2B}$.

To find the equilibrium points, we consider

$$\frac{du}{d\eta} = 0 \quad \text{and} \quad \frac{dz}{d\eta} = 0.$$

This yields the equilibrium points $E_0(0,0)$ and $E_1(\frac{C}{D}, 0)$. The Jacobian matrix for the system (3.2) is given by:

$$J = \begin{bmatrix} 0 & 1 \\ C - 2Du & 0 \end{bmatrix}.$$

$E_0(0,0)$: The eigenvalues of the matrix J are $\sqrt{C}$ and $-\sqrt{C}$. Since C is always positive, λ_1 and λ_2 are real eigenvalues with opposite signs, indicating that E_0 is a saddle point.

$E_1(\frac{C}{D}, 0)$: The eigenvalues of the matrix J are given by $\sqrt{-C}$ and $-\sqrt{-C}$. Since C is always positive, the eigenvalues λ_1 and λ_2 are complex and have opposite signs. Therefore, E_1 is a center. Phase portrait of the system (3.2) is depicted by Figure 1.

The phase portrait in Figure 1 shows that the system has two equilibrium points $E_0(0,0)$ and $E_1(\frac{C}{D}, 0)$, where E_0 is a saddle point (which is an unstable equilibrium point) and E_1 is a center (which is a stable equilibrium point).

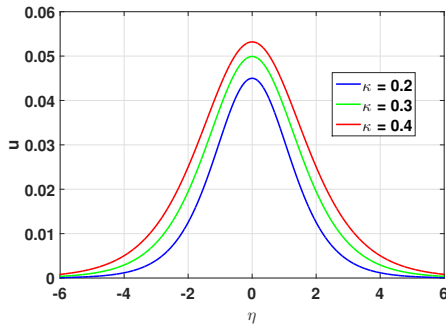


Figure 2. Impact of κ on the solitary wave characteristics of system (2.1) with $\beta = 0.4$, $h = 0.1$, $l = 0.2$, $a = 0.5$ and, $V = 0.1$.

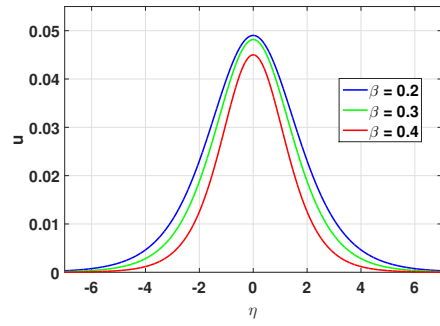


Figure 3. Impact of β on the solitary wave characteristics of system (2.1) with $\kappa = 0.2$, $h = 0.1$, $l = 0.2$, $a = 0.5$ and, $V = 0.1$.

4 Solitary and periodic wave features

The system (3.2) is a Hamiltonian system with Hamiltonian function

$$H(u, z) = \frac{z^2}{2} - \left(C \frac{u^2}{2} - D \frac{u^3}{3} \right) = h(\text{say}). \quad (4.1)$$

Any orbit in the phase portrait corresponding to an initial condition is given by

$$H(u, z) = h. \quad (4.2)$$

At E_0 , $H(0, 0) = 0$.

So, we get $H(u, z) = 0$ for E_0 , which gives

$$\frac{z^2}{2} - \left(C \frac{u^2}{2} - D \frac{u^3}{3} \right) = 0. \quad (4.3)$$

Substituting z from system (3.2), we get

$$\frac{du}{u\sqrt{1 - \frac{2D}{3C}u}} = \pm\sqrt{C}d\eta$$

Substituting $\frac{2D}{3C}u$ by k^2 , we get

$$2\frac{dk}{k\sqrt{1 - k^2}} = \pm\sqrt{C}d\eta$$

Integrating and substituting k , C and D we obtain the solitary wave solution of (2.1) as:

$$u = \frac{3C}{2D} \text{sech}^2\left(\frac{1}{2}\sqrt{C}\eta\right). \quad (4.4)$$

The figures 2-7 shows the impact of the parameters material constants (κ and β), characteristic axial displacement (h), characteristic wavelength (l), radius of the rod (a) and speed of traveling wave (V) on the solitary wave solution of system (3.2) in correspond to the homoclinic orbit traversing the saddle point E_0 .

Figures 2 demonstrate that both the amplitude as well as the width of the solitary wave increase with an increase in the material constant κ . Conversely, Figure 3 indicates that an increase in the material constant β leads to a decrease in both the amplitude and width of the solitary wave. From Figures 4-6, it is observed that the characteristic axial displacement (h), characteristic wavelength (l), and rod radius (a) do not affect the wave's amplitude. However, the width decreases with increasing h and l , while it increases with a larger a . Finally, Figure 7 reveals that a higher value of V results in greater wave amplitude, though the width remains unchanged.

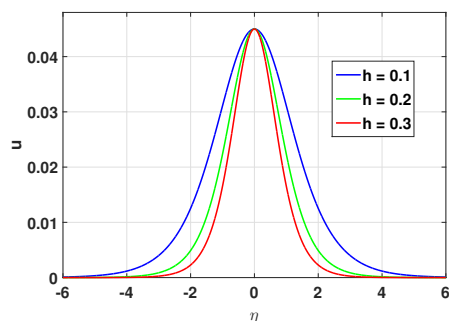


Figure 4. Impact of h on the solitary wave characteristics of system (2.1) with $\kappa = 0.2$, $\beta = 0.4$, $l = 0.2$, $a = 0.5$ and, $V = 0.1$.

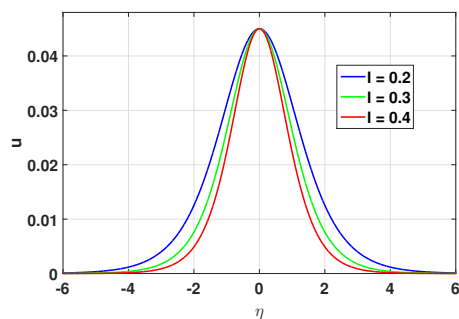


Figure 5. Impact of l on the solitary wave characteristics of system (2.1) with $\kappa = 0.2$, $\beta = 0.4$, $h = 0.1$, $a = 0.5$ and, $V = 0.1$.

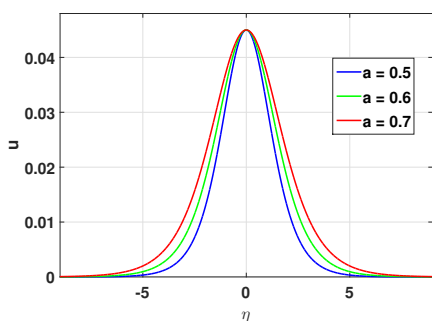


Figure 6. Impact of a on the solitary wave characteristic of system (2.1) with $\kappa = 0.2$, $\beta = 0.4$, $h = 0.1$, $l = 0.2$ and, $V = 0.1$.

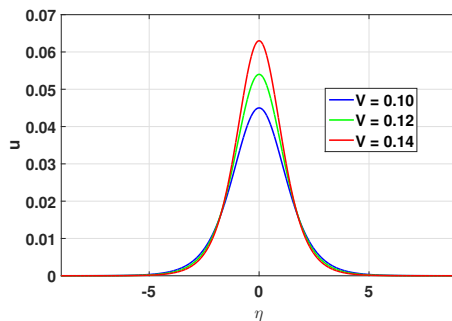


Figure 7. Impact of V on the solitary wave characteristics of system (2.1) with $\kappa = 0.2$, $\beta = 0.4$, $h = 0.1$, $l = 0.2$ and $a = 0.5$.

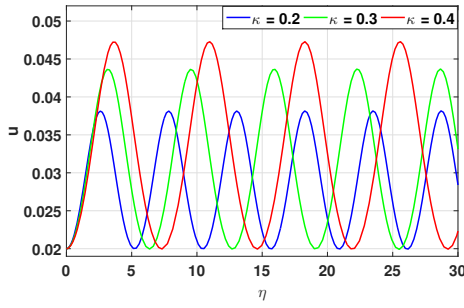


Figure 8. Impact of κ on the periodic wave characteristics of system (2.1) with $\beta = 0.4$, $h = 0.1$, $l = 0.2$, $a = 0.5$ and, $V = 0.1$.

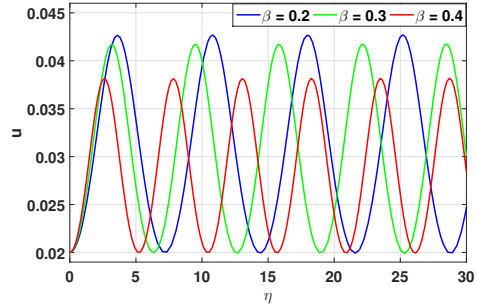


Figure 9. Impact of β on the periodic wave characteristics of system (2.1) with $\kappa = 0.2$, $h = 0.1$, $l = 0.2$, $a = 0.5$ and, $V = 0.1$.

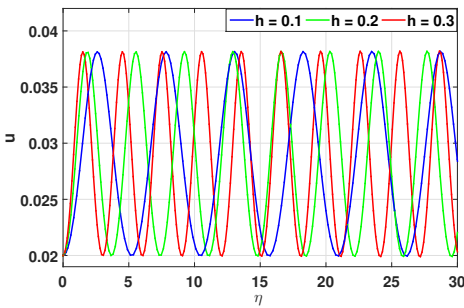


Figure 10. Impact of h on the periodic wave characteristics of system (2.1) with $\kappa = 0.2$, $\beta = 0.4$, $l = 0.2$, $a = 0.5$ and, $V = 0.1$.

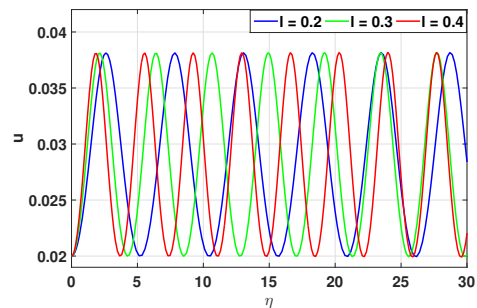


Figure 11. Impact of l on the periodic wave characteristics of system (2.1) with $\kappa = 0.2$, $\beta = 0.4$, $h = 0.1$, $a = 0.5$ and, $V = 0.1$.

The figures 8-13 shows the impact of the parameters material constants (κ and β), characteristic axial displacement (h), characteristic wavelength (l), speed of traveling wave (V) and radius of the rod (a) on the periodic wave solutions of the system (3.2) in correspond to the periodic orbit encircling the center E_1 .

Figure 8 clearly shows that as the material constant κ increases, both the amplitude as well as the width of the periodic wave grow. In contrast, as depicted in Figure 9, both the amplitude as well as width of the wave decrease as the material constant β increases. Figures 10 and 11 indicate that the characteristic axial displacement (h) and characteristic wavelength (l) have no effect on the wave amplitude; however, the wave width decreases as h and l increase. Additionally, Figure 12 demonstrates that the radius (a) has no impact on the wave amplitude, but the wave width increases with a larger a . Finally, Figure 13 reveals that increasing the value of V raises the amplitude of the wave while reducing its width.

5 Two-soliton solution and the phase shift

The Equation (2.1) can also be expressed as

$$\frac{\partial u}{\partial \tau} - \hat{C}u \frac{\partial u}{\partial \xi} + D \frac{\partial^3 u}{\partial \xi^3} = 0, \quad (5.1)$$

where $\hat{C} = -C$. Let us think of ξ as $\bar{\xi}D^{\frac{1}{3}}$, u as $6\bar{u}\hat{C}^{-1}D^{\frac{1}{3}}$ and τ as $\bar{\tau}$, then equation (5.1) can be written in the standard form as

$$\frac{\partial \bar{u}}{\partial \bar{\tau}} - 6\bar{u} \frac{\partial \bar{u}}{\partial \bar{\xi}} + \frac{\partial^3 \bar{u}}{\partial \bar{\xi}^3} = 0. \quad (5.2)$$

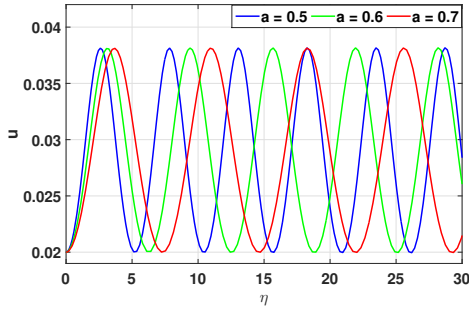


Figure 12. Impact of a on the periodic wave characteristics of system (2.1) with $\kappa = 0.2$, $\beta = 0.4$, $h = 0.1$, $l = 0.2$ and, $V = 0.1$.

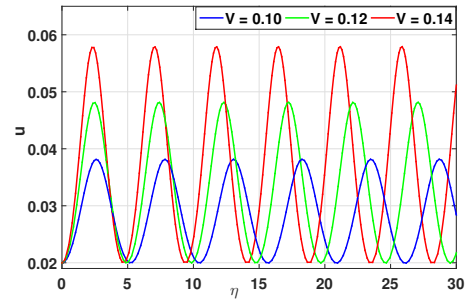


Figure 13. Impact of V on the periodic wave characteristics of system (2.1) with $\kappa = 0.2$, $\beta = 0.4$, $h = 0.1$, $l = 0.2$ and $a = 0.5$.

We now apply the transformation $\bar{u} = -2(\log f)_{\bar{\xi}\bar{\xi}}$, which transforms equation (5.2) into the Hirota bilinear form, as presented in [36, 37].

$$(H_{\bar{\xi}}H_{\bar{\tau}} + H_{\bar{\xi}}^4)[f \cdot f] = 0, \quad (5.3)$$

Here, H is known as Hirota's operator. To get the two-soliton solution, substitute $g = 1 + \epsilon(e^{\bar{\theta}_1} + e^{\bar{\theta}_2}) + \epsilon^2 g_2$ in equation (5.3), where $\bar{\theta}_i = k_i \bar{\xi} + \omega_i \bar{\tau}$, $i = 1, 2$. Comparing different powers of ϵ we obtain $\omega_i = -k_i^3$, $g_2 = a_{12}e^{\bar{\theta}_1 + \bar{\theta}_2}$, where a_{12} is the phase shift given by $a_{12} = \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2}$.

Finally, by setting $\epsilon = 1$, we can derive the two-soliton solution as follows:

$$\bar{u} = -2 \frac{k_1^2 e^{\bar{\theta}_1} + k_2^2 e^{\bar{\theta}_2} + a_{12}(k_2^2 e^{\bar{\theta}_1} + k_1^2 e^{\bar{\theta}_2})e^{\bar{\theta}_1 + \bar{\theta}_2} + 2(k_1 - k_2)^2 e^{\bar{\theta}_1 + \bar{\theta}_2}}{1 + e^{\bar{\theta}_1} + e^{\bar{\theta}_2} + a_{12}e^{\bar{\theta}_1 + \bar{\theta}_2}}. \quad (5.4)$$

Hence, the two-soliton solution of equation (2.1) is given by

$$u = 12 \frac{D^{1/3}}{\delta_1} \frac{k_1^2 e^{\theta_1} + k_2^2 e^{\theta_2} + a_{12}(k_2^2 e^{\theta_1} + k_1^2 e^{\theta_2})e^{\theta_1 + \theta_2} + 2(k_1 - k_2)^2 e^{\theta_1 + \theta_2}}{1 + e^{\theta_1} + e^{\theta_2} + a_{12}e^{\theta_1 + \theta_2}}, \quad (5.5)$$

where $\theta_i = \frac{k_i}{D^{1/3}}\xi - k_i^3\tau$.

As $\tau \gg 1$, the solution presented in equation (5.5) evolves into a combination of two separate soliton solutions:

$$u_{0(2)} \sim - \sum_{i=1}^2 A_i \text{sech}^2 \left[\frac{k_i}{2D^{1/3}} (\xi - k_i^2 D^{1/3} \tau + \Delta_i) \right], \quad (5.6)$$

Here A_i represents the amplitude defined by $A_i = \frac{3D^{1/3}k_i^2}{2\hat{C}}$, $i = 1, 2$. As a consequence of the interaction, the phase shifts of the solitons are given by $\Delta_{1,2} = \frac{2D^{1/3} \log \left| \frac{k_2 - k_1}{k_2 + k_1} \right|}{k_{1,2}}$.

Figure 14 shows the propagation of two-soliton as τ changes from -200 to 0 to 200 . It is evident that the soliton with the larger amplitude travels more quickly than the one with the smaller amplitude. Additionally, the solitons preserve their shape and size as they propagate.

Figure 15 demonstrates that as the material constant (κ) increases, the amplitude of the two-soliton wave rises. Conversely, from Figure 16, increasing the material constant (β) results in a decrease in the wave's amplitude. Figures 17 and 18 indicate that the amplitude of the two-soliton wave diminishes as the characteristic axial displacement (h) and characteristic wavelength (l) increase. Similarly, Figures 19 and 20 demonstrate that the amplitude of the two-soliton wave increases with the radius (a), while it decreases as the speed of the traveling wave

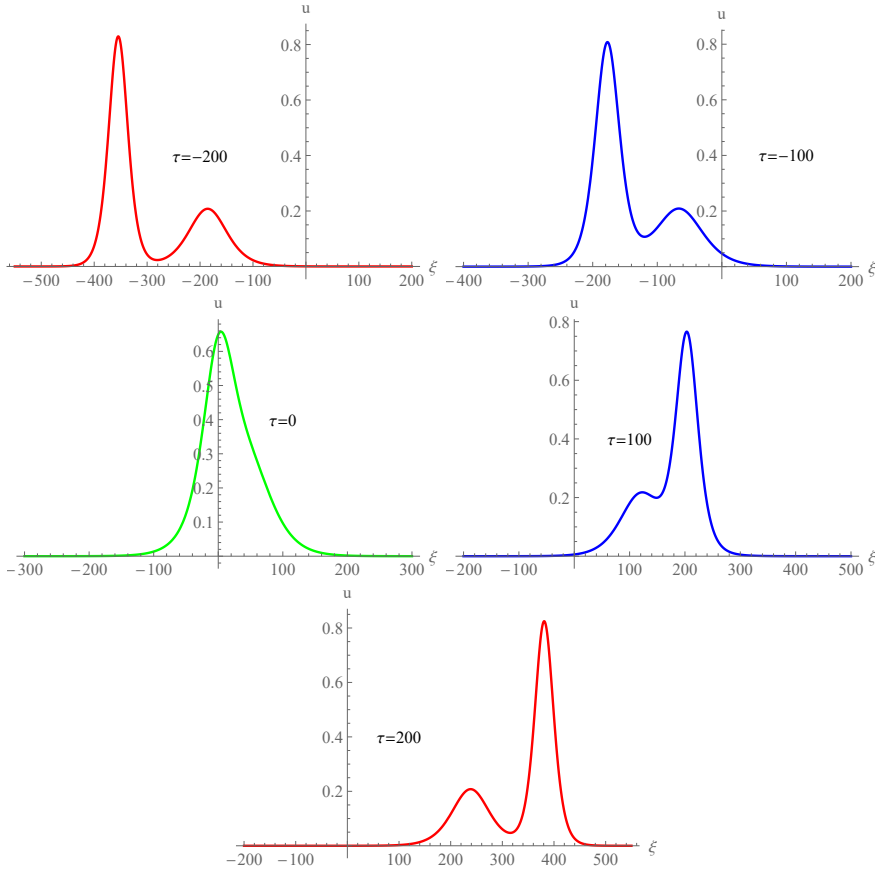


Figure 14. Comprehensive two-soliton profile for various values of τ with $k_1 = 0.2$, $k_2 = 0.4$, $\kappa = 0.1$, $\beta = 0.4$, $h = 0.1$, $l = 0.2$, $a = 0.5$ and $V = 0.1$.

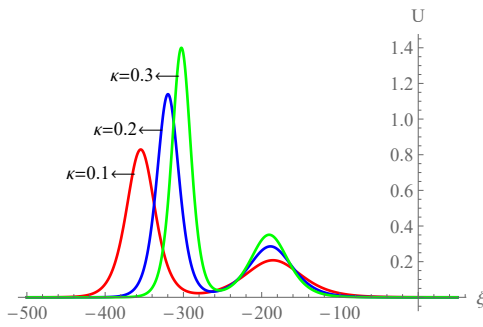


Figure 15. Impact of κ on two-soliton profile with $\tau = -200$, $k_1 = 0.2$, $k_2 = 0.4$, $\beta = 0.4$, $h = 0.1$, $l = 0.2$, $a = 0.5$ and $V = 0.1$.

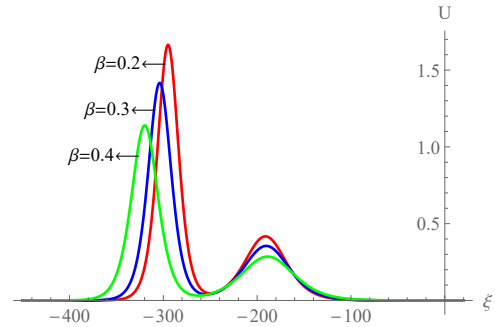


Figure 16. Impact of β on two-soliton profile with $\tau = -200$, $k_1 = 0.2$, $k_2 = 0.4$, $\kappa = 0.2$, $h = 0.1$, $l = 0.2$, $a = 0.5$ and $V = 0.1$.

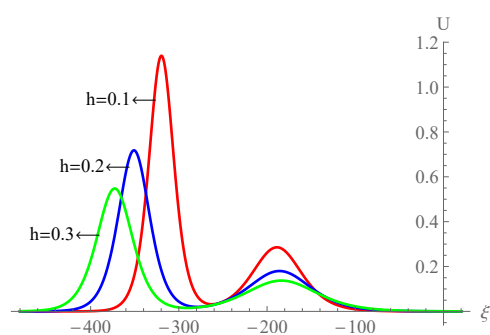


Figure 17. Impact of h on two-soliton profile with $\tau = -200$, $k_1 = 0.2$, $k_2 = 0.4$, $\kappa = 0.2$, $\beta = 0.4$, $l = 0.2$, $a = 0.5 = 1$ and $V = 0.1$.

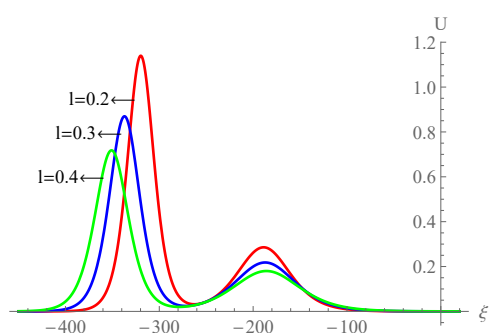


Figure 18. Impact of l on two-soliton profile with $\tau = -200$, $k_1 = 0.2$, $k_2 = 0.4$, $\kappa = 0.2$, $\beta = 0.4$, $h = 0.1$, $a = 0.5 = 1$ and $V = 0.1$.

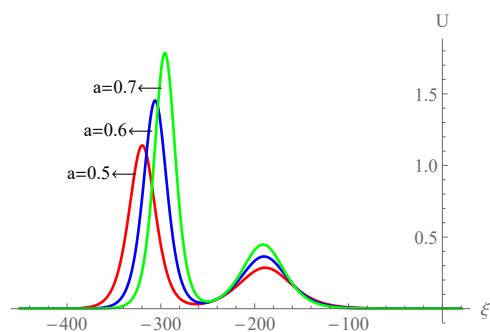


Figure 19. Impact of a on two-soliton profile with $\tau = -200$, $k_1 = 0.2$, $k_2 = 0.4$, $\kappa = 0.2$, $\beta = 0.4$, $h = 0.1$, $l = 0.2$ and $V = 0.1$.

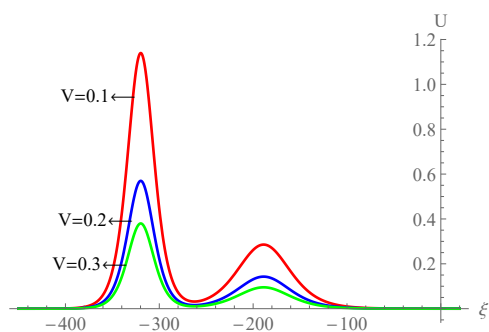


Figure 20. Impact of V on two-soliton profile with $\tau = -200$, $k_1 = 0.2$, $k_2 = 0.4$, $\kappa = 0.2$, $\beta = 0.4$, $h = 0.1$, $l = 0.2$ and $a = 0.5$.

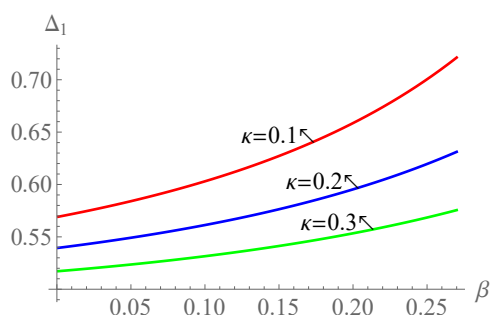


Figure 21. Variation of phase shifts for two-soliton interactions with respect to β , for different values of κ , with $k_1 = 2$, $k_2 = 0.2$, $h = 0.1$, $l = 0.2$, and $a = 0.5$.

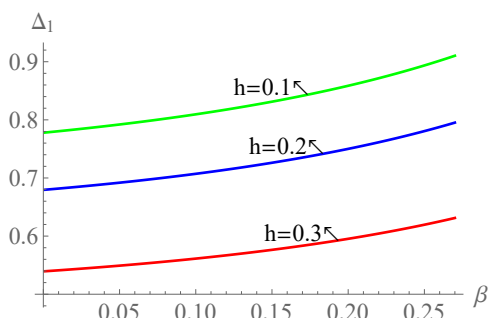


Figure 22. variation of Phase Shifts for two-soliton with respect to β for different values of h with $k_1 = 2$, $k_2 = 0.2$, $\kappa = 0.2$, $l = 0.2$ and $a = 0.5$.

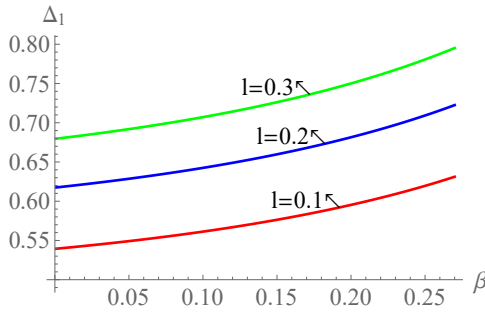


Figure 23. variation of Phase Shifts for two-soliton with respect to β for different values of l with $k_1 = 2$, $k_2 = 0.2$, $\kappa = 0.2$, $h = 0.1$ and $a = 0.5$.

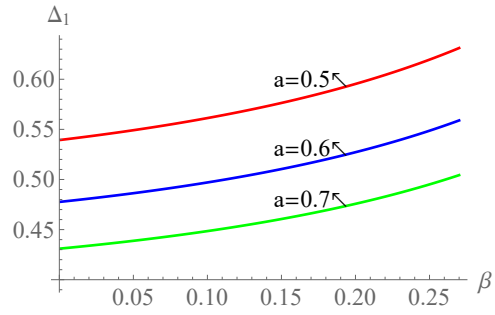


Figure 24. variation of Phase Shifts for two-soliton with respect to β for different values of a with $k_1 = 2$, $k_2 = 0.2$, $\kappa = 0.2$, $h = 0.1$ and $l = 0.2$.

(V) rises. It can also be observed that all the parameters have no effect on the width of the two-soliton.

Figures 21 to 24 depict the impact to the phase shifts of the two-soliton system with respect to β was illustrated for distinct values of κ , h , l , and a .

6 Conclusion

In this manuscript, we explored the dynamics of long finite-amplitude waves within a compressible rod made of Mooney-Rivlin materials. First, we applied a traveling wave transformation to convert the basic governing equations and the evolution equation for finite-amplitude waves in the compressible rod into a dynamical system. We then conducted a phase plane analysis to examine the qualitative behavior of this system. The phase portrait revealed that there exists two fixed points at E_0 and E_1 . We also determined the periodic and solitary wave solutions corresponding to homoclinic and periodic orbits, respectively, and investigated the effects of parameters material constants (κ and β), characteristic axial displacement (h), characteristic wavelength (l), radius of the rod (a) and speed of traveling wave (V) on these solutions. Additionally, the propagation and impact of different parameters on the two-soliton were illustrated. The phase shift and its variation for distinct values of parameters were also shown.

It was found that both the amplitude and width of the solitary wave increase as the material constant κ rises, while an increase in the material constant β contributes to a decrease in both amplitude and width. The characteristic axial displacement (h), wavelength (l), and rod radius (a) have no effect on the wave's amplitude. However, the wave's width decreases with increasing h and l , and increases with a larger a . Additionally, a higher wave speed (V) results in greater wave amplitude, but the width remains unchanged. For periodic waves, both amplitude and width increase with the material constant κ , whereas increasing β causes a reduction in both properties. The axial displacement (h) and wavelength (l) do not affect the amplitude, but the wave's width decreases as these parameters increase. The radius (a) does not impact the amplitude, but a larger radius increases the width. A higher wave speed (V) increases the amplitude but reduces the width. For the two-soliton wave, an increase in κ raises the amplitude, while an increase in β reduces it. The amplitude also decreases with higher values of axial displacement (h) and wavelength (l), but increases with a larger radius (a) and decreases with greater wave speed (V). Notably, none of these parameters affect the width of the two-soliton wave. Additionally, the impact on the phase shifts of the two-soliton system with respect to β was illustrated for distinct values of κ , h , l , and a . This study will help in understanding the dynamics of nonlinear waves compressible rod composed of Mooney-Rivlin material.

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