

Lie ideals and generalized Jordan triple derivations associated with Hochschild 2-cocycles in rings

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Abstract In this paper, we prove: If R is a prime ring of characteristic different from two and S is a non-central square-closed Lie ideal of R and f is a generalized Jordan triple derivation on S associated with Hochschild 2-cocycle β , then f is a generalized derivation on S associated with β .

1 Introduction

Let R represents an associative ring with Z as its center. The primeness (semiprimeness) of R gives if for $w, s \in R, wRs = (0)$ ($w \in R, wRw = (0)$), then $w = 0$ or $s = 0$ ($w = 0$). Herstein in [7] demonstrated that if δ is a Jordan derivation on a prime ring R and $\text{char}(R) \neq 2$, then δ is a derivation. If R is semiprime rings, Brešar demonstrated the same outcome in [3]. Additionally, in [4], he demonstrated that if δ is a Jordan triple derivation of a 2-torsion free semiprime ring, then δ is a derivation. Awtar showed in [1] Herstein's result in case of square-closed Lie ideal. More generalization can be found in [8] and [11].

Now a bi-additive map $\beta: R \times R \rightarrow W$ is a Hochschild 2-cocycle, where W is an R -bimodule, if $t\beta(l, r) - \beta(tl, r) + \beta(t, lr) - \beta(t, l)r = 0 \quad \forall t, l, r \in R$, and β is symmetric (skew symmetric) if $\beta(t, l) = \beta(l, t)$ (resp. $\beta(t, l) = -\beta(l, t)$). The following definitions were introduced by Nakajima [9]: An additive map $f: R \rightarrow W$ and $f(ed) = f(e)d + ef(d) + \beta(e, d) \quad \forall e, d \in R$ will be generalized derivation associated with Hochschild 2-cocycles β , additionally if $f(l^2) = f(l)l + lf(l) + \beta(l, l) \quad \forall l \in R$, then f is a generalized Jordan derivation associated with β . At $\beta = 0$, we find f denotes the derivation and Jordan derivation. Also, Nakajima demonstrated in [9] that: Every generalized Jordan derivation of R associated with a Hochschild 2-cocycle β is a generalized derivation on R associated with β under some assumptions.

In [10] Rehman and Hongan obtained Nakajima's result in the case of square-closed Lie ideal.

Generalized Jordan triple derivations associated with Hochschild 2-cocycles were defined by Ezzat and Nabel in [5] as follows: The additive map $f: R \rightarrow W$ satisfies $f(l dl) = f(l)dl + lf(d)l + ldf(l) + \beta(l, d)l + \beta(ld, l) \quad \forall l, d \in R$

In this work, we prove the following result: Let R be a prime ring with characteristic not two and S a non-central square-closed Lie ideal of R . If $f: R \rightarrow R$ is an additive map and $f(pep) = f(p)ep + pf(e)p + pef(p) + \beta(p, e)p + \beta(pe, p) \quad \forall p, e \in S$ where β is a Hochschild 2-cocycle, then $f(pe) = f(p)e + pf(e) + \beta(p, e) \quad \forall p, e \in S$.

2 Main results

In this section, f is an additive map from R into R and $A, B: R \times R \times R \rightarrow R$, $A(p, e, g) = f(peg) - f(p)eg - pf(e)g - pef(g) - \beta(p, e)g - \beta(pe, g)$ where β be a bi-additive map from $R \times R$ into R , and $B(d, h, l) = dhl - lhd$.

We start by proving the lemmas listed below, which are utilized in our main conclusion.

