

Intuitionistic fuzzy $\mathfrak{G}$ -semiprimary ideals in rings

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Abstract This study explores the algebraic characteristics of intuitionistic fuzzy (IF) ideals and their associated radicals under $\mathfrak{G}$ -homomorphisms on rings. In addition, it examines how image and inverse image of IF $\mathfrak{G}$ -primary and IF $\mathfrak{G}$ -semiprimary ideals behave under $\mathfrak{G}$ -homomorphism. Furthermore, conditions under which an IF semiprimary ideal becomes an IF prime ideal, along with the resulting implications, are investigated.

1 Introduction

After Zadeh's seminal introduction of fuzzy sets [14], a generalization of crisp sets, there was a major development in the field of mathematical reasoning and modelling. Rosenfeld [7] expanded and developed the idea of fuzzy subgroups of a group in 1971 and used the concept of fuzzy sets to group theory. Subsequently, Liu defined fuzzy rings, fuzzy ideals and provided several fundamental ideas of fuzzy algebra such as fuzzy invariant subgroups as well as he proved certain associated facts in [12]. The new notions of fuzzy rings and fuzzy ideals were developed in [6, 16]. The concepts of prime L-fuzzy ideals and primary L-fuzzy ideals were introduced in [13], where some of their basic properties were also proven. In 1992, Kumar [5] generalized the established notion of fuzzy primary ideals by introducing the more encompassing class of fuzzy semiprimary ideals. In [17], fuzzy points have been used to redefine semiprimary ideals and it has been demonstrated that such a definition is very similar to Kumar's definition [5]. In 1992, Zahedi [15] defined the L-fuzzy primary ideal in terms of the L-fuzzy points. Furthermore, Ali et al. [11] studied the notion of fuzzy primary, fuzzy semiprimary ideals of a ring $\mathfrak{R}$ under group actions.

Unfortunately, the fuzzy set theory failed because inadequate information was known about the negative membership degree of the function. To solve this challenge, Atanassov [1] developed and presented the concept of the IF set. This set employed the function's negative membership degree in fuzzy set theory, ensuring that the sum of the positive and negative membership degrees did not exceed 1. Moreover, Atanassov defined a few new operations on IF set and examined their characteristics in [2, 3]. In 2007, Jun et al. [10] introduced the notion of nil radicals of an IF ideal and characterized notion of IF semiprime ideals.

In this paper, we introduce a generalization of IF primary and IF semiprimary ideals of a ring $\mathfrak{R}$ under group action and say IF $\mathfrak{G}$ -primary and IF $\mathfrak{G}$ -semiprimary ideals of $\mathfrak{R}$. Moreover, we investigate the conservation properties concerning both the images and the inverse images of an IF $\mathfrak{G}$ -semiprimary ideal of a ring $\mathfrak{R}$. Further, we show that when an IF semiprimary ideal is an IF prime. We also establish the relation between IF primary ideals and IF prime ideals of a ring $\mathfrak{R}$. Throughout the paper $\mathfrak{R}$ is a commutative ring unless and otherwise stated.

2 Preliminaries

Definition 2.1 ([11]). A group $\mathfrak{G}$ is said to act on a set $\mathfrak{X}$ if there is a function $\phi : \mathfrak{G} \times \mathfrak{X} \rightarrow \mathfrak{X}$. This function, which is commonly written as $g * x$ instead of $\phi(g, x)$, satisfy the following two

properties:

$$(i) \quad \mathfrak{e}_1 * (\mathfrak{e}_2 * \mathfrak{x}) = (\mathfrak{e}_1 \mathfrak{e}_2) * \mathfrak{x}, \text{ for all } \mathfrak{e}_1, \mathfrak{e}_2 \in \mathfrak{G}, \mathfrak{x} \in \mathfrak{X}$$

$$(ii) \quad \mathfrak{e} * \mathfrak{z} = \mathfrak{z}, \mathfrak{e} \text{ is the identity of } \mathfrak{G}.$$

The function ϕ is called the action of $\mathfrak{G}$ on X , and the set X equipped with this operation is called a $\mathfrak{G}$ -set.

Definition 2.2 ([11]). A ring homomorphism $\vartheta : \mathfrak{R}_1 \rightarrow \mathfrak{R}_2$ is said to be a $\mathfrak{G}$ -homomorphism, if it is compatible with the action of group $\mathfrak{G}$ on the rings. For any $g \in \mathfrak{G}, r \in \mathfrak{R}_1$, $\vartheta(g * r) = g \diamond \vartheta(r)$, where $*$ and $\diamond$ are the respective group actions on rings $\mathfrak{R}_1$ and $\mathfrak{R}_2$ respectively.

Definition 2.3 ([4]). Assume that $\mathfrak{A} = (\zeta_{\mathfrak{A}}, \eta_{\mathfrak{A}})$ and $\mathfrak{B} = (\zeta_{\mathfrak{B}}, \eta_{\mathfrak{B}})$ are IF subsets of a set $\mathfrak{X}$ and $\mathfrak{Y}$ respectively and $\vartheta : \mathfrak{X} \rightarrow \mathfrak{Y}$ is a map. Then the image of $\mathfrak{A}$ under ϑ denoted by $\vartheta(\mathfrak{A}) = (\zeta_{\vartheta(\mathfrak{A})}, \eta_{\vartheta(\mathfrak{A})})$ is defined as follows:

$$\zeta_{\vartheta(\mathfrak{A})}(y) = \begin{cases} \sup \{ \zeta_{\mathfrak{A}}(z) \mid z \in \vartheta^{-1}(y) \} & \text{if } \vartheta^{-1}(y) \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

$$\eta_{\vartheta(\mathfrak{A})}(y) = \begin{cases} \inf \{ \eta_{\mathfrak{A}}(z) \mid z \in \vartheta^{-1}(y) \} & \text{if } \vartheta^{-1}(y) \neq \emptyset \\ 1 & \text{otherwise} \end{cases}$$

for all $y \in \mathfrak{Y}$ and inverse image of $\mathfrak{B} = (\zeta_{\mathfrak{B}}, \eta_{\mathfrak{B}})$ is defined as $\vartheta^{-1}(\mathfrak{B}) = (\zeta_{\vartheta^{-1}(\mathfrak{B})}, \eta_{\vartheta^{-1}(\mathfrak{B})})$, where $\zeta_{\vartheta^{-1}(\mathfrak{B})}(z) = \zeta_{\mathfrak{B}}(\vartheta(z))$ and $\eta_{\vartheta^{-1}(\mathfrak{B})}(z) = \eta_{\mathfrak{B}}(\vartheta(z))$ for all $z \in \mathfrak{X}$.

Definition 2.4 ([4]). An (α, β) -cut of an IF subset $\mathfrak{A} = (\zeta_{\mathfrak{A}}, \eta_{\mathfrak{A}})$ of a set $\mathfrak{X}$ is defined as

$$\mathfrak{A}^{(\alpha, \beta)} = \{ z \in \mathfrak{X} \mid \zeta_{\mathfrak{A}}(z) \geq \alpha, \eta_{\mathfrak{A}}(z) \leq \beta \},$$

where $\alpha, \beta \in [0, 1]$ such that $\alpha + \beta \leq 1$.

Definition 2.5 ([10]). Let $\mathfrak{X}$ and $\mathfrak{Y}$ be sets, and let $\vartheta : \mathfrak{X} \rightarrow \mathfrak{Y}$ be a map. An IF set $\mathfrak{A} = (\zeta_{\mathfrak{A}}, \eta_{\mathfrak{A}})$ of $\mathfrak{X}$ is called ϑ -invariant if $\vartheta(x_1) = \vartheta(x_2)$ implies that $\zeta_{\mathfrak{A}}(x_1) = \zeta_{\mathfrak{A}}(x_2)$ and $\eta_{\mathfrak{A}}(x_1) = \eta_{\mathfrak{A}}(x_2)$, where $x_1, x_2 \in \mathfrak{X}$.

Let $\mathfrak{G}$ be a finite group that acts on a ring $\mathfrak{R}$. In this case, group action on an IF ideal $\mathfrak{A} = (\zeta_{\mathfrak{A}}, \eta_{\mathfrak{A}})$ of $\mathfrak{R}$ is define as follows:

Definition 2.6 ([8]). An action of a group $\mathfrak{G}$ on an IF set $\mathfrak{A} = (\zeta_{\mathfrak{A}}, \eta_{\mathfrak{A}})$ of the ring $\mathfrak{R}$ is defined as $\mathfrak{A}^g = (\zeta_{\mathfrak{A}^g}, \eta_{\mathfrak{A}^g})$, where $\zeta_{\mathfrak{A}^g}(x) = \zeta_{\mathfrak{A}}(x^g)$, $\eta_{\mathfrak{A}^g}(x) = \eta_{\mathfrak{A}}(x^g)$, $g \in \mathfrak{G}, x \in \mathfrak{R}$ and x^g means g acts on x .

