

Ellis Fluid Flow in a Finite Length Tube Driven by Peristalsis

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Abstract This study investigates the flow of an Ellis fluid within a finite-length tube driven by both systematic peristaltic waves and single waves, under the assumptions of low Reynolds number and wave number. Under these assumptions, the expressions for both axial and transverse velocities, pressure distribution, volume flow rate, and time-averaged volume flow rate are obtained analytically while the pressure gradient is evaluated numerically. The impact of varying wave amplitudes and fluid parameters on the axial velocity, velocity vector field, pressure within the tube, volume flow rate, and particle trajectories is analysed for single and multiple peristaltic wave trains. This analysis reveals that, quantitatively, the pressure distribution for the Ellis fluid deviates from the Newtonian case, exhibiting lower pressure values within the tube. Consequently, the velocities are also affected, leading to a non-linear relationship between the time-averaged volume flow rate and the pressure head for non-Newtonian fluids. The parameters of the Ellis fluid have a remarkable impact both qualitatively and quantitatively on axial velocity. These findings have potential applications in biomedical device design, physiological modeling of non-Newtonian fluid transport (such as flows in the gastrointestinal tract, ureter, urethra, oviduct, and lymphatic ducts) and industrial peristaltic systems involving shear-thinning fluids.

1 Introduction

Peristaltic transport, a fundamental mechanism in both biological and industrial systems, plays a crucial role in various physiological functions. Due to its significance, this pumping process has been extensively studied. Early models, such as those by Shapiro et al. [1], focused on steady peristaltic transport in infinitely long tubes. However, subsequent work by Li and Brasseur [2] expanded this understanding to include non-steady peristaltic transport in finite-length tubes by addressing the complexities introduced by finite tube lengths and arbitrary wave shapes. Using the approaches of Shapiro et al. [1] or Li and Brasseur [2] or numerical methods, various authors studied the flow analysis of peristalsis by considering various flow conditions and geometries for both Newtonian and non-Newtonian models.

In many physiological and industrial applications, the working fluid exhibits non-Newtonian behavior, where viscosity varies with shear rate. Such fluids include blood, synovial fluid, polymer solutions, and food slurries. Modeling these flows accurately is essential for predicting transport efficiency, wall shear stress, and pressure profiles. Among various rheological models, the Ellis fluid model offers a versatile framework for shear-thinning behavior without yield stress. Unlike power-law fluids, which lack a Newtonian plateau, or Casson and Bingham fluids, which exhibit yield stress behavior, Ellis fluids offer a continuous transition from Newtonian to shear-thinning regimes without a yield threshold. This makes them particularly suitable for modeling biological and industrial fluids where gradual viscosity reduction is observed. The present study extends peristaltic transport analysis to Ellis fluids in finite-length tubes, a configuration not previously explored in this context. It captures the transition from Newtonian behavior at low shear rates to power-law behavior at high shear rates, governed by three key parameters: the zero shear viscosity μ_0 , τ_0 the shear stress at which the viscosity is half μ_0 and the shear-thinning index α . When, $\alpha = 1$ or $\tau_0 \rightarrow \infty$ the Ellis model reduces to a Newtonian fluid, and it approximates a power-law fluid under some special cases. This flexibility makes the Ellis model particularly suitable for analyzing fluid transport in biological conduits and engineered systems where fluid properties vary dynamically.

Shapiro et al. [1] investigated peristaltic pumping with long wavelengths at low Reynolds numbers. This study examined peristaltic pumping under conditions where the Reynolds number is low and the wavelength to diameter ratio is large. The authors used analytical techniques to derive solutions for the flow fields and pressure distributions, providing insights into physiological implications such as reflux and trapping. Srivastava and Srivastava [3] studied the peristaltic transport of non-Newtonian fluids, specifically power-law fluids, in the vas deferens and small intestine under the assumptions of long wavelength and zero Reynolds number. The study compared the results with Newtonian fluid models under various conditions and also with suitable experimental results. To overcome the assumptions made by previous analytical studies, Takabatake et al. [4] conducted a numerical analysis of peristaltic pumping in circular cylindrical tubes, employing finite difference methods to solve the Navier-Stokes equations and analyze peristaltic mixing and transport. Greater attention was given to analysing the trapping and reflux phenomena. This study finds that the peristaltic mixing and transport are greater in the cylinder case compared to the channel case. Misra and Pandey [5] considered peristaltic transport in tapered tubes, using perturbation techniques to derive approximate solutions for the flow fields. This study analysed flow for various cases, such as low and high Reynolds numbers and wave numbers. Li and Brasseur [2] generalized the classical lubrication theory model of steady peristaltic transport to arbitrary wave shapes and wavenumbers in finite-length tubes, using simulations to analyze the effects of tube length, wave number, and occlusion on pumping performance.

Chu and Fang [6] investigated the peristaltic transport in slip flow, examining the effects of slip conditions on peristaltic transport in microchannels. This study indicated that reflux is high due to slip-flow effects in the case of free pumping. They used analytical techniques to solve the governing equations, providing insights into the behavior of fluids under slip flow conditions. Mishra and Rao [7] studied the peristaltic transport of Newtonian fluids in channels with asymmetric geometries, highlighting the impact of channel shape on fluid flow. This study showed that the trapping and reflux of the fluid are less in the case of an asymmetric channel compared to the symmetric channel, and occur maximum in the case of a symmetric channel. Peristaltic transport of a Herschel-Bulkley fluid in an inclined tube was studied by [8], using perturbation techniques to derive approximate solutions and considering the effects of fluid properties and tube inclination on flow behavior using the approach of Shapiro et al. [1]. Later, Hariharan et al. [9] examined the peristaltic transport of non-Newtonian fluid in a diverging tube with different waveforms, considering various waveforms and their effects on fluid dynamics. This study has shown that the non-Newtonian nature of the flow increases the flow reversal behaviour, and among all the peristaltic waves considered, the triangular waves have shown the best pumping performance.

Nagarani [10] studied the peristaltic transport of a Casson fluid in an inclined channel, using analytical techniques to derive solutions for the flow fields and focusing on the rheological properties and flow behavior. This study investigated the combined effects of yield stress and angle of inclination on the peristaltic transport. Pandey and Chaube [11] investigated the peristaltic transport of a viscoelastic fluid in a tube of non-uniform cross-section, using perturbation methods to solve the governing equations and providing a detailed mathematical analysis. This model is analysed for the cases of low Reynolds number and low wave number. The study showed that the more elastic the fluid is, the lower the flow rate, and hence the pump has to work more to maintain the same flow rate compared to the Newtonian fluid. Tripathi et al. [12] considered the influence of slip conditions on the peristaltic transport of a viscoelastic fluid with a fractional Burger's model, using analytical techniques to derive solutions using the approach of Shapiro et al. [1] and focusing on the effects of various parameters on flow behavior. Chrispell and Fauci [13] analyzed the peristaltic pumping of solid particles immersed in a viscoelastic fluid, using simulations to study the interactions between particles and fluid flow. This study has been particularly applied to flow in biological situations, such as the transport of urine and the movement of ova.

Nagarani and Lewis [14] investigated the peristaltic flow of a Casson fluid in an annulus, using analytical techniques to derive solutions for the flow fields and analyzing the effects of fluid properties on flow behavior. Husseny et al. [15] examined the flow separation of peristaltic transport for Maxwell fluid between two coaxial tubes. This analysis discussed the effects of the endoscope on peristalsis. This study is developed under the assumption of small wave numbers.

Tripathi et al. [16] studied non-steady peristaltic propulsion with exponential variable viscosity, focusing on transport through the digestive system. This model considered the effects of variable viscosity on the efficiency and effectiveness of peristaltic transport, providing insights into the complex interactions between fluid properties and peristaltic motion, with implications for understanding digestive processes and designing biomedical devices. Peristaltic pumping of an Ellis fluid in an inclined asymmetric channel was investigated by [17], focusing on the effects of inclination and fluid properties on flow characteristics. The authors used simulations to analyze flow behavior, considering various parameters such as the Ellis fluid index, inclination angle, and channel asymmetry. Kamalakkannan et al. [18] studied the peristaltic transport of a couple stress fluid through a horizontal tapered asymmetric channel, focusing on the effects of fluid parameters on velocity and pressure. The results showed that the pressure distribution and wall shear stress are influenced by the fluid parameters and the shape of the peristaltic wave.

This study investigates the flow of an Ellis fluid within a finite-length tube, with a focus on the effects of the fluid parameters and peristaltic wave amplitude on velocity, pressure distribution, wall shear stress, time-averaged volume flow rate, and the particle trajectories in the cases of single and multiple wave trains. The flow is studied under the assumption of low Reynolds number and low wave number. Under these assumptions, the expressions for axial and transverse velocities, time-averaged flow, and shear stress at the wall are obtained analytically, and the pressure gradient is calculated numerically. The effect of peristaltic wave amplitude and the fluid parameters on the axial velocity, pressure distribution, wall shear stress, time-averaged volume flow rate and velocity field are analysed in section 3. These results are relevant for designing biomedical devices and understanding physiological processes involving non-Newtonian fluids.

2 Mathematical Formulation and Solution

We considered the flow of an incompressible non-Newtonian fluid (Ellis fluid) in a finite length tube. The flow is driven by peristaltic waves along the tube walls as shown in figure 1 and is described by the equation

$$\hat{h} = \epsilon + 0.5\hat{A} \left(1 - \cos 2\pi \left(\frac{\hat{x} - c\hat{t}}{\lambda} \right) \right), \quad (2.1)$$

where ϵ is the occlusion, a is the mean radius of the tube, $\hat{A}$ the amplitude of the peristaltic wave, c the speed of the wave and λ is the wavelength of the peristaltic wave. For a single peristaltic wave we have

$$\hat{h} = \begin{cases} \epsilon + 0.5\hat{A} \left(1 - \cos 2\pi \left(\frac{\hat{x} - c\hat{t}}{\lambda} \right) \right), & \hat{t} \leq \hat{x} \leq \hat{t} + \lambda, \\ \epsilon, & \text{elsewhere,} \end{cases} \quad (2.2)$$

where in both 2.1 and 2.2, $0 \leq \hat{x} \leq L$ and L is the tube length.

The governing flow equations are given as

$$\frac{1}{\hat{r}} \frac{\partial}{\partial \hat{r}} (\hat{r} \hat{v}) + \frac{1}{\hat{r}} \frac{\partial \hat{w}}{\partial \hat{\theta}} + \frac{\partial \hat{u}}{\partial \hat{x}} = 0, \quad (2.3)$$

$$\begin{aligned} \rho \left(\frac{\partial \hat{v}}{\partial \hat{t}} + \hat{v} \frac{\partial \hat{v}}{\partial \hat{r}} + \frac{\hat{w}}{\hat{r}} \frac{\partial \hat{v}}{\partial \hat{\theta}} + \hat{u} \frac{\partial \hat{v}}{\partial \hat{x}} - \frac{\hat{w}^2}{\hat{r}} \right) &= -\frac{\partial \hat{p}}{\partial \hat{r}} + \frac{1}{\hat{r}} \frac{\partial}{\partial \hat{r}} (\hat{r} \hat{\tau}_{rr}) + \frac{1}{\hat{r}} \frac{\partial \hat{\tau}_{r\theta}}{\partial \hat{\theta}} + \frac{\partial \hat{\tau}_{rx}}{\partial \hat{x}} + \frac{\hat{\tau}_{\theta\theta}}{\hat{r}} - \frac{\hat{\tau}_{\theta r}}{\hat{r}}, \\ \rho \left(\frac{\partial \hat{w}}{\partial \hat{t}} + \hat{v} \frac{\partial \hat{w}}{\partial \hat{r}} + \frac{\hat{w}}{\hat{r}} \frac{\partial \hat{w}}{\partial \hat{\theta}} + \hat{u} \frac{\partial \hat{w}}{\partial \hat{x}} + \frac{\hat{v} \hat{w}}{\hat{r}} \right) &= -\frac{1}{\hat{r}} \frac{\partial \hat{p}}{\partial \hat{\theta}} + \frac{1}{\hat{r}} \frac{\partial}{\partial \hat{r}} (\hat{r} \hat{\tau}_{\theta r}) + \frac{1}{\hat{r}} \frac{\partial \hat{\tau}_{\theta\theta}}{\partial \hat{\theta}} + \frac{\partial \hat{\tau}_{\theta x}}{\partial \hat{x}} + \frac{\hat{\tau}_{r\theta}}{\hat{r}} - \frac{\hat{\tau}_{rr}}{\hat{r}}, \\ \rho \left(\frac{\partial \hat{u}}{\partial \hat{t}} + \hat{v} \frac{\partial \hat{u}}{\partial \hat{r}} + \frac{\hat{w}}{\hat{r}} \frac{\partial \hat{u}}{\partial \hat{\theta}} + \hat{u} \frac{\partial \hat{u}}{\partial \hat{x}} \right) &= -\frac{\partial \hat{p}}{\partial \hat{x}} - \frac{1}{\hat{r}} \frac{\partial}{\partial \hat{r}} (\hat{r} \hat{\tau}_{xr}) + \frac{1}{\hat{r}} \frac{\partial \hat{\tau}_{x\theta}}{\partial \hat{\theta}} + \frac{\partial \hat{\tau}_{xx}}{\partial \hat{x}}. \end{aligned} \quad (2.4)$$

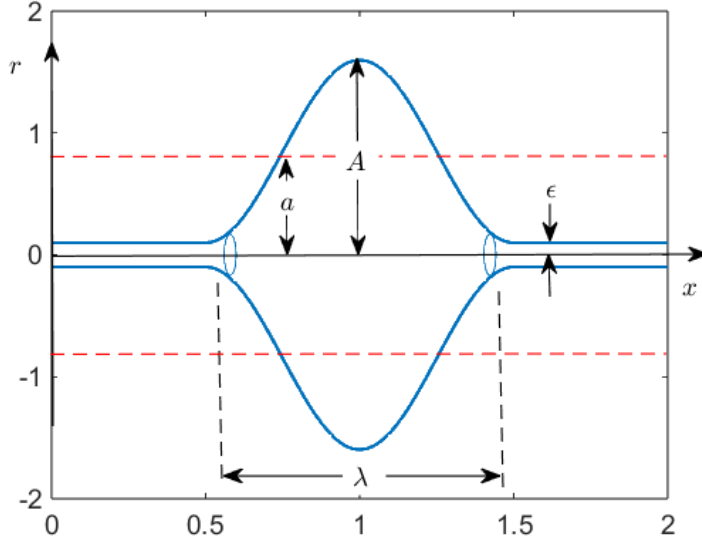


Figure 1. Wall shape for single peristaltic wave.

The Ellis fluid constitutive relation is given by

$$\hat{\tau}_{ij} = \left(\frac{\mu_0}{1 + \left| \frac{II_{\hat{\tau}}}{2\tau_0} \right|^{\frac{\alpha-1}{2}}} \right) \hat{\gamma}_{ij}, \quad (2.5)$$

subject to:

$$\tau_{\hat{r}\hat{x}}|_{\hat{r}=0} = finite, \quad \hat{u}|_{\hat{r}=\hat{h}} = 0, \quad \frac{\partial \hat{u}}{\partial \hat{r}}|_{\hat{r}=0} = 0, \quad \hat{v}|_{\hat{r}=0} = finite, \quad \hat{v}|_{\hat{r}=\hat{h}} = \frac{\partial \hat{h}}{\partial \hat{t}}, \quad (2.6)$$

where $\langle \hat{v}, \hat{w}, \hat{u} \rangle$ is the velocity vector, ρ is the density of the fluid, $\hat{\tau}$ is the stress tensor μ_0 is the viscosity, α and τ_0 are fluid parameters, $\hat{\gamma}$ is the rate of deformation tensor and $II_{\hat{\tau}}$ is the second invariant of the stress tensor.

The dimensionless quantities used are

$$x = \frac{\hat{x}}{\lambda}, \quad r = \frac{\hat{r}}{a}, \quad u = \frac{\hat{u}}{c}, \quad v = \frac{\hat{v}}{c\delta}, \quad t = \frac{c\hat{t}}{\lambda}, \quad A = \frac{\hat{A}}{a}, \quad p = \frac{a\delta\hat{p}}{\mu_0 c}, \quad \delta = \frac{a}{\lambda},$$

$$Re = \frac{\rho a c}{\mu_0}, \quad \tau_{rx} = \frac{a\tau_{\hat{r}\hat{x}}}{\mu_0 c}, \quad \tau_{rr} = \frac{a\tau_{\hat{r}\hat{r}}}{\mu_0 c}, \quad \tau_{xx} = \frac{\lambda\tau_{\hat{r}\hat{x}}}{\mu_0 c}. \quad (2.7)$$

We apply the conditions that the flow is axi-symmetric and fully developed enough to neglect the entrance effects. Also, the flow is studied under the assumptions of low Reynolds number and low wavenumber. With these assumptions, the flow equations 2.4-2.5 and constitutive relation 2.6 are reduced to

$$\frac{1}{r} \frac{\partial}{\partial r} \{ r v \} + \frac{\partial u}{\partial x} = 0, \quad (2.8)$$

$$\frac{\partial p}{\partial r} = 0, \quad (2.9)$$

$$\frac{\partial p}{\partial x} = -\frac{1}{r} \frac{\partial}{\partial r} \{ r \tau_{rx} \}, \quad (2.10)$$

$$\tau_{rx} = \frac{1}{1 + \left| \frac{\tau_{rx}}{\tau_0} \right|^{\alpha-1}} \left\{ -\frac{\partial u}{\partial r} \right\}. \quad (2.11)$$

The dimensionless wall shape is now expressed as

$$h = \frac{\epsilon}{a} + 0.5A(1 - \cos 2\pi(x - t)). \quad (2.12)$$

To maintain a constant volume within each peristaltic wave the occlusion is related to the amplitude by

$$A = \frac{4}{3} \left(\sqrt{\frac{3}{2} - \frac{1}{2} \left(\frac{\epsilon}{a} \right)^2} - \frac{\epsilon}{a} \right). \quad (2.13)$$

The corresponding conditions in the non-dimensional form are

$$\tau_{rx}|_{r=0} = \text{finite}, \quad u|_{r=h} = 0, \quad \frac{\partial u}{\partial r} \Big|_{r=0} = 0, \quad v|_{r=0} = \text{finite}, \quad v|_{r=h} = \frac{\partial h}{\partial t}. \quad (2.14)$$

From 2.10 we substitute τ_{rx} into equation 2.11 and solve for u using the boundary conditions in 2.14 to obtain

$$u = P \left\{ \frac{h^2 - r^2}{4} + k|P|^{\alpha+1} (h^{\alpha+3} - r^{\alpha+3}) \right\}, \quad (2.15)$$

where $P = -\frac{\partial p}{\partial x}$ and $k = \frac{1}{(\alpha+3)2^{\alpha+2}\tau_0^{\alpha+1}}$.

From 2.15 and 2.8 we solve for v subject to conditions in 2.14 to obtain

$$v = -\frac{\partial P}{\partial x} \left\{ \frac{1}{4} \left(\frac{h^2 r}{2} - \frac{r^3}{4} \right) + k|P|^{\alpha+1} \left(\frac{h^{\alpha+3} r}{2} - \frac{r^{\alpha+4}}{\alpha+5} \right) \right\} - P \left\{ \frac{hr}{4} \frac{\partial h}{\partial x} + \right. \\ \left. (\alpha+1)k|P|^\alpha \frac{\partial P}{\partial x} \text{sign}(P) \left(\frac{h^{\alpha+3} r}{2} - \frac{r^{\alpha+4}}{\alpha+5} \right) + (\alpha+3)k|P|^{\alpha+1} \frac{h^{\alpha+2} r}{2} \frac{\partial h}{\partial x} \right\}. \quad (2.16)$$

The volume flow rate is given by the expression

$$Q = 2\beta \int_0^h u r dr,$$

where $\beta = 1$ for multiple train waves and $\frac{L}{\lambda}$ for single waves [2]. Therefore, we have

$$Q = 2\beta P \left\{ \frac{h^4}{16} + \frac{k|P|^{\alpha+1}(\alpha+3)}{2(\alpha+5)} h^{\alpha+5} \right\}. \quad (2.17)$$

An important quantity of interest is the time averaged volume flow rate. We may now numerically evaluate the time averaged volume flow rate as using the expression

$$\bar{Q} = \frac{1}{T} \int_0^T Q dt. \quad (2.18)$$

The shear stress at the wall can be evaluated from the roots of the equation

$$G(\tau_{wall}) = \tau_{wall} \left(1 + \left| \frac{\tau_{wall}}{\tau_0} \right|^{\alpha+1} \right) + \frac{\partial u}{\partial r} \Big|_{r=h} = 0. \quad (2.19)$$

Here we solve for τ_{wall} numerically using Newton's method. Hence

$$\tau_{wall(i+1)} = \tau_{wall i} - \frac{G(\tau_{wall i})}{G'(\tau_{wall i})}, \quad (2.20)$$

where

$$G'(\tau_{wall}) = 1 + (\alpha+2) \left| \frac{\tau_{wall}}{\tau_0} \right|^{\alpha+1}, \quad (2.21)$$

and the initial approximation is

$$\tau_{wall 0} = P \left(-\frac{h}{2} - k|P|^{\alpha+1}(\alpha+3)h^{\alpha+2} \right). \quad (2.22)$$

In the expressions above, the pressure gradient P is unknown and must be evaluated. Using the final condition in 2.14 in 2.16, we get

$$-h \frac{\partial h}{\partial t} = \frac{\partial}{\partial x} \left\{ \frac{Ph^4}{16} + \frac{\alpha+3}{2(\alpha+5)} kP |P|^{\alpha+1} h^{\alpha+5} \right\}. \quad (2.23)$$

We now solve this differential equation for P by first integrating both sides of equation 2.23 with respect to x and numerically solving the resulting nonlinear equation for P . Therefore, we define the functions

$$F(P) = \frac{Ph^4}{16} + \frac{\alpha+3}{2(\alpha+5)} kP |P|^{\alpha+1} h^{\alpha+5} + \int_0^x h \frac{\partial h}{\partial t} ds + \gamma(t), \quad (2.24)$$

$$F'(P) = \frac{h^4}{16} + \frac{(\alpha+3)(\alpha+2)}{2(\alpha+5)} k |P|^{\alpha+1} h^{\alpha+5}. \quad (2.25)$$

Here,

$$\gamma(t) = \frac{\Delta p - \int_0^l \left\{ \frac{\int_0^x h \frac{\partial h}{\partial t} ds}{\frac{h^4}{16} + \frac{(\alpha+3)k |P|^{\alpha+1} h^{\alpha+5}}{2(\alpha+5)}} \right\} dx}{\int_0^l \left\{ \frac{1}{\frac{h^4}{16} + \frac{(\alpha+3)k |P|^{\alpha+1} h^{\alpha+5}}{2(\alpha+5)}} \right\} dx}. \quad (2.26)$$

is a constant of integration which is only dependent on t and Δp is the pressure head. From $F'(P)$ we see that $F(P)$ is increasing monotonically and hence has a single root. Using Newton's method, we have

$$P_{i+1} = P_i - \frac{F(P_i)}{F'(P_i)}, \quad (2.27)$$

where the initial approximation to the pressure gradient is taken as the Newtonian result obtained by [2], and is expressed as

$$P_0 = -\frac{16}{h^4} \left\{ \gamma_0(t) + \int_0^x h \frac{\partial h}{\partial t} ds \right\}, \quad (2.28)$$

where

$$\gamma_0(t) = \frac{\Delta p - \int_0^l \frac{16}{h^4} \int_0^x h \frac{\partial h}{\partial t} ds dx}{\int_0^l \frac{16}{h^4} dx}. \quad (2.29)$$

Newton's method was iterated until convergence was achieved with a tolerance of 10^{-4} , assessed with the absolute error in the Euclidean norm of the pressure gradient matrix.

Now

$$p(x, t) = p(0) - \int_0^x P dx \quad (2.30)$$

is the pressure distributed over the tube and $\Delta p = p(l) - p(0) = -\int_0^l P dx$ is the pressure head mentioned above which was prescribed by keeping the pressure at the inlet $p(0)$, and the pressure at the outlet $p(l)$, constant with respect to time. By taking the limiting case $\tau_0 \rightarrow \infty$, the Ellis fluid model reduces to Newtonian behavior. Under this condition, the derived expressions for pressure distribution, wall shear stress, particle trajectories, volume flow rate, and axial velocity coincide with the results reported by Li and Brasseur [2], thereby validating the present analysis against established Newtonian benchmarks.

3 Results and Discussion

The effects of the Ellis fluid parameters α , τ_0 , and the occlusion ϵ on the pressure distribution in the tube, wall shear stress, and volume flow rate in the case of a single peristaltic wave and multiple wave trains are analysed. Also, the particle trajectories were drawn and analysed in these cases. Here α is taken from 2 to 4, τ_0 is given $\{2, 4, \infty\}$, the occlusion $\epsilon = \{0.01, 0.05, 0.1, 0.2, 0.3, 0.6\}$ and the corresponding amplitudes of the peristaltic waves are $\{1.6196, 1.5656, 1.4969, 1.3554, 1.2083, 0.7319\}$.

3.1 Pressure Distribution

It is important to note that in each figure in which the case $\tau_0 \rightarrow \infty$ is included, the Newtonian limit which were produced by [2]. This deliberate inclusion enables direct visual and quantitative comparison between Newtonian and Ellis fluid behavior, highlighting the impact of shear-thinning effects.

Figures 2 and 3 show the pressure distributed along the tube with varying occlusions and time, for multiple wave trains as well as a single peristaltic wave present within the tube. Each plot has superimposed the wall shape at that instant of time and the Newtonian pressure profile for comparison with the profiles of varying values of the fluid parameters α and τ_0 . The purpose of imposing the wall shape is to understand the locations where the pressure distribution takes its peak values. The pressure distributed along the tube shows the highest pressure variations in the neighborhood of the occlusion for all instances of time. From these figures we observed that the higher pressure values are seen in the case of Newtonian fluid compared to the Ellis fluid. This is due to the shear-thinning behavior of Ellis fluids, which exhibit a reduction in apparent viscosity with increasing shear stress. As a result, less pressure is required to drive the flow through regions of high shear, such as near occlusions, compared to Newtonian fluids that maintain constant viscosity. Consequently, the pressure peaks observed in the Ellis fluid case are lower than those in the Newtonian case, with the magnitude of this reduction depending on the degree of deviation from Newtonian rheology.

Figures 2a and 2b show that with a decrease in the fluid parameter τ_0 the pressure variations, or pressure peaks observed near the occlusion, decrease for both cases of $\epsilon = 0.01$ and $\epsilon = 0.1$ respectively. This means that for shear-thinning fluids that deviate in rheology from Newtonian fluids, the pressure variation in the tube is less depending on the degree of this deviation. The occlusion of the tube also influences the scale of this difference in the pressure profile, since for smaller occluded tubes the difference in the pressure variation is more significant. The magnitude of the pressure in the tube decreased with an increase in the amplitude; however, the distribution profile of the pressure distribution expanded. This observation was made for single-wave and multiple-wave peristaltic pumping as seen in Figure 3. We also observed from Figures 2b and 2c that increasing the fluid parameter α reduces the pressure peaks observed, as this also indicates that the rheology of the fluid is divergent from the Newtonian case. A similar response is observed for single wave peristaltic pumping where the pressure peaks are highest for Newtonian fluids and reduce with a reduction in the fluid parameter τ_0 and an increase in the fluid parameter α as seen in Figure 3.

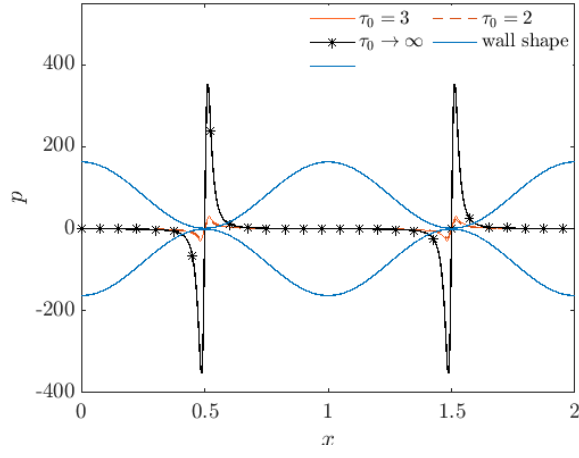
3.2 Wall Shear Stress

The variation of wall shear stress at the wall in the case of multiple wave trains was projected in Figure 4(a-b), which has the wall shape (multiple train waves) superimposed. It is noted that the shear stress at the walls shows a similar response as in the case of pressure distribution as the wall shear stress has its peaks at the occlusions of the tube. The effects of the fluid parameter τ_0 is a reduction of these peaks in shear stress with an increase in τ_0 , and a simultaneous vertical rise profile of the shear stress. An increase in the occlusion of the tube, results in a reduction in the size of these peaks and also in the magnitude of the shear stress. An increase in α results in a downward shift of the entire profile as the fluid takes on a non-Newtonian nature. This is also true for single-wave pumping.

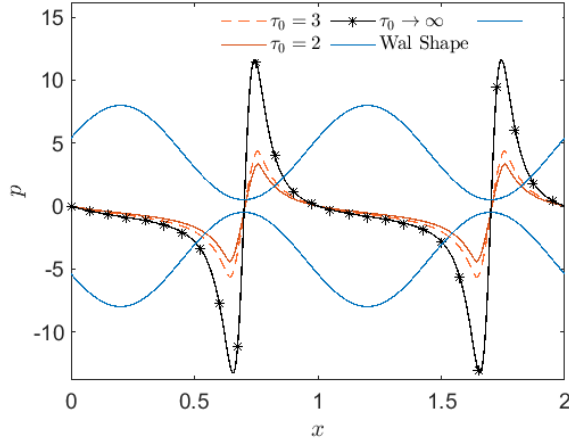
3.3 Axial Velocity

The axial velocity, as illustrated in Figures 5 and 6 show an apparent parabolic profile. The axial velocity profile for the Newtonian case for comparison is also placed in each plot. We see here that the velocity near the center of the tube is less for lower values of τ_0 and greater near the wall. Lower values of velocity imply a divergence of the fluid from the Newtonian case. This was deduced from figures 5a and 5b which illustrate the effects of τ_0 and α respectively. These plots are developed at different axial locations within the tube at varying values of occlusion.

(a)



(b)



(c)

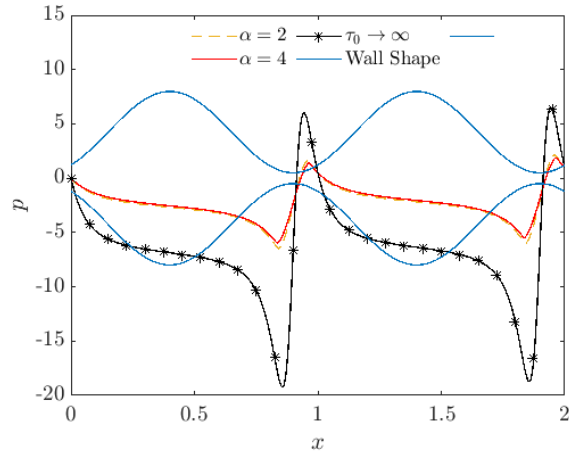
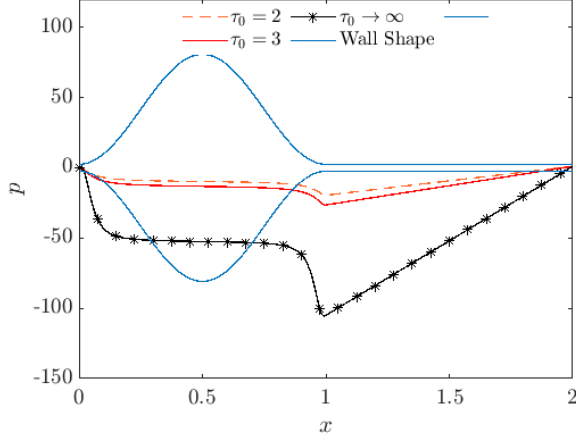


Figure 2. Pressure along the tube (Multiple wave train with $L = 2$) a) $t = 0.5$, $\epsilon = 0.01$, $A = 1.6196$, $\Delta p = 0$ and $\alpha = 2$. b) $t = 0.7$, $\epsilon = 0.1$, $A = 1.4969$, $\Delta p = 0$ and $\alpha = 2$. c) $t = 0.9$, $\epsilon = 0.1$, $A = 1.4969$, $\Delta p = 1$ and $\tau_0 = 2$.

(a)



(b)

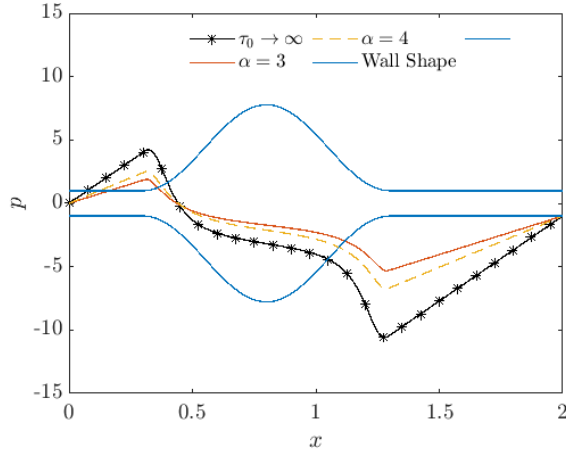
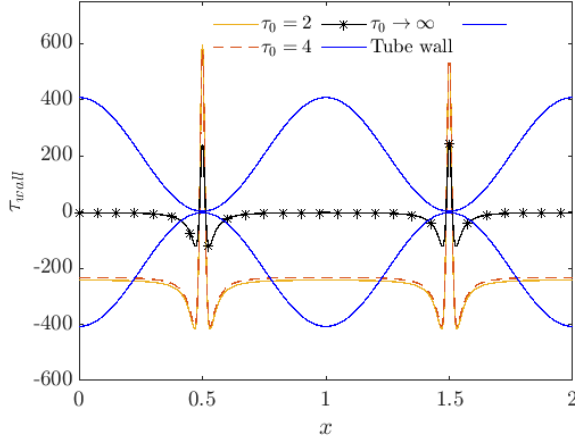


Figure 3. Pressure along the tube (single wave with $L = 2$) a) $t = 0$, $\epsilon = 0.05$, $A = 1.5656$, $\Delta p = 1$ and $\alpha = 4$. b) $t = 0.3$, $\epsilon = 0.2$, $A = 1.3554$, $\Delta p = -1$ and $\tau_0 = 3$.

(a)



(b)

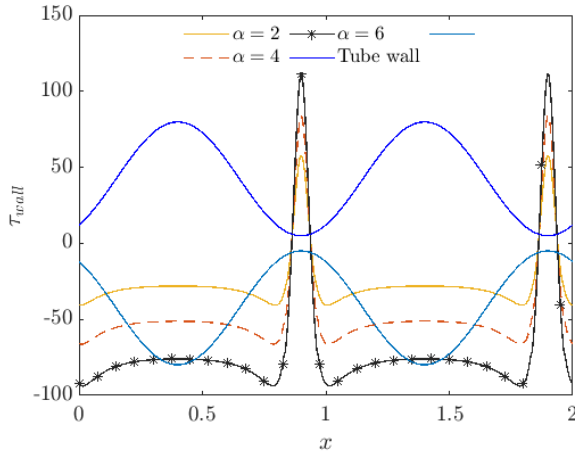
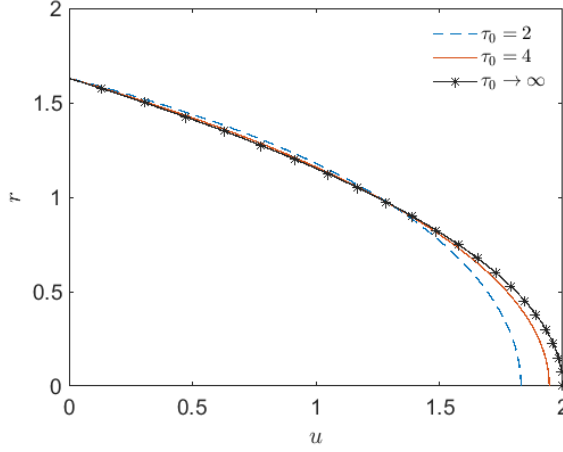


Figure 4. Shear stress at the walls for (Multiple train waves with $L = 2$) a) $t = 0.5$, $\epsilon = 0.01$, $A = 1.6196$, $\Delta p = 0$, $\alpha = 2$ b) $t = 0.9$, $\epsilon = 0.1$, $A = 1.4969$, $\Delta p = 1$, $\tau_0 = 2$

Figures 6a and 6b show cases in which the velocity of the fluid is in the retrograde direction in non-Newtonian cases. However, the Newtonian case provides a positive velocity profile. These plots are projected for various α at $x = 0$.

(a)



(b)

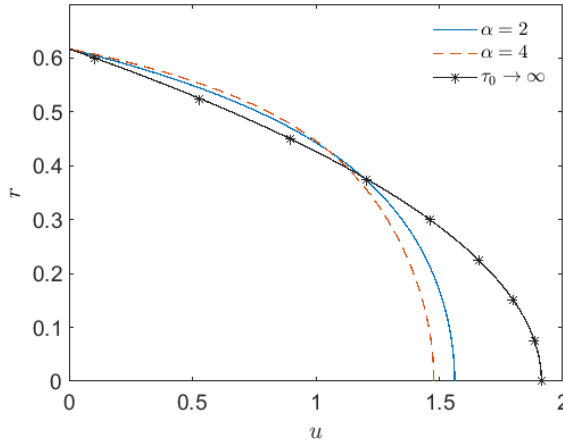
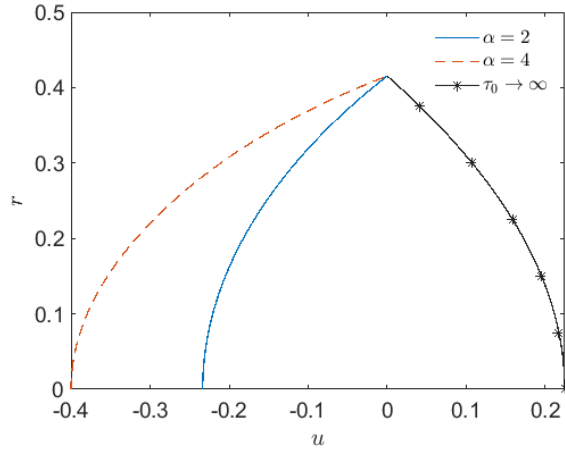


Figure 5. Axial velocity (Multiple train waves) with $L = 2$ a) $x = 1$, $t = 0.5$, $\epsilon = 0.01$, $A = 1.6196$, $\Delta p = 0$, $\alpha = 2$. b) $x = 0.5$, $t = 0.3$, $\epsilon = 0.1$, $A = 1.4969$, $\Delta p = 1$, $\tau_0 = 2$.

3.4 Time Averaged Volume Flow Rate

An important quantity of interest in this study is the time-averaged volume flow rate. Figure 7(a-c) illustrates this quantity with the linear Newtonian relation in each case. It is seen here that the time-averaged volume flow rate displays a nonlinear relation with the pressure head in the case of Ellis fluid. From figures 7a and 7c it is observed that decreasing the fluid parameter τ_0 results in a downward divergence of the plot from the Newtonian case and the same is observed for larger values of Δp (figure 7b). Increasing τ_0 results in the plot becoming more linear and converging to the Newtonian case. Increasing the occlusion ϵ exaggerates this divergence. We observe an inverse effect on the time averaged volume flow rate, with the fluid parameter α in figure 7b, as decreasing α results in the plot becoming more linear and converging to the Newtonian case. For multiple wave pumping, $\bar{Q}$ is independent of the position within the tube.

(a)



(b)

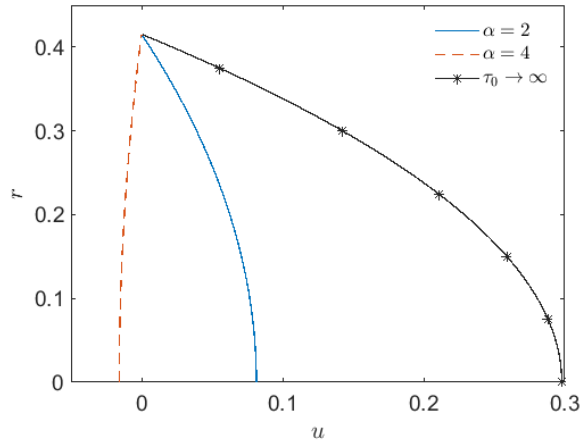
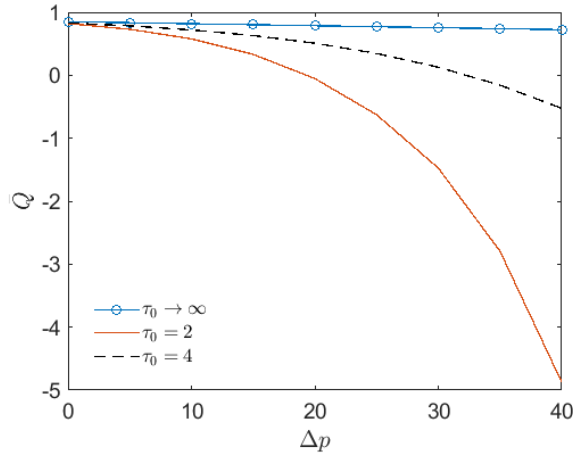
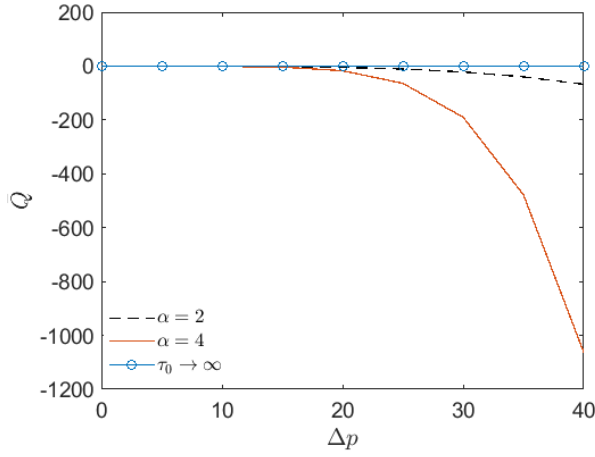


Figure 6. Axial velocity (single wave) with $L = 2$, $x = 0$, $t = 0.9$, $\epsilon = 0.3$, $A = 1.2083$, $\tau_0 = 2$
a) $\Delta p = 1$, b) $\Delta p = -1$.

(a)



(b)



(c)

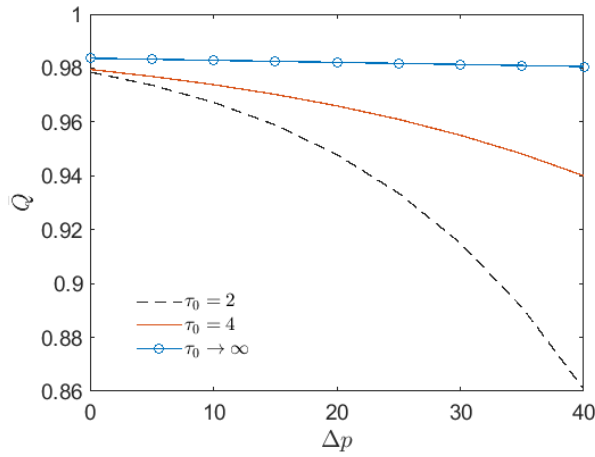
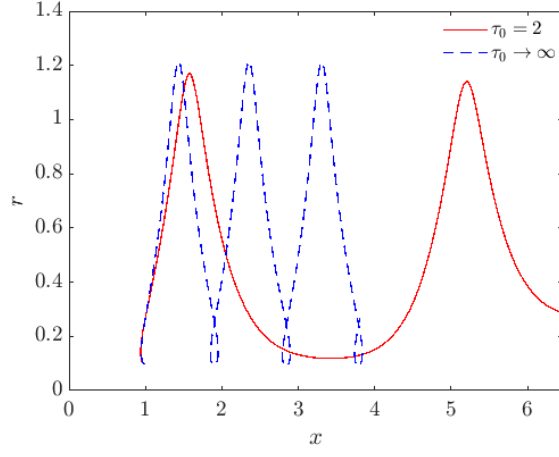
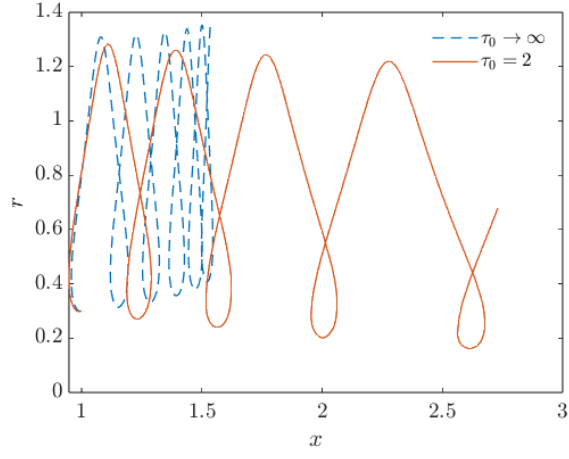


Figure 7. The time averaged volume flow rate over the pressure head for (Multiple train waves)
a) $\epsilon = 0.3$, $\alpha = 2$, $A = 1.2083$. b) $\epsilon = 0.6$, $\tau_0 = 2$, $A = 0.7319$. c) $\epsilon = 0.1$, $\alpha = 2$, $A = 1.4969$.

(a)



(b)



(c)

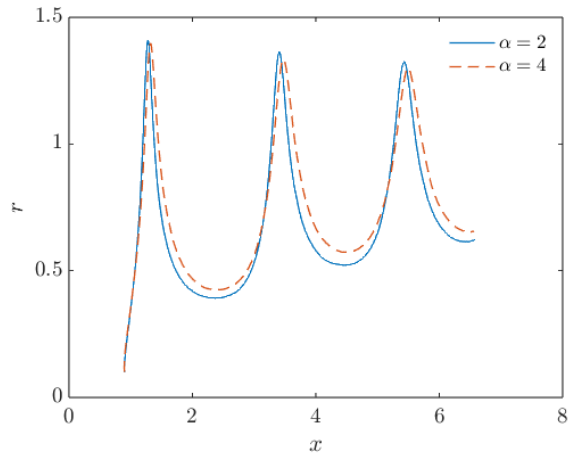


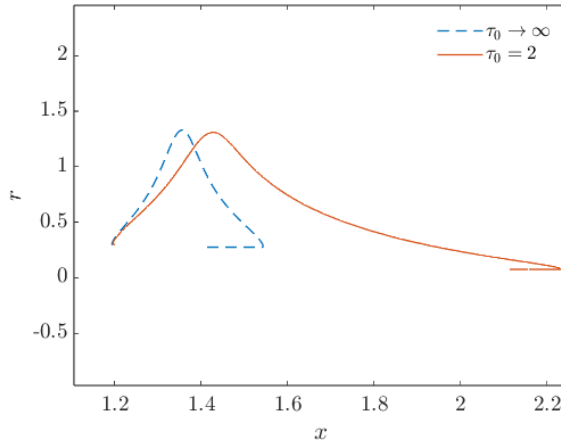
Figure 8. Particle trajectory a) $\xi = 1$, $\eta = 0.1$, $\alpha = 2$, $\Delta p = 0$, $\epsilon = 0.5$, $A = 0.8968$ b) $\xi = 1$, $\eta = 0.3$, $\alpha = 2$, $\Delta p = 0$, $\epsilon = 0.5$, $A = 0.8968$ c) $\xi = 0.9$, $\eta = 0.1$, $\tau_0 = 2$, $\Delta p = 3$, $\epsilon = 0.3$, $A = 1.2083$ (Multiple train waves)

3.5 Particle Trajectories

The particle trajectory was determined using the velocity field and the initial position of the particle $(\xi(0), \eta(0))$, to determine the future positions of the particle $(\xi(t), \eta(t))$. Figure 8(a-c) provides the particle trajectory of a particle that is initially located at a point in the tube and later the path it takes for multiple train waves. Figure 9(a-b) gives the same information for particles in single-wave pumping. These plots are done for varying starting positions of the particle, tube occlusion, and pressure head. It is observed that by changing the fluid parameters α and τ for particles that start at the same initial position, the path traversed may be very different. Depending on the initial position of the particle and the tube occlusion, the particle may travel in a path that illustrates that it is trapped with the peristaltic wave.

The Newtonian limit of the Ellis fluid model, obtained by taking $\tau_0 \rightarrow \infty$, consistently reproduces the benchmark results of [5] across all flow metrics, including pressure distribution, axial velocity, and volume flow rate.

(a)



(b)

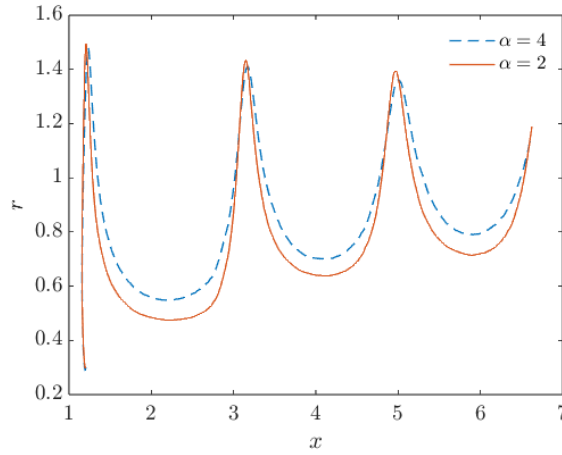


Figure 9. Particle trajectory a) $\xi = 1.2, \eta = 0.3, \alpha = 4, \Delta p = 0, \epsilon = 0.5, A = 0.8968$ b) $\xi = 1.2, \eta = 0.3, \tau_0 = 2, \Delta p = 0, \epsilon = 0.2, A = 1.3554$ (Single wave).

4 Conclusion

The following conclusions can therefore be made from this study:

- **Pressure Distribution:** Highest pressure variations occur near the occlusions. Decreasing the fluid parameter τ_0 reduces pressure variation near occlusions. Smaller occluded tubes show more significant pressure variation differences. Increasing fluid parameter α reduces pressure peaks, indicating divergence from Newtonian behavior.
- **Shear Stress:** Increasing τ_0 reduces shear stress peaks and raises the entire profile vertically. Increasing occlusion ϵ , reduces peak sizes. Increasing α shifts the profile downward, indicating more non-Newtonian behavior.
- **Axial Velocity:** Lower τ_0 values result in lower center tube velocity and higher near-wall velocity. Non-Newtonian fluids can exhibit retrograde flow, even when for similar tube condition Newtonian fluids does not.
- **Time Averaged Volume Flow Rate:** Increasing α causes divergence from the Newtonian case for larger pressure heads. Increasing ϵ exaggerates this divergence. Increasing τ_0 makes the plot more linear, converging to the Newtonian case. For multiple wave pumping, the flow rate is independent of tube position.
- **Particle Trajectory:** Varying fluid parameters τ_0 and α affects the path of particles starting from the same initial position.

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