Lemma 2.1. *Let R be ring and $\beta: R \times R \rightarrow R$ is a Hochschild 2-cocycle. Then we have the following relations $\forall p, e, q, g, c \in R$*

- (i) $\beta(pe, q) - p\beta(e, q) = \beta(p, eq) - \beta(p, e)q.$
- (ii) $\beta(peq, pe) - \beta((peq)p, e) = \beta(peq, p)e - peq\beta(p, e).$
- (iii) $\beta((pe)g, c) - pe\beta(g, c) = \beta(pe, gc) - \beta(pe, g)c.$
- (iv) $\beta(pegc, (ge)p) - \beta((pegc)(ge), p) = \beta(pegc, ge)p - pegc\beta(ge, p).$
- (v) $\beta((ge)p, c) - ge\beta(p, c) = \beta(ge, pc) - \beta(ge, p)c.$
- (vi) $\beta(gepc, (pe)g) - \beta((gepc)(pe), g) = \beta(gepc, pe)g - gepc\beta(pe, g).$
- (vii) $\beta(pe, gc) = p\beta(e, gc) + \beta(p, e(gc)) - \beta(p, e)gc.$
- (viii) $\beta(pegc, ge) = -gepc\beta(g, e) + \beta((pegc)g, e) + \beta(pegc, g)e.$
- (ix) $\beta(ge, pc) = g\beta(e, pc) + \beta(g, e(pc)) - \beta(g, e)pc.$
- (x) $\beta(gepc, pe) = -gepc\beta(p, e) + \beta((gepc)p, e) + \beta(gepc, p)e.$
- (xi) $\beta(e, gc)g - e\beta(gc, g) - \beta(e, (gc)g) = -\beta(e(gc), g).$
- (xii) $\beta(p(egcg), e) - p\beta(egcg, e) - \beta(p, (egcg)e) = -\beta(p, egcg)e.$
- (xiii) $\beta(e, pc)p - e\beta(pc, p) - \beta(e, (pc)p) = -\beta(e(pc), p).$
- (xiv) $\beta(g(epcp), e) - g\beta(epcp, e) - \beta(g, (epcp)e) = -\beta(g, epcp)e.$

Lemma 2.2. *Let R be a 2-torsion free semiprime ring and S be a Lie ideal of R . Then $B(d, h, l) = 0 \quad \forall d, h, l \in S$ if and only if $S \subseteq Z$.*

Proof. If $S \subseteq Z$, then $B(d, h, l) = dhl - lhd = 0 \quad \forall d, h, l \in S$. Conversely, let $B(d, h, l) = 0 \quad \forall d, h, l \in S$, that is

$$dhl = lhd \quad (2.1)$$

Putting $[l, r]$ instead of l in (2.1), where $r \in R$, then

$$dh[l, r] = [l, r]hd \quad (2.2)$$

Substituting rl for r in (2.2), we obtain

$$dh[l, r]l = [l, r]lhd \quad (2.3)$$

From (2.2) and (2.3), we find

$$[l, r]hdl = [l, r]lhd \quad (2.4)$$

Hence,

$$[l, r][l, hd] = 0 \quad (2.5)$$

Putting hdr instead of r in (2.5) implies that

$$[l, hd]r[l, hd] = 0$$

Hence $[l, hd] = 0$, i.e.,

$$[l, h]d = -h[l, d] \quad (2.6)$$

Since $lhd = dhl \quad \forall d, h, l \in S$, then (2.4) lies to $[l, r]hdl = [l, r]dhl$ that is

$$[l, r][d, h]l = 0 \quad (2.7)$$

By (2.6), we have $[d, h]l = -h[d, l]$. Thus (2.7) becomes

$$[l, r]h[d, l] = 0 \quad (2.8)$$

Writting dr instead of r in (2.8), we find $[l, d]rhd[d, l] = 0 \quad \forall d, h, l \in S, r \in R$. Therefore

$$h[d, l] = 0 \quad (2.9)$$

Upon substituting $[d, r]$ for h in (2.9), we obtain

$$[d, r][d, l] = 0 \quad (2.10)$$

Substituting lr for r in (2.10), we arrive at $[d, l] = 0$ for every $d, l \in S$. Using [6, Lemma 1] implies that $S \subseteq Z$. $\square$

Lemma 2.3. *Let S be a Lie ideal of R . If $f(pep) = f(p)ep + pf(e)p + pef(p) + \beta(p, e)p + \beta(pe, p) \quad \forall p, e \in S$ where β is a Hochschild 2-cocycle, then we have $\forall p, e, g \in S$*
(i) $f(peg + gep) = f(p)eg + pf(e)g + pef(g) + f(g)ep + gf(e)p + gef(p) + \beta(p, e)g + \beta(pe, g) + \beta(g, e)p + \beta(ge, p)$.
(ii) $A(p, e, g) = -A(g, e, p)$.