In Definition 2.6, if group $\mathfrak{G}$ is trivial group, then IF ideal under group action coincide with itself.

3 IF $\mathfrak{G}$ -Primary Ideals

Definition 3.1 ([9]). An IF ideal $\mathfrak{A} = (\zeta_{\mathfrak{A}}, \eta_{\mathfrak{A}})$ of a ring $\mathfrak{R}$ is said to be IF primary ideal of $\mathfrak{R}$ if for any $r_1, r_2 \in \mathfrak{R}$, either $\zeta_{\mathfrak{A}}(r_1 r_2) = \zeta_{\mathfrak{A}}(r_1)$ and $\eta_{\mathfrak{A}}(r_1 r_2) = \eta_{\mathfrak{A}}(r_1)$ or $\zeta_{\mathfrak{A}}(r_1 r_2) = \zeta_{\mathfrak{A}}(r_2^m)$ and $\eta_{\mathfrak{A}}(r_1 r_2) = \eta_{\mathfrak{A}}(r_2^m)$ for some positive integer m .

Definition 3.2. Let $\mathfrak{G}$ be a group acting on a ring $\mathfrak{R}$. Then an IF primary ideal $\mathfrak{A}$ of $\mathfrak{R}$ is said to be an IF $\mathfrak{G}$ -primary ideal of $\mathfrak{R}$ if $\mathfrak{A}$ is $\mathfrak{G}$ -invariant.

Example 3.1. Consider the IF ideal of the ring of integers $\mathbb{Z}$, denoted by $\mathfrak{A} = (\zeta_{\mathfrak{A}}, \eta_{\mathfrak{A}})$. This ideal is formally specified as:

$$\zeta_{\mathfrak{A}}(\mathfrak{x}) = \begin{cases} 0.8 & \mathfrak{x} = 0 \\ 0.1 & \mathfrak{x} \neq 0, \end{cases} \quad \eta_{\mathfrak{A}}(\mathfrak{x}) = \begin{cases} 0.1 & \mathfrak{x} = 0 \\ 0.7 & \mathfrak{x} \neq 0. \end{cases}$$

Hence, $\mathfrak{A}$ is an IF primary ideal of $\mathbb{Z}$.

Definition 3.3. Let $\mathfrak{A} = (\zeta_{\mathfrak{A}}, \eta_{\mathfrak{A}})$ be an IF ideal of a ring $\mathfrak{R}$. Then IF nil radical of $\mathfrak{A}$ denoted by $\sqrt{\mathfrak{A}} = (\zeta_{\sqrt{\mathfrak{A}}}, \eta_{\sqrt{\mathfrak{A}}})$ is defined as

$$\zeta_{\sqrt{\mathfrak{A}}}(r) = \sup_{n \geq 1} \{\zeta_{\mathfrak{A}}(r^n)\}, \quad \eta_{\sqrt{\mathfrak{A}}}(r) = \inf_{n \geq 1} \{\eta_{\mathfrak{A}}(r^n)\},$$

for all $r \in \mathfrak{R}$ and some $n \in \mathbb{N}$, the set of natural numbers.

Example 3.2. Suppose that ring $\mathbb{Z}_8$ of integers modulo 8. Then an IF ideal $\mathfrak{A} = (\zeta_{\mathfrak{A}}, \eta_{\mathfrak{A}})$ of $\mathbb{Z}_8$ can be, represented by ordered triplet $\langle x, \zeta_{\mathfrak{A}}(x), \eta_{\mathfrak{A}}(x) \rangle$, defined by

$$\mathfrak{A} = \{\langle \bar{0}, 1, 0 \rangle, \langle \bar{1}, 0, 1 \rangle, \langle \bar{2}, 0.5, 0.4 \rangle, \langle \bar{3}, 0, 1 \rangle, \langle \bar{4}, 0.6, 0.3 \rangle, \langle \bar{5}, 0, 1 \rangle, \langle \bar{6}, 0.5, 0.4 \rangle, \langle \bar{7}, 0, 1 \rangle\}.$$

Hence, IF radical of $\mathfrak{A}$ can be, represented by ordered triplet $\sqrt{\mathfrak{A}} = \langle x, \zeta_{\sqrt{\mathfrak{A}}}(x), \eta_{\sqrt{\mathfrak{A}}}(x) \rangle$, given by

$$\sqrt{\mathfrak{A}} = \{\langle \bar{0}, 1, 0 \rangle, \langle \bar{1}, 0, 1 \rangle, \langle \bar{2}, 1, 0 \rangle, \langle \bar{3}, 0, 1 \rangle, \langle \bar{4}, 1, 0 \rangle, \langle \bar{5}, 0, 1 \rangle, \langle \bar{6}, 1, 0 \rangle, \langle \bar{7}, 0, 1 \rangle\}.$$

Proposition 3.1. Let $\vartheta : \mathfrak{R}_1 \rightarrow \mathfrak{R}_2$ be a homomorphism from a ring $\mathfrak{R}_1$ to a ring $\mathfrak{R}_2$ and $\mathfrak{A} = (\zeta_{\mathfrak{A}}, \eta_{\mathfrak{A}})$ an IF ideal of $\mathfrak{R}_1$. If $\mathfrak{A}$ is constant on $\ker \vartheta$, then $\vartheta(\mathfrak{A})(\vartheta(\mathfrak{r})) = \mathfrak{A}(\mathfrak{r}_1)$, for all $\mathfrak{r}_1 \in \mathfrak{R}_1$.

Proof. Let us assume that $\mathfrak{A}$ is constant on $\ker \vartheta$, that is, $\zeta_{\mathfrak{A}}(x) = \zeta_{\mathfrak{A}}(y)$, $\eta_{\mathfrak{A}}(x) = \eta_{\mathfrak{A}}(y)$ for all $x, y \in \ker \vartheta$ and $\mathfrak{r}_1 \in \mathfrak{R}_1$. Consequently, for $\mathfrak{r}_2 \in \mathfrak{R}_2$, we have $\vartheta(\mathfrak{r}_1) = \mathfrak{r}_2$. Furthermore, for any $\mathfrak{t} \in \vartheta^{-1}(\mathfrak{r}_2)$, we have $\vartheta(\mathfrak{t}) = \mathfrak{r}_2 = \vartheta(\mathfrak{r}_1)$,

$$\begin{aligned} \vartheta(\mathfrak{t} - \mathfrak{r}_1) &= 0 \\ \mathfrak{t} - \mathfrak{r}_1 &\in \ker \vartheta \\ \zeta_{\mathfrak{A}}(\mathfrak{t} - \mathfrak{r}_1) &= \zeta_{\mathfrak{A}}(0) \\ \zeta_{\mathfrak{A}}(\mathfrak{t}) &= \zeta_{\mathfrak{A}}(\mathfrak{r}_1). \end{aligned}$$

Now,

$$\begin{aligned} \zeta_{\vartheta(\mathfrak{A})}(\vartheta(\mathfrak{r}_1)) &= \zeta_{\vartheta(\mathfrak{A})}(\mathfrak{r}_2) \\ &= \sup\{\zeta_{\mathfrak{A}}(\mathfrak{t}) \mid \mathfrak{t} \in \vartheta^{-1}(\mathfrak{r}_2)\} \\ &= \sup\{\zeta_{\mathfrak{A}}(\mathfrak{r}_1)\} = \zeta_{\mathfrak{A}}(\mathfrak{r}_1). \end{aligned}$$

Moreover,

$$\begin{aligned} \eta_{\mathfrak{A}}(\mathfrak{t} - \mathfrak{r}_1) &= \eta_{\mathfrak{A}}(0) \\ \eta_{\mathfrak{A}}(\mathfrak{t}) &= \eta_{\mathfrak{A}}(\mathfrak{r}_1). \end{aligned}$$

Now,

$$\begin{aligned} \eta_{\vartheta(\mathfrak{A})}(\vartheta(\mathfrak{r}_1)) &= \vartheta(\eta_{\mathfrak{A}})(\mathfrak{r}_2) \\ &= \inf\{\eta_{\mathfrak{A}}(\mathfrak{t}) \mid \mathfrak{t} \in \vartheta^{-1}(\mathfrak{r}_2)\} \\ &= \inf\{\eta_{\mathfrak{A}}(\mathfrak{r}_1)\} = \eta_{\mathfrak{A}}(\mathfrak{r}_1). \end{aligned}$$

Lemma 3.1. Let a group $\mathfrak{G}$ act on rings $\mathfrak{R}_1$ and $\mathfrak{R}_2$, and let $\vartheta : \mathfrak{R}_1 \rightarrow \mathfrak{R}_2$ a $\mathfrak{G}$ -epimorphism. Then

- (i) If $\mathfrak{A}$ is an IF $\mathfrak{G}$ -primary ideal of $\mathfrak{R}_1$ that is constant on $\ker \vartheta$, then $\vartheta(\mathfrak{A})$ is an IF $\mathfrak{G}$ -primary ideal of $\mathfrak{R}_2$.
- (ii) If $\mathfrak{B}$ is an IF $\mathfrak{G}$ -primary ideal of $\mathfrak{R}_2$, then $\vartheta^{-1}(\mathfrak{B})$ is an IF $\mathfrak{G}$ -primary ideal of $\mathfrak{R}_1$.

Proof. (i) Assume that $\mathfrak{A}$ is an IF $\mathfrak{G}$ -primary ideal of the ring $\mathfrak{R}_1$ which is constant on $\ker \vartheta$ and $x, y \in \mathfrak{R}_2$. If $x = \vartheta(u)$ and $y = \vartheta(v)$ for some $u, v \in \mathfrak{R}_1$, then $u - v \in \vartheta^{-1}(x - y)$ and

$$\begin{aligned}\zeta_{\vartheta(\mathfrak{A})}(x - y) &= \sup\{\zeta_{\mathfrak{A}}(z) \mid z \in \vartheta^{-1}(x - y)\} \\ &= \sup\{\zeta_{\mathfrak{A}}(u - v) \mid u - v \in \vartheta^{-1}(x - y)\} \\ &\geq \sup\{\min\{\zeta_{\mathfrak{A}}(u), \zeta_{\mathfrak{A}}(v)\} \mid u \in \vartheta^{-1}(x), v \in \vartheta^{-1}(y)\} \\ &= \min\{\sup\{\zeta_{\mathfrak{A}}(u) \mid u \in \vartheta^{-1}(x)\}, \sup\{\zeta_{\mathfrak{A}}(v) \mid v \in \vartheta^{-1}(y)\}\} \\ &= \min\{\zeta_{\vartheta(\mathfrak{A})}(x), \zeta_{\vartheta(\mathfrak{A})}(y)\}.\end{aligned}\tag{3.1}$$

Since $uv \in \vartheta^{-1}(xy)$ for all $u \in \vartheta^{-1}(x)$ and $v \in \vartheta^{-1}(y)$, then

$$\begin{aligned}\zeta_{\vartheta(\mathfrak{A})}(xy) &= \sup\{\zeta_{\mathfrak{A}}(z) \mid z \in \vartheta^{-1}(xy)\} \\ &= \sup\{\zeta_{\mathfrak{A}}(uv) \mid uv \in \vartheta^{-1}(xy)\} \\ &\geq \sup\{\max\{\zeta_{\mathfrak{A}}(u), \zeta_{\mathfrak{A}}(v)\} \mid u \in \vartheta^{-1}(x), v \in \vartheta^{-1}(y)\} \\ &= \max\{\sup\{\zeta_{\mathfrak{A}}(u) \mid u \in \vartheta^{-1}(x)\}, \sup\{\zeta_{\mathfrak{A}}(v) \mid v \in \vartheta^{-1}(y)\}\} \\ &= \max\{\zeta_{\vartheta(\mathfrak{A})}(x), \zeta_{\vartheta(\mathfrak{A})}(y)\}.\end{aligned}\tag{3.2}$$

Furthermore,

$$\begin{aligned}\eta_{\vartheta(\mathfrak{A})}(x - y) &= \inf\{\eta_{\mathfrak{A}}(z) \mid z \in \vartheta^{-1}(x - y)\} \\ &= \inf\{\eta_{\mathfrak{A}}(u - v) \mid u - v \in \vartheta^{-1}(x - y)\} \\ &\leq \inf\{\max\{\eta_{\mathfrak{A}}(u), \eta_{\mathfrak{A}}(v)\} \mid u \in \vartheta^{-1}(x), v \in \vartheta^{-1}(y)\} \\ &= \max\{\inf\{\eta_{\mathfrak{A}}(u) \mid u \in \vartheta^{-1}(x)\}, \inf\{\eta_{\mathfrak{A}}(v) \mid v \in \vartheta^{-1}(y)\}\} \\ &= \max\{\eta_{\vartheta(\mathfrak{A})}(x), \eta_{\vartheta(\mathfrak{A})}(y)\}.\end{aligned}\tag{3.3}$$

Now

$$\begin{aligned}\eta_{\vartheta(\mathfrak{A})}(xy) &= \inf\{\eta_{\mathfrak{A}}(z) \mid z \in \vartheta^{-1}(xy)\} \\ &= \inf\{\eta_{\mathfrak{A}}(uv) \mid uv \in \vartheta^{-1}(xy)\} \\ &\leq \inf\{\min\{\eta_{\mathfrak{A}}(u), \eta_{\mathfrak{A}}(v)\} \mid u \in \vartheta^{-1}(x), v \in \vartheta^{-1}(y)\} \\ &= \min\{\inf\{\eta_{\mathfrak{A}}(u) \mid u \in \vartheta^{-1}(x)\}, \inf\{\eta_{\mathfrak{A}}(v) \mid v \in \vartheta^{-1}(y)\}\} \\ &= \min\{\eta_{\vartheta(\mathfrak{A})}(x), \eta_{\vartheta(\mathfrak{A})}(y)\}.\end{aligned}\tag{3.4}$$

Equations (3.1) – (3.4) show that $\vartheta(\mathfrak{A})$ is an IF ideal of $\mathfrak{R}_2$.

Now, we will show that $\vartheta(\mathfrak{A})$ is an IF $\mathfrak{G}$ -primary ideal of $\mathfrak{R}_2$. Since ϑ is an epimorphism, then for $r, s \in \mathfrak{R}_2$, there exists $t \in \mathfrak{R}_1$ such that $\vartheta(t) = rs$ and also there exist $r_1, s_1 \in \mathfrak{R}_1$ such that $\vartheta(r_1) = r$ and $\vartheta(s_1) = s$. Therefore, $\zeta_{\vartheta(\mathfrak{A})}(rs) = \sup_{\vartheta(t)=rs} \zeta_{\mathfrak{A}}(t)$. Since $\mathfrak{A}$ is an IF $\mathfrak{G}$ -primary

ideal and $\mathfrak{A}$ is constant on $\ker \vartheta$, then by Proposition 3.1 either $\zeta_{\vartheta(\mathfrak{A})}(rs) = \zeta_{\mathfrak{A}}(r_1) = \zeta_{\vartheta(\mathfrak{A})}(r)$

and $\eta_{\vartheta(\mathfrak{A})}(rs) = \eta_{\mathfrak{A}}(r_1) = \eta_{\vartheta(\mathfrak{A})}(r)$ or $\zeta_{\vartheta(\mathfrak{A})}(rs) = \zeta_{\mathfrak{A}}(s_1^n) = \zeta_{\vartheta(\mathfrak{A})}(s^n)$ and $\eta_{\vartheta(\mathfrak{A})}(rs) =$

$\eta_{\mathfrak{A}}(s_1^n) = \eta_{\vartheta(\mathfrak{A})}(s^n)$, for some positive integer n . Thus, $\vartheta(\mathfrak{A})$ is an IF primary ideal of $\mathfrak{R}_1$.

Moreover, we need to prove that $\vartheta(\mathfrak{A})$ is $\mathfrak{G}$ -invariant. Let $r \in \mathfrak{R}_1$. Then

$$\begin{aligned}\zeta_{\vartheta(\mathfrak{A})}(r^{\mathfrak{g}}) &= \sup_{\vartheta(z)=r^{\mathfrak{g}}} \zeta_{\mathfrak{A}}(z) = \sup_{(\vartheta(z))^{\mathfrak{g}^{-1}}=r} \zeta_{\mathfrak{A}}(z) \\ &= \sup_{\vartheta(z^{\mathfrak{g}^{-1}})=r} \zeta_{\mathfrak{A}}(z^{\mathfrak{g}^{-1}}) \\ &= \zeta_{\vartheta(\mathfrak{A})}(r)\end{aligned}$$

and

$$\begin{aligned}\eta_{\vartheta(\mathfrak{A})}(r^{\mathfrak{g}}) &= \inf_{\vartheta(z)=r^{\mathfrak{g}}} \eta_{\mathfrak{A}}(z) = \inf_{(\vartheta(z))^{\mathfrak{g}^{-1}}=r} \eta_{\mathfrak{A}}(z) \\ &= \inf_{\vartheta(z^{\mathfrak{g}^{-1}})=r} \eta_{\mathfrak{A}}(z^{\mathfrak{g}^{-1}}) \\ &= \eta_{\vartheta(\mathfrak{A})}(r).\end{aligned}$$

Thus, $\vartheta(\mathfrak{A})$ is an IF $\mathfrak{G}$ -primary ideal of $\mathfrak{R}_2$.

(ii) Let $\mathfrak{B}$ be an IF $\mathfrak{G}$ -primary ideal of $\mathfrak{R}_2$. Then for $s, t \in \mathfrak{R}_2$, either

$$\zeta_{\vartheta^{-1}(\mathfrak{B})}(st) = \zeta_{\mathfrak{B}}(\vartheta(st)) = \zeta_{\mathfrak{B}}(\vartheta(s)\vartheta(t)) = \zeta_{\mathfrak{B}}(\vartheta(s)) = \zeta_{\vartheta^{-1}(\mathfrak{B})}(s)$$

and

$$\eta_{\vartheta^{-1}(\mathfrak{B})}(st) = \eta_{\mathfrak{B}}(\vartheta(st)) = \eta_{\mathfrak{B}}(\vartheta(s)\vartheta(t)) = \eta_{\mathfrak{B}}(\vartheta(s)) = \eta_{\vartheta^{-1}(\mathfrak{B})}(s)$$

or

$$\zeta_{\vartheta^{-1}(\mathfrak{B})}(st) = \zeta_{\mathfrak{B}}(\vartheta(st)) = \zeta_{\mathfrak{B}}(\vartheta(s)\vartheta(t)) = \zeta_{\mathfrak{B}}(\vartheta(t)^n) = \zeta_{\vartheta^{-1}(\mathfrak{B})}(t^n)$$

and

$$\eta_{\vartheta^{-1}(\mathfrak{B})}(st) = \eta_{\mathfrak{B}}(\vartheta(st)) = \eta_{\mathfrak{B}}(\vartheta(s)\vartheta(t)) = \eta_{\mathfrak{B}}(\vartheta(t)^n) = \eta_{\vartheta^{-1}(\mathfrak{B})}(t^n),$$

for some positive integer n . Therefore, $\vartheta^{-1}(\mathfrak{B})$ is an IF primary ideal. Now our claim is that $\vartheta^{-1}(\mathfrak{B})$ is $\mathfrak{G}$ -invariant. Since $\mathfrak{B}$ is $\mathfrak{G}$ -invariant, then for $s \in \mathfrak{R}_2$, we have

$$\begin{aligned}\zeta_{\vartheta^{-1}(\mathfrak{B})}(s^g) &= \zeta_{\mathfrak{B}}(\vartheta(s^g)) = \zeta_{\mathfrak{B}}((\vartheta(s))^g) \\ &= \zeta_{\mathfrak{B}}(\vartheta(s)) \\ &= \zeta_{\vartheta^{-1}(\mathfrak{B})}(s).\end{aligned}$$

Similarly, $\eta_{\vartheta^{-1}(\mathfrak{B})}(s^g) = \eta_{\vartheta^{-1}(\mathfrak{B})}(s)$. Hence $\vartheta^{-1}(\mathfrak{B})$ is an IF $\mathfrak{G}$ -primary ideal of $\mathfrak{R}_1$.

Proposition 3.2. If $\mathfrak{A}$ is an IF $\mathfrak{G}$ -primary ideal of a ring $\mathfrak{R}$, then each of its (α, β) -cut $\mathfrak{A}_{\alpha, \beta}$ is an IF $\mathfrak{G}$ -primary ideal of $\mathfrak{R}$.

Proof. Let $\mathfrak{A}$ be an IF $\mathfrak{G}$ -primary ideal of $\mathfrak{R}$. Then $\eta_{\mathfrak{A}} \in \mathfrak{A}_{\alpha, \beta}$, i.e., $\zeta_{\mathfrak{A}}(\eta_{\mathfrak{A}}) \geq \alpha$, $\eta_{\mathfrak{A}}(\eta_{\mathfrak{A}}) \leq \beta$, for any $\eta, \mathfrak{z} \in \mathfrak{R}$. Since $\mathfrak{A}$ is an IF primary ideal, then

$$\begin{aligned}\zeta_{\mathfrak{A}}(\eta_{\mathfrak{A}}) &= \zeta_{\mathfrak{A}}(\eta) \geq \alpha \text{ and } \eta_{\mathfrak{A}}(\eta_{\mathfrak{A}}) = \eta(\eta) \leq \beta, \\ &\text{or} \\ \zeta_{\mathfrak{A}}(\eta_{\mathfrak{A}}) &= \zeta(\mathfrak{z}^n) \geq \alpha \text{ and } \eta_{\mathfrak{A}}(\eta_{\mathfrak{A}}) = \eta_{\mathfrak{A}}(\mathfrak{z}^n) \leq \beta.\end{aligned}$$

Therefore, $\eta \in \mathfrak{A}_{\alpha, \beta}$ or $\mathfrak{z}^n \in \mathfrak{A}_{\alpha, \beta}$. Subsequently $\mathfrak{A}_{\alpha, \beta}$ is a primary ideal of $\mathfrak{R}$.

Moreover, $\mathfrak{A}_{\alpha, \beta}$ is $\mathfrak{G}$ -invariant, since

$$\begin{aligned}\mathfrak{A}_{\alpha, \beta}^{\mathfrak{g}} &= \{x^{\mathfrak{g}} \in \mathfrak{R} \mid \zeta_{\mathfrak{A}}(x^{\mathfrak{g}}) \geq \alpha, \eta_{\mathfrak{A}}(x^{\mathfrak{g}}) \leq \beta\} \\ &= \{x^{\mathfrak{g}} \in \mathfrak{R} \mid \zeta_{\mathfrak{A}^{\mathfrak{g}}}(x) \geq \alpha, \eta_{\mathfrak{A}^{\mathfrak{g}}}(x) \leq \beta\} \\ &= \{x \in \mathfrak{R} \mid \zeta_{\mathfrak{A}}(x) \geq \alpha, \eta_{\mathfrak{A}}(x) \leq \beta\} \\ &= \mathfrak{A}_{\alpha, \beta}.\end{aligned}$$

Theorem 3.1. Suppose that $\mathfrak{G}$ is a group acting on rings $\mathfrak{R}_1$ and $\mathfrak{R}_2$. If ϑ is a $\mathfrak{G}$ -homomorphism from $\mathfrak{R}_1$ onto $\mathfrak{R}_2$, $\mathfrak{A}_1, \mathfrak{A}_2$ and $\mathfrak{B}_1, \mathfrak{B}_2$ are IF ideals of rings $\mathfrak{R}_1$ and $\mathfrak{R}_2$ respectively, then

- (i) $(\vartheta^{-1}(\mathfrak{B}_1)\vartheta^{-1}(\mathfrak{B}_2))^{\mathfrak{G}} \subseteq (\vartheta^{-1}(\mathfrak{B}_1\mathfrak{B}_2))^{\mathfrak{G}}$. Equality holds if ϑ is one-one.
- (ii) $(\vartheta(\mathfrak{A}_1)\vartheta(\mathfrak{A}_2))^{\mathfrak{G}} \subseteq (\vartheta(\mathfrak{A}_1\mathfrak{A}_2))^{\mathfrak{G}}$
- (iii) If $\mathfrak{B}_1 \subseteq \mathfrak{B}_2$, then $(\vartheta^{-1}(\mathfrak{B}_1))^{\mathfrak{G}} \subseteq \vartheta^{-1}(\mathfrak{B}_2^{\mathfrak{G}})$.

Proof. (i) Let ϑ be a $\mathfrak{G}$ -homomorphism from $\mathfrak{R}_1$ onto $\mathfrak{R}_2$ and $\mathfrak{B}_1, \mathfrak{B}_2$ are IF ideals of the ring $\mathfrak{R}_2$. Then

$$\begin{aligned}
 \zeta_{(\vartheta^{-1}(\mathfrak{B}_1)\vartheta^{-1}(\mathfrak{B}_2))^\mathfrak{G}}(s) &= \bigcap_{g \in \mathfrak{G}} (\zeta_{(\vartheta^{-1}(\mathfrak{B}_1)\vartheta^{-1}(\mathfrak{B}_2))}(s^g)) \\
 &= \bigcap_{g \in \mathfrak{G}} \left[\sup_{s^g = ab} \{ \min(\zeta_{\vartheta^{-1}(\mathfrak{B}_1)}(a), \zeta_{\vartheta^{-1}(\mathfrak{B}_2)}(b)) \} \right] \\
 &\leq \bigcap_{g \in \mathfrak{G}} \{ \min(\zeta_{\mathfrak{B}_1}(\vartheta(a)), \zeta_{\mathfrak{B}_2}(\vartheta(b))) \} \\
 &\leq \bigcap_{g \in \mathfrak{G}} \zeta_{\mathfrak{B}_1\mathfrak{B}_2}(\vartheta(s^g)) \\
 &= \bigcap_{g \in \mathfrak{G}} (\zeta_{(\mathfrak{B}_1\mathfrak{B}_2)^g})(\vartheta(s)) \\
 &= (\zeta_{(\mathfrak{B}_1\mathfrak{B}_2)^\mathfrak{G}})(\vartheta(s)) \\
 &= (\zeta_{(\vartheta^{-1}(\mathfrak{B}_1)\vartheta^{-1}(\mathfrak{B}_2))^\mathfrak{G}})(s)
 \end{aligned}$$

and

$$\begin{aligned}
 \eta_{(\vartheta^{-1}(\mathfrak{B}_1)\vartheta^{-1}(\mathfrak{B}_2))^\mathfrak{G}}(s) &= \bigcap_{g \in \mathfrak{G}} (\eta_{(\vartheta^{-1}(\mathfrak{B}_1)\vartheta^{-1}(\mathfrak{B}_2))}(s^g)) \\
 &= \bigcap_{g \in \mathfrak{G}} \left[\inf_{s^g = ab} \{ \max(\eta_{\vartheta^{-1}(\mathfrak{B}_1)}(a), \eta_{\vartheta^{-1}(\mathfrak{B}_2)}(b)) \} \right] \\
 &\geq \bigcap_{g \in \mathfrak{G}} \{ \max(\eta_{\mathfrak{B}_1}(\vartheta(a)), \eta_{\mathfrak{B}_2}(\vartheta(b))) \} \\
 &\geq \bigcap_{g \in \mathfrak{G}} \eta_{\mathfrak{B}_1\mathfrak{B}_2}(\vartheta(s^g)) \\
 &= \bigcap_{g \in \mathfrak{G}} (\eta_{(\mathfrak{B}_1\mathfrak{B}_2)^g})(\vartheta(s)) \\
 &= \eta_{(\mathfrak{B}_1\mathfrak{B}_2)^\mathfrak{G}}(\vartheta(s)) \\
 &= \eta_{(\vartheta^{-1}(\mathfrak{B}_1)\vartheta^{-1}(\mathfrak{B}_2))^\mathfrak{G}}(s),
 \end{aligned}$$

where $a, b, s \in \mathfrak{R}_1$.

Thus, $(\vartheta^{-1}(\mathfrak{B}_1)\vartheta^{-1}(\mathfrak{B}_2))^\mathfrak{G} \subseteq (\vartheta^{-1}(\mathfrak{B}_1\mathfrak{B}_2))^\mathfrak{G}$. Now, assume that ϑ is one-one. Then for $s \in \mathfrak{R}_1$,

$$\begin{aligned}
 \zeta_{(\vartheta^{-1}(\mathfrak{B}_1\mathfrak{B}_2))^\mathfrak{G}}(s) &= \bigcap_{g \in \mathfrak{G}} (\zeta_{(\mathfrak{B}_1\mathfrak{B}_2)^g})(\vartheta(s)) \\
 &= \bigcap_{g \in \mathfrak{G}} \zeta_{(\mathfrak{B}_1\mathfrak{B}_2)}(\vartheta(s))^g \\
 &= \bigcap_{g \in \mathfrak{G}} (\zeta_{\mathfrak{B}_1\mathfrak{B}_2})(\vartheta(s^g)) \\
 &= \bigcap_{g \in \mathfrak{G}} \left[\sup_{\vartheta(s^g) = a_1 b_1} \{ \min((\zeta_{\mathfrak{B}_1})(a_1), (\zeta_{\mathfrak{B}_2})(b_1)) \} \right].
 \end{aligned}$$

Since ϑ is onto and $a_1, b_1 \in \mathfrak{R}_2$, then there exist $a, b \in \mathfrak{R}_1$ such that $\vartheta(a) = a_1$ and $\vartheta(b) = b_1$. Also given that ϑ is one-one, then $\vartheta(s^g) = \vartheta(ab)$ implies that $s^g = ab$. Thus,

$$\begin{aligned}
 &\bigcap_{g \in \mathfrak{G}} \left[\sup_{\vartheta(s^g) = a_1 b_1} \{ \min((\zeta_{\mathfrak{B}_1})(a), (\zeta_{\mathfrak{B}_2})(b)) \} \right] \\
 &= \bigcap_{g \in \mathfrak{G}} \left[\sup_{\vartheta(s^g) = \vartheta(a)\vartheta(b)} \{ \min((\zeta_{\mathfrak{B}_1})(\vartheta(a)), (\zeta_{\mathfrak{B}_2})(\vartheta(b))) \} \right] \\
 &= \bigcap_{g \in \mathfrak{G}} \left[\sup_{s^g = ab} \{ \min((\zeta_{\vartheta^{-1}(\mathfrak{B}_1)})(a), (\zeta_{\vartheta^{-1}(\mathfrak{B}_2)})(b)) \} \right] \\
 &= \bigcap_{g \in \mathfrak{G}} (\zeta_{(\vartheta^{-1}(\mathfrak{B}_1)\vartheta^{-1}(\mathfrak{B}_2))}(s^g)) \\
 &= \bigcap_{g \in \mathfrak{G}} (\zeta_{(\vartheta^{-1}(\mathfrak{B}_1)\vartheta^{-1}(\mathfrak{B}_2))^g}(s)) \\
 &= \zeta_{(\vartheta^{-1}(\mathfrak{B}_1)\vartheta^{-1}(\mathfrak{B}_2))^\mathfrak{G}}(s).
 \end{aligned}$$

$\zeta_{(\vartheta^{-1}(\mathfrak{B}_1\mathfrak{B}_2)^\mathfrak{G})} = \zeta_{(\vartheta^{-1}(\mathfrak{B}_1)\vartheta^{-1}(\mathfrak{B}_2))^\mathfrak{G}}$. Similarly, $\eta_{(\vartheta^{-1}(\mathfrak{B}_1\mathfrak{B}_2)^\mathfrak{G})} = \eta_{(\vartheta^{-1}(\mathfrak{B}_1)\vartheta^{-1}(\mathfrak{B}_2))^\mathfrak{G}}$. Therefore, $(\vartheta^{-1}(\mathfrak{B}_1)\vartheta^{-1}(\mathfrak{B}_2))^\mathfrak{G} \supseteq (\vartheta^{-1}(\mathfrak{B}_1\mathfrak{B}_2)^\mathfrak{G})$.

(ii) Assume that ϑ is a $\mathfrak{G}$ -homomorphism from ring $\mathfrak{R}_1$ onto ring $\mathfrak{R}_2$ and $\mathfrak{A}_1, \mathfrak{A}_2$ are IF ideals of $\mathfrak{R}_1$. Then for $\mathfrak{r} \in \mathfrak{R}_2$,

$$\begin{aligned} \zeta_{(\vartheta(\mathfrak{A}_1)\vartheta(\mathfrak{A}_2))^\mathfrak{G}}(\mathfrak{r}) &= \bigcap_{\mathfrak{g} \in \mathfrak{G}} \zeta_{(\vartheta(\mathfrak{A}_1)\vartheta(\mathfrak{A}_2))^\mathfrak{G}}(\mathfrak{r}) \\ &= \bigcap_{\mathfrak{g} \in \mathfrak{G}} \zeta_{(\vartheta(\mathfrak{A}_1)\vartheta(\mathfrak{A}_2))^\mathfrak{G}}(\mathfrak{r}^\mathfrak{g}) \\ &= \bigcap_{\mathfrak{g} \in \mathfrak{G}} \left[\sup_{\mathfrak{r}^\mathfrak{g} = \mathfrak{a}\mathfrak{b}} \{ \min(\zeta_{\vartheta(\mathfrak{A}_1)}(\mathfrak{a}), \zeta_{\vartheta(\mathfrak{A}_2)}(\mathfrak{b})) \} \right] \\ &= \bigcap_{\mathfrak{g} \in \mathfrak{G}} \left[\sup_{\mathfrak{r}^\mathfrak{g} = \mathfrak{a}\mathfrak{b}} \{ \min(\sup_{\mathfrak{a}_1 \in \vartheta^{-1}(\mathfrak{a})} \zeta_{\mathfrak{A}_1}(\mathfrak{a}_1), \sup_{\mathfrak{b}_1 \in \vartheta^{-1}(\mathfrak{b})} \zeta_{\mathfrak{A}_2}(\mathfrak{b}_1)) \} \right]. \end{aligned}$$

Now,

$$\begin{aligned} \bigcap_{\mathfrak{g} \in \mathfrak{G}} \left[\sup_{\mathfrak{r}^\mathfrak{g} = \mathfrak{a}\mathfrak{b}} \{ \min(\sup_{\mathfrak{a}_1 \in \vartheta^{-1}(\mathfrak{a})} \zeta_{\mathfrak{A}_1}(\mathfrak{a}_1), \sup_{\mathfrak{b}_1 \in \vartheta^{-1}(\mathfrak{b})} \zeta_{\mathfrak{A}_2}(\mathfrak{b}_1)) \} \right] &\leq \bigcap_{\mathfrak{g} \in \mathfrak{G}} \{ \min(\zeta_{\mathfrak{A}_1}(\mathfrak{a}_2), \zeta_{\mathfrak{A}_2}(\mathfrak{b}_2)) \} \\ &\leq \bigcap_{\mathfrak{g} \in \mathfrak{G}} (\zeta_{\mathfrak{A}_1\mathfrak{A}_2})(\mathfrak{a}_2\mathfrak{b}_2) \\ &\leq \sup_{\mathfrak{c} \in \vartheta^{-1}(\mathfrak{r}^\mathfrak{g})} \bigcap_{\mathfrak{g} \in \mathfrak{G}} (\zeta_{\mathfrak{A}_1\mathfrak{A}_2})(\mathfrak{c}) \\ &= \sup_{\mathfrak{c} \in \vartheta^{-1}(\mathfrak{r}^\mathfrak{g})} (\zeta_{(\vartheta(\mathfrak{A}_1)\vartheta(\mathfrak{A}_2))^\mathfrak{G}})(\mathfrak{r}) \\ &= \zeta_{(\vartheta(\mathfrak{A}_1)\vartheta(\mathfrak{A}_2))^\mathfrak{G}}(\mathfrak{r}) \end{aligned}$$

for $\mathfrak{a}_2 \in \vartheta^{-1}(\mathfrak{a})$, $\mathfrak{b}_2 \in \vartheta^{-1}(\mathfrak{b})$, $\mathfrak{a}_2\mathfrak{b}_2 = \mathfrak{c}$. This shows that $\zeta_{(\vartheta(\mathfrak{A}_1)\vartheta(\mathfrak{A}_2))^\mathfrak{G}} \subseteq \zeta_{(\vartheta(\mathfrak{A}_1\mathfrak{A}_2))^\mathfrak{G}}$. Moreover,

$$\begin{aligned} \eta_{(\vartheta(\mathfrak{A}_1)\vartheta(\mathfrak{A}_2))^\mathfrak{G}}(\mathfrak{r}) &= \bigcap_{\mathfrak{g} \in \mathfrak{G}} \eta_{(\vartheta(\mathfrak{A}_1)\vartheta(\mathfrak{A}_2))^\mathfrak{G}}(\mathfrak{r}) \\ &= \bigcap_{\mathfrak{g} \in \mathfrak{G}} (\eta_{(\vartheta(\mathfrak{A}_1)\vartheta(\mathfrak{A}_2))^\mathfrak{G}})(\mathfrak{r}^\mathfrak{g}) \\ &= \bigcap_{\mathfrak{g} \in \mathfrak{G}} \left[\inf_{\mathfrak{r}^\mathfrak{g} = \mathfrak{a}\mathfrak{b}} \{ \max(\eta_{\vartheta(\mathfrak{A}_1)}(\mathfrak{a}), \eta_{\vartheta(\mathfrak{A}_2)}(\mathfrak{b})) \} \right] \\ &= \bigcap_{\mathfrak{g} \in \mathfrak{G}} \left[\inf_{\mathfrak{r}^\mathfrak{g} = \mathfrak{a}\mathfrak{b}} \{ \max(\inf_{\mathfrak{a}_1 \in \vartheta^{-1}(\mathfrak{a})} \eta_{\mathfrak{A}_1}(\mathfrak{a}_1), \inf_{\mathfrak{b}_1 \in \vartheta^{-1}(\mathfrak{b})} \eta_{\mathfrak{A}_2}(\mathfrak{b}_1)) \} \right]. \end{aligned}$$

Now,

$$\begin{aligned} \bigcap_{\mathfrak{g} \in \mathfrak{G}} \left[\inf_{\mathfrak{r}^\mathfrak{g} = \mathfrak{a}\mathfrak{b}} \{ \max(\inf_{\mathfrak{a}_1 \in \vartheta^{-1}(\mathfrak{a})} \eta_{\mathfrak{A}_1}(\mathfrak{a}_1), \inf_{\mathfrak{b}_1 \in \vartheta^{-1}(\mathfrak{b})} \eta_{\mathfrak{A}_2}(\mathfrak{b}_1)) \} \right] &\geq \bigcap_{\mathfrak{g} \in \mathfrak{G}} \{ \max(\eta_{\mathfrak{A}_1}(\mathfrak{a}_2), \eta_{\mathfrak{A}_2}(\mathfrak{b}_2)) \} \\ &\geq \bigcap_{\mathfrak{g} \in \mathfrak{G}} (\eta_{\mathfrak{A}_1\mathfrak{A}_2})(\mathfrak{a}_2\mathfrak{b}_2) \\ &\geq \inf_{\mathfrak{c} \in \vartheta^{-1}(\mathfrak{r}^\mathfrak{g})} \bigcap_{\mathfrak{g} \in \mathfrak{G}} (\eta_{\mathfrak{A}_1\mathfrak{A}_2})(\mathfrak{c}) \\ &= \inf_{\mathfrak{c} \in \vartheta^{-1}(\mathfrak{r}^\mathfrak{g})} \eta_{(\vartheta(\mathfrak{A}_1)\vartheta(\mathfrak{A}_2))^\mathfrak{G}}(\mathfrak{r}) \\ &= (\eta_{(\vartheta(\mathfrak{A}_1)\vartheta(\mathfrak{A}_2))^\mathfrak{G}})(\mathfrak{r}) \end{aligned}$$

for some $\mathfrak{a}_2 \in \vartheta^{-1}(\mathfrak{a})$, $\mathfrak{b}_2 \in \vartheta^{-1}(\mathfrak{b})$, $\mathfrak{a}_2\mathfrak{b}_2 = \mathfrak{c}$. This shows that $\eta_{(\vartheta(\mathfrak{A}_1)\vartheta(\mathfrak{A}_2))^\mathfrak{G}} \supseteq \eta_{(\vartheta(\mathfrak{A}_1\mathfrak{A}_2))^\mathfrak{G}}$.

(iii) Let $\mathfrak{s} \in \mathfrak{R}_1$. Then

$$\begin{aligned}\zeta_{(\vartheta^{-1}(\mathfrak{B}_1))^\mathfrak{G}}(\mathfrak{s}) &= \bigcap_{\mathfrak{g} \in \mathfrak{G}} \zeta_{(\vartheta^{-1}(\mathfrak{B}_1))^\mathfrak{g}}(\mathfrak{s}) \\ &= \bigcap_{\mathfrak{g} \in \mathfrak{G}} (\zeta_{\vartheta^{-1}(\mathfrak{B}_1)}(\mathfrak{s}^\mathfrak{g})) \\ &= \bigcap_{\mathfrak{g} \in \mathfrak{G}} \zeta_{\mathfrak{B}_1}((\vartheta(\mathfrak{s}))^\mathfrak{g}) \\ &\leq \bigcap_{\mathfrak{g} \in \mathfrak{G}} \zeta_{\mathfrak{B}_2}((\vartheta(\mathfrak{s}))^\mathfrak{g}) \\ &= \zeta_{\mathfrak{B}_2^\mathfrak{G}}((\vartheta(\mathfrak{s}))) \\ &= \zeta_{\vartheta^{-1}(\mathfrak{B}_2^\mathfrak{G})}(\mathfrak{s}).\end{aligned}$$

Therefore, $\zeta_{(\vartheta^{-1}(\mathfrak{B}_1))^\mathfrak{G}} \subseteq \zeta_{\vartheta^{-1}(\mathfrak{B}_2^\mathfrak{G})}$. Furthermore,

$$\begin{aligned}(\eta_{(\vartheta^{-1}(\mathfrak{B}_1))^\mathfrak{G}})(\mathfrak{s}) &= \bigcap_{\mathfrak{g} \in \mathfrak{G}} (\eta_{(\vartheta^{-1}(\mathfrak{B}_1))^\mathfrak{g}})(\mathfrak{s}) = \bigcap_{\mathfrak{g} \in \mathfrak{G}} (\eta_{\vartheta^{-1}(\mathfrak{B}_1)})(\mathfrak{s}^\mathfrak{g}) \\ &= \bigcap_{\mathfrak{g} \in \mathfrak{G}} \eta_{\mathfrak{B}_1}(\vartheta(\mathfrak{s}^\mathfrak{g})) = \bigcap_{\mathfrak{g} \in \mathfrak{G}} \eta_{\mathfrak{B}_1}((\vartheta(\mathfrak{s}))^\mathfrak{g}) \\ &\geq \bigcap_{\mathfrak{g} \in \mathfrak{G}} \eta_{\mathfrak{B}_2}((\vartheta(\mathfrak{s}))^\mathfrak{g}) = \bigcap_{\mathfrak{g} \in \mathfrak{G}} \eta_{\mathfrak{B}_2^\mathfrak{G}}((\vartheta(\mathfrak{s}))) \\ &= \eta_{\mathfrak{B}_2^\mathfrak{G}}((\vartheta(\mathfrak{s}))) = \eta_{\vartheta^{-1}(\mathfrak{B}_2^\mathfrak{G})}(\mathfrak{s}).\end{aligned}$$

Therefore, $(\vartheta^{-1}(\mathfrak{B}_1))^\mathfrak{G} \subseteq \vartheta^{-1}(\mathfrak{B}_2^\mathfrak{G})$.

4 IF $\mathfrak{G}$ -Semiprimary ideals

Definition 4.1 ([9]). An IF ideal $\mathfrak{A} = (\zeta_{\mathfrak{A}}, \eta_{\mathfrak{A}})$ of a ring $\mathfrak{R}$ is said to be IF semiprimary if for any $x, y \in \mathfrak{R}$, $\zeta_{\mathfrak{A}}(xy) \leq \zeta_{\mathfrak{A}}(x^m)$ and $\eta_{\mathfrak{A}}(xy) \geq \eta_{\mathfrak{A}}(x^n)$ or $\zeta_{\mathfrak{A}}(xy) \leq \zeta_{\mathfrak{A}}(y^m)$ and $\eta_{\mathfrak{A}}(xy) \geq \eta_{\mathfrak{A}}(y^n)$ for some positive integers m and n .

Definition 4.2. Let $\mathfrak{G}$ be a group acting on a ring $\mathfrak{R}$. Then an IF semiprimary ideal $\mathfrak{A}$ of $\mathfrak{R}$ is said to be an IF $\mathfrak{G}$ -semiprimary ideal of $\mathfrak{R}$ if $\mathfrak{A}$ is $\mathfrak{G}$ -invariant.

Lemma 4.1. Every IF primary ideal of a ring $\mathfrak{R}$ is an IF semiprimary ideal of $\mathfrak{R}$.

Remark 4.1. In a general context, the converse of Lemma 4.1 does not hold true.

Example 4.1. Consider an IF ideal $\mathfrak{A} = (\zeta_{\mathfrak{A}}, \eta_{\mathfrak{A}})$ of the ring of integers $\mathbb{Z}$, defined as follows:

$$\zeta_{\mathfrak{A}}(\mathfrak{z}) = \begin{cases} 0 & \text{for } \mathfrak{z} \notin \langle p \rangle, \\ \frac{n}{n+1} & \text{for } \mathfrak{z} \in \langle p^n \rangle \sim \langle p^{n+1} \rangle, \text{ and } \\ 1 & \mathfrak{z} = 0 \end{cases} \quad \text{and} \quad \eta_{\mathfrak{A}}(\mathfrak{z}) = \begin{cases} 1 & \text{for } \mathfrak{z} \notin \langle p \rangle, \\ \frac{1}{n+2} & \text{for } \mathfrak{z} \in \langle p^n \rangle \sim \langle p^{n+1} \rangle, \\ 0 & \mathfrak{z} = 0, \end{cases}$$

where n is any positive integer. Then $\zeta_{\mathfrak{A}}(p^2 p^3) = \zeta_{\mathfrak{A}}(p^5) = \frac{5}{6} \neq \frac{2}{3} = \zeta_{\mathfrak{A}}(p^2)$ and $\eta_{\mathfrak{A}}(p^2 p^3) = \eta_{\mathfrak{A}}(p^5) = \frac{1}{7} \neq \frac{1}{4} = \eta_{\mathfrak{A}}(p^2)$. Also for any positive integer m , $\zeta_{\mathfrak{A}}((p^3)^m) \neq \frac{5}{6} = \zeta_{\mathfrak{A}}(p^2 p^3)$ and $\eta_{\mathfrak{A}}((p^3)^m) \neq \eta_{\mathfrak{A}}((p^5)) = \frac{1}{7}$. Thus $\mathfrak{A}$ is not an IF primary ideal. However, it is an IF semiprimary ideal of $\mathbb{Z}$.

Lemma 4.2. Let $\mathfrak{G}$ be a group acting on rings $\mathfrak{R}_1, \mathfrak{R}_2$ and $\vartheta : \mathfrak{R}_1 \rightarrow \mathfrak{R}_2$ be a $\mathfrak{G}$ -epimorphism. Then

- (i) If $\mathfrak{A}$ is an IF $\mathfrak{G}$ -semiprimary ideal of $\mathfrak{R}_1$, which is constant on $\ker \vartheta$, then $\vartheta(\mathfrak{A})$ is an IF $\mathfrak{G}$ -semiprimary ideal of $\mathfrak{R}_2$.
- (ii) If $\mathfrak{B}$ is an IF $\mathfrak{G}$ -semiprimary ideal of $\mathfrak{R}_2$, then $\vartheta^{-1}(\mathfrak{B})$ is an IF $\mathfrak{G}$ -semiprimary ideal of $\mathfrak{R}_1$.

Proof. Proof is similar to that of Lemma 3.1.

The behavior of IF ideals under a group action is determined by a key identity stated in the following theorem. An important structural result shows that the $\mathfrak{G}$ -invariant radical operator can be distributed over the intersection of ideals.

Theorem 4.1. Let $\mathfrak{G}$ be a group acting on a ring $\mathfrak{R}$, and $\mathfrak{A}_1 = (\zeta_{\mathfrak{A}_1}, \eta_{\mathfrak{A}_1})$ and $\mathfrak{A}_2 = (\zeta_{\mathfrak{A}_2}, \eta_{\mathfrak{A}_2})$ be IF ideals of $\mathfrak{R}$. Then $(\sqrt{\mathfrak{A}_1 \cap \mathfrak{A}_2})^{\mathfrak{G}} = (\sqrt{\mathfrak{A}_1})^{\mathfrak{G}} \cap (\sqrt{\mathfrak{A}_2})^{\mathfrak{G}}$.

Proof. Since $\zeta_{\mathfrak{A}_1} \subseteq \zeta_{\sqrt{\mathfrak{A}_1}}$ and $\zeta_{\mathfrak{A}_2} \subseteq \zeta_{\sqrt{\mathfrak{A}_2}}$, then $\zeta_{\mathfrak{A}_1}^{\mathfrak{G}} \subseteq \zeta_{(\sqrt{\mathfrak{A}_1})^{\mathfrak{G}}}$ and $\zeta_{\mathfrak{A}_2}^{\mathfrak{G}} \subseteq \zeta_{(\sqrt{\mathfrak{A}_2})^{\mathfrak{G}}}$. Thus $\zeta_{(\sqrt{\mathfrak{A}_1})^{\mathfrak{G}} \cap (\sqrt{\mathfrak{A}_2})^{\mathfrak{G}}} \subseteq \zeta_{\sqrt{\mathfrak{A}_1} \cap \sqrt{\mathfrak{A}_2}} \subseteq \zeta_{(\sqrt{\mathfrak{A}_1 \cap \mathfrak{A}_2})^{\mathfrak{G}}}$. For other side assume that $x \in \mathfrak{R}$. Then

$$\begin{aligned}
 \zeta_{(\sqrt{\mathfrak{A}_1 \cap \mathfrak{A}_2})^{\mathfrak{G}}}(x) &= \bigcap_{g \in \mathfrak{G}} \zeta_{(\sqrt{\mathfrak{A}_1 \cap \mathfrak{A}_2})^{\mathfrak{G}}}(x^g) \\
 &= \bigcap_{g \in \mathfrak{G}} \zeta_{(\sqrt{\mathfrak{A}_1 \cap \mathfrak{A}_2})}(x^g) \\
 &= \bigcap_{g \in \mathfrak{G}} [\sup\{(\zeta_{\mathfrak{A}_1 \cap \mathfrak{A}_2})(x^g)^l \mid l > 0\}] \\
 &= \bigcap_{g \in \mathfrak{G}} [\sup\{\min(\zeta_{\mathfrak{A}_1}(x^g)^l, \zeta_{\mathfrak{A}_2}(x^g)^l) \mid l > 0\}] \\
 &\leq \bigcap_{g \in \mathfrak{G}} [\min\{\sup(\zeta_{\mathfrak{A}_1}(x^g)^l, \mid l > 0), \sup(\zeta_{\mathfrak{A}_2}(x^g)^l \mid l > 0)\}] \\
 &= \min\left[\bigcap_{g \in \mathfrak{G}} \zeta_{\sqrt{\mathfrak{A}_1}}(x^g), \bigcap_{g \in \mathfrak{G}} \zeta_{\sqrt{\mathfrak{A}_2}}(x^g)\right] \\
 &= \min\{\zeta_{(\sqrt{\mathfrak{A}_1})^{\mathfrak{G}}}(x), \zeta_{(\sqrt{\mathfrak{A}_2})^{\mathfrak{G}}}(x)\} \\
 &= \zeta_{(\sqrt{\mathfrak{A}_1})^{\mathfrak{G}} \cap (\sqrt{\mathfrak{A}_2})^{\mathfrak{G}}}(x).
 \end{aligned} \tag{4.1}$$

Moreover, $\eta_{\mathfrak{A}_1} \supseteq \eta_{\sqrt{\mathfrak{A}_1}}$ and $\eta_{\mathfrak{A}_2} \supseteq \eta_{\sqrt{\mathfrak{A}_2}}$, then $\eta_{\mathfrak{A}_1}^{\mathfrak{G}} \supseteq \eta_{(\sqrt{\mathfrak{A}_1})^{\mathfrak{G}}}$ and $\eta_{\mathfrak{A}_2}^{\mathfrak{G}} \supseteq \eta_{(\sqrt{\mathfrak{A}_2})^{\mathfrak{G}}}$. This implies that $\eta_{(\sqrt{\mathfrak{A}_1})^{\mathfrak{G}} \cap (\sqrt{\mathfrak{A}_2})^{\mathfrak{G}}} \supseteq \eta_{\sqrt{\mathfrak{A}_1} \cap \sqrt{\mathfrak{A}_2}} \supseteq \eta_{(\sqrt{\mathfrak{A}_1 \cap \mathfrak{A}_2})^{\mathfrak{G}}}$. For other side,

$$\begin{aligned}
 \eta_{(\sqrt{\mathfrak{A}_1 \cap \mathfrak{A}_2})^{\mathfrak{G}}}(x) &= \bigcap_{g \in \mathfrak{G}} \eta_{(\sqrt{\mathfrak{A}_1 \cap \mathfrak{A}_2})^{\mathfrak{G}}}(x^g) \\
 &= \bigcap_{g \in \mathfrak{G}} (\eta_{\sqrt{\mathfrak{A}_1 \cap \mathfrak{A}_2}})(x^g) \\
 &= \bigcap_{g \in \mathfrak{G}} [\inf\{(\eta_{\mathfrak{A}_1 \cap \mathfrak{A}_2})(x^g)^l \mid l > 0\}] \\
 &= \bigcap_{g \in \mathfrak{G}} [\inf\{\min(\eta_{\mathfrak{A}_1}(x^g)^l, \eta_{\mathfrak{A}_2}(x^g)^l) \mid l > 0\}] \\
 &\geq \bigcap_{g \in \mathfrak{G}} [\min\{\inf(\eta_{\mathfrak{A}_1}(x^g)^l, \mid l > 0), \inf(\eta_{\mathfrak{A}_2}(x^g)^l \mid l > 0)\}] \\
 &= \min\left[\bigcap_{g \in \mathfrak{G}} \eta_{\sqrt{\mathfrak{A}_1}}(x^g), \bigcap_{g \in \mathfrak{G}} \eta_{\sqrt{\mathfrak{A}_2}}(x^g)\right] \\
 &= \eta_{(\sqrt{\mathfrak{A}_1})^{\mathfrak{G}} \cap (\sqrt{\mathfrak{A}_2})^{\mathfrak{G}}}(x).
 \end{aligned} \tag{4.2}$$

Thus, $(\sqrt{\mathfrak{A}_1 \cap \mathfrak{A}_2})^{\mathfrak{G}} = (\sqrt{\mathfrak{A}_1})^{\mathfrak{G}} \cap (\sqrt{\mathfrak{A}_2})^{\mathfrak{G}}$. The ideas of IF prime, IF semiprime, and IF semiprimary ideals play a key role in studying the structure of rings. The following theorem connects these concepts directly, showing that the combination of IF semiprime and IF semiprimary properties leads to the important condition of being IF prime.

Theorem 4.2. If $\mathfrak{A}$ is an IF semiprime as well as IF semiprimary ideal of a ring $\mathfrak{R}$, then $\mathfrak{A}$ is an IF prime ideal of $\mathfrak{R}$.

Proof. Let $\mathfrak{A} = (\zeta_{\mathfrak{A}}, \eta_{\mathfrak{A}})$ be an IF semiprime as well as semiprimary ideal of the ring $\mathfrak{R}$. Then

$$\zeta_{\mathfrak{A}}(x^n) = \zeta_{\mathfrak{A}}(x) \quad \text{or} \quad \eta_{\mathfrak{A}}(x^n) = \eta_{\mathfrak{A}}(x) \tag{4.3}$$

as well as

$$\begin{aligned}
 &\zeta_{\mathfrak{A}}(xy) \leq \zeta_{\mathfrak{A}}(x^n) \quad \text{and} \quad \eta_{\mathfrak{A}}(xy) \geq \eta_{\mathfrak{A}}(x^n) \\
 \text{or} \quad &\zeta_{\mathfrak{A}}(xy) \leq \zeta_{\mathfrak{A}}(y^m) \quad \text{and} \quad \eta_{\mathfrak{A}}(xy) \geq \eta_{\mathfrak{A}}(y^m)
 \end{aligned} \tag{4.4}$$

for $x, y \in \mathfrak{R}$, $m, n \in \mathbb{Z}^+$. From Equations (4.3) and (4.4),

$$\begin{aligned}
 &\zeta_{\mathfrak{A}}(xy) \leq \zeta_{\mathfrak{A}}(x) \quad \text{and} \quad \eta_{\mathfrak{A}}(xy) \geq \eta_{\mathfrak{A}}(x) \\
 \text{or} \quad &\zeta_{\mathfrak{A}}(xy) \leq \zeta_{\mathfrak{A}}(y) \quad \text{and} \quad \eta_{\mathfrak{A}}(xy) \geq \eta_{\mathfrak{A}}(y).
 \end{aligned}$$

Since $\mathfrak{A}$ is an IF ideal, then

$$\begin{aligned} & \zeta_{\mathfrak{A}}(x) \leq \zeta_{\mathfrak{A}}(xy) \leq \zeta_{\mathfrak{A}}(x) \quad \text{and} \quad \eta_{\mathfrak{A}}(x) \geq \eta_{\mathfrak{A}}(xy) \geq \eta_{\mathfrak{A}}(x) \\ \text{or} \quad & \zeta_{\mathfrak{A}}(y) \leq \zeta_{\mathfrak{A}}(xy) \leq \zeta_{\mathfrak{A}}(y) \quad \text{and} \quad \eta_{\mathfrak{A}}(y) \geq \eta_{\mathfrak{A}}(xy) \geq \eta_{\mathfrak{A}}(y). \end{aligned}$$

Thus, $\zeta_{\mathfrak{A}}(xy) = \zeta_{\mathfrak{A}}(x)$ and $\eta_{\mathfrak{A}}(xy) = \eta_{\mathfrak{A}}(x)$ or $\zeta_{\mathfrak{A}}(xy) = \zeta_{\mathfrak{A}}(y)$ and $\eta_{\mathfrak{A}}(xy) = \eta_{\mathfrak{A}}(y)$. Therefore, $\mathfrak{A}$ is an IF prime ideal of $\mathfrak{R}$.

The following corollary is direct from Theorem 4.2.

Corollary 4.1. If $\mathfrak{A}$ is an IF primary as well as IF semiprime ideal of a ring $\mathfrak{R}$, then $\mathfrak{A}$ is an IF prime ideal of $\mathfrak{R}$.

5 Conclusion

In this paper we studied how intuitionistic fuzzy ideals work together with $\mathfrak{G}$ -homomorphisms. We have explained how certain properties, like IF $\mathfrak{G}$ -primary and semiprimary, are kept under these mappings. We also proved an important result that connects IF semiprimary and IF prime ideals. These findings help us better understand the structure of such systems and point to new research directions. In the future, this work could be extended to other algebraic systems, such as semirings, near rings or used in fields where fuzzy algebraic ideas are applied.

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