Proof. (i) We have

$$f(pep) = f(p)ep + pf(e)p + pef(p) + \beta(p, e)p + \beta(pe, p) \quad \forall p, e \in S. \quad (2.11)$$

Substituting $p + g$ for p in (2.11), we get

$$\begin{aligned} f((p + g)e(p + g)) &= f((p + g))e(p + g) + (p + g)f(e)(p + g) + (p + g)ef(p + g) \\ &\quad + \beta(p + g, e)(p + g) + \beta((p + g)e, p + g) \quad \forall p, e, g \in S. \end{aligned} \quad (2.12)$$

Also, we have

$$f((p + g)e(p + g)) = f(pep) + f(peg + gep) + f(geg) \quad \forall p, e, g \in S. \quad (2.13)$$

From (2.12) and (2.13), we obtain the needed result.

(ii) Follows easily from (i). $\square$

Lemma 2.4. *Let R be a prime ring with characteristic not two and $S \not\subseteq Z$ square-closed Lie ideal of R . If $A(p, e, g) = 0 \quad \forall p, e, g \in S$, then $f(pe) = f(p)e + pf(e) + \beta(p, e) \quad \forall p, e \in S$.*

Proof. We have $A(p, e, g) = 0 \quad \forall p, e, g \in S$, that is

$$f(peg) = f(p)eg + pf(e)g + pef(g) + \beta(p, e)g + \beta(pe, g) \quad (2.14)$$

Now, for $p, e, q \in S$ let $M = f(4peqp)$, then $0 = M - M = f((2pe)q(2pe)) - f(p(4eqp)e)$. By (2.14) and our assumption, we find $0 = 2f(2pe)qpe + 4pef(q)pe + 2\beta(2pe, q)pe + 2peqf(2pe) + \beta(2peq, 2pe) - 4f(p)eqpe - 4pf(e)qpe - 4pef(q)pe - 4peqf(p)e - 4p\beta(e, q)pe - 4p\beta(eq, p)e - \beta(p, 4eqp)e - 4peqp f(e) - \beta(4peqp, e)$. Since f additive and β bi-additive, then $0 = 4(f(pe)qpe + pef(q)pe + \beta(pe, q)pe + peqf(pe) + \beta(peq, pe) - f(p)eqpe - pf(e)qpe - pef(q)pe - peqf(p)e - p\beta(e, q)pe - p\beta(eq, p)e - \beta(p, eqp)e - peqp f(e) - \beta(peqp, e))$.

Since R has characteristic not 2, then $0 = f(pe)qpe + pef(q)pe + \beta(pe, q)pe + peqf(pe) + \beta(peq, pe) - f(p)eqpe - pf(e)qpe - pef(q)pe - peqf(p)e - p\beta(e, q)pe - p\beta(eq, p)e - \beta(p, eqp)e - peqp f(e) - \beta(peqp, e)$ So,

$$\begin{aligned} 0 &= (f(pe) - f(p)e - pf(e))qpe + peq(f(pe) - f(p)e - pf(e)) + (\beta(pe, q) - p\beta(e, q))pe \\ &\quad + (\beta(peq, pe) - \beta(peqp, e)) - p\beta(eq, p)e - \beta(p, eqp)e. \end{aligned} \quad (2.15)$$

By Lemma 2.1 (i) and (ii), we obtain $(f(pe) - f(p)e - pf(e) - \beta(p, e))qpe + peq(f(pe) - f(p)e - pf(e) - \beta(p, e)) + \beta(p, eq)pe + \beta(peq, p)e - p\beta(eq, p)e - \beta(p, eqp)e = 0$. Also, we have $\beta(p, eq)p + \beta(peq, p) - p\beta(eq, p) - \beta(p, eqp) = 0$. Therefore $(f(pe) - f(p)e - pf(e) - \beta(p, e))q(pe) + (pe)q(f(pe) - f(p)e - pf(e) - \beta(p, e)) = 0 \quad \forall p, e, q \in S$. By [10, Lemma 2.6], we find $(f(pe) - f(p)e - pf(e) - \beta(p, e))q(pe) = 0 \quad \forall p, e, q \in S$. [10, Lemma 2.7] gives $(f(pe) - f(p)e - pf(e) - \beta(p, e))q(ws) = 0 \quad \forall p, e, q, w, s \in S$. Using [2, Lemma 4] and our hypothesis $S \not\subseteq Z$ imply that $f(pe) = f(p)e + pf(e) + \beta(p, e)$ for any $p, e \in S$. $\square$

Lemma 2.5. *Let S be a square-closed Lie ideal of a ring R . If $f(pkp) = f(p)kp + pf(k)p + pkf(p) + \beta(p, k)p + \beta(pk, p) \quad \forall p, k \in S$ where β is a Hochschild 2-cocycle, then $A(p, k, g)cB(p, k, g) + B(p, k, g)cA(p, k, g) = 0 \quad \forall c, p, k, g \in S$.*

Proof. For $p, k, g, c \in S$, we let $T = f(16pkgcgkp + 16gkpcpkp)$. Then $0 = T - T = f((4pkg)c(4gkp) + (4gkp)c(4pkp)) - f(p(16kgcgk)p + g(16kpcpk)g)$.

By Lemma 2.3(i) and since f additive and β bi-additive, we find $0 = 16[f(pkg)cgkp + pkgf(c)gkp + \beta(pkg, c)gkp + pkgcf(gkp) + \beta(pkgc, gkp) + f(gkp)cpkg + gkpf(c)pkg + \beta(gkp, c)pkg + gkpcf(pkg) + \beta(gkpc, pkg) - (f(p)kgcgkp + pf(k)gcgkp + pkf(g)cgkp + pkgf(c)gkp + pk\beta(g, c)gkp + pkgcf(g)kp + pk\beta(gc, g)kp + p\beta(k, gcg)kp + pkgcgf(k)p + p\beta(kgcg, k)p + \beta(p, kgcgk)p + pkgcgkf(p) + \beta(pkgcgk, p) + f(g)kpcpkp + gf(k)pcpkp + gkf(p)cpkg + gkpf(c)pkg + gk\beta(p, c)pkg + gkpcf(p)kg + gk\beta(pc, p)kg + g\beta(k, pcg)kg + g\beta(kpcp, k)g + \beta(g, kpcpk)g + gkpcpkf(g) + \beta(gkpcpk, g)]$. Since R has characteristic not 2, then

$$\begin{aligned} 0 = & f(pkg)cgkp + pkgf(c)gkp + \beta(pkg, c)gkp + pkgcf(gkp) + \beta(pkgc, gkp) \\ & + f(gkp)cpkg + gkpf(c)pkg + \beta(gkp, c)pkg + gkpcf(pkg) + \beta(gkpc, pkg) \\ & - (f(p)kgcgkp + pf(k)gcgkp + pkf(g)cgkp + pkgf(c)gkp + pk\beta(g, c)gkp \\ & + pkgcf(g)kp + pk\beta(gc, g)kp + p\beta(k, gcg)kp + pkgcgf(k)p + p\beta(kgcg, k)p \\ & + \beta(p, kgcgk)p + pkgcgkf(p) + \beta(pkgcgk, p) + f(g)kpcpkp + gf(k)pcpkp \\ & + gkf(p)cpkg + gkpf(c)pkg + gk\beta(p, c)pkg + gkpcf(p)kg + gk\beta(pc, p)kg \\ & + g\beta(k, pcg)kg + g\beta(kpcp, k)g + \beta(g, kpcpk)g + gkpcpkf(g) + \beta(gkpcpk, g) \\ & + \beta(gkpcpk, g) \end{aligned} \quad (2.16)$$

Therefore,

$$\begin{aligned} 0 = & (f(pkg) - f(p)kg - pf(k)g - pkf(g))cgkp + pkgc(f(gkp) - f(g)kp - gf(k)p \\ & - gkf(p)) + (\beta(pkg, c) - pk\beta(g, c))gkp + (\beta(pkgc, gkp) - \beta(pkgcgk, p)) \\ & - (pk\beta(gc, g)kp + p\beta(k, gcg)kp + p\beta(kgcg, k)p + \beta(p, kgcgk)p) + (f(gkp) \\ & - f(g)kp - gf(k)p - gkf(p))cpkg + gkpc(f(pkg) - f(p)kg - pf(k)g - pkf(g)) \\ & + (\beta(gkp, c) - gk\beta(p, c))pkg + (\beta(gkpc, pkg) - \beta(gkpcpk, g)) - (gk\beta(pc, p)kg \\ & + g\beta(k, pcg)kg + g\beta(kpcp, k)g + \beta(g, kpcpk)g) \end{aligned} \quad (2.17)$$

From (iii-vi) in Lemma 2.1, we get

$$\begin{aligned} 0 = & (f(pkg) - f(p)kg - pf(k)g - pkf(g))cgkp + pkgc(f(gkp) - f(g)kp - gf(k)p \\ & - gkf(p)) + (\beta(pk, gc) - \beta(pk, g)c)gkp + (\beta(pkgc, gk)p - pkgc\beta(gk, p)) \\ & - (pk\beta(gc, g)kp + p\beta(k, gcg)kp + p\beta(kgcg, k)p + \beta(p, kgcgk)p) + (f(gkp) \\ & - f(g)kp - gf(k)p - gkf(p))cpkg + gkpc(f(pkg) - f(p)kg - pf(k)g - pkf(g)) \\ & + (\beta(gk, pc) - \beta(gk, p)c)pkg + (\beta(gkpc, pk)g - gkpc\beta(pk, g)) - (gk\beta(pc, p)kg \\ & + g\beta(k, pcg)kg + g\beta(kpcp, k)g + \beta(g, kpcpk)g) \end{aligned} \quad (2.18)$$

By (vii-x) in Lemma 2.1 and (2.18), we obtain

$$\begin{aligned} 0 = & (f(pkg) - f(p)kg - pf(k)g - pkf(g) - \beta(pk, g) - \beta(p, k)g)cgkp \\ & + pkgc((f(gkp) - f(g)kp - gf(k)p - gkf(p)) - \beta(gk, p) - \beta(g, k)p) \\ & + (p\beta(k, gc)gkp - pk\beta(gc, g)kp - p\beta(k, gcg)kp) + (\beta(pkgcg, k)p \\ & - p\beta(kgcg, k)p - \beta(p, kgcgk)p) + \beta(p, kgcg)gkp + \beta(pkgc, g)kp \\ & + (f(gkp) - f(g)kp - gf(k)p - gkf(p) - \beta(gk, p) - \beta(g, k)p)cpkg \\ & + gkpc(f(pkg) - f(p)kg - pf(k)g - pkf(g) - \beta(pk, g) - \beta(p, k)g) \\ & + (g\beta(k, pc)pkg - gk\beta(pc, p)kg - g\beta(k, pcg)kg) + (\beta(gkpcp, k)g \\ & - g\beta(kpcp, k)g - \beta(g, kpcpk)g) + \beta(g, kpc)pkg + \beta(gkpc, p)kg \end{aligned} \quad (2.19)$$

From (xi-xiv) in Lemma 2.1 and (2.19), we obtain

$$\begin{aligned} 0 = & A(p, k, g)cgkp + pkgcA(g, k, p) - p\beta(kgc, g)kp - \beta(p, kgc)kp \\ & + \beta(p, kgc)gkp + \beta(pkgc, g)kp + A(g, k, p)cpkg + gkpcA(p, k, g) \\ & - g\beta(kpc, p)kg - \beta(g, kpcp)kg + \beta(g, kpc)pkg + \beta(gkpc, p)kg \end{aligned} \quad (2.20)$$

From our hypothesis, we have $-p\beta(kgc, g) - \beta(p, (kgc)g) + \beta(p, kgc)g + \beta(p(kgc), g) = 0$ and $-g\beta(kpc, p) - \beta(g, (kpc)p) + \beta(g, kpc)p + \beta(g(kpc), p) = 0$. By (2.20), we find $A(p, k, g)cgkp + pkgcA(g, k, p) + A(g, k, p)cpkg + gkpcA(p, k, g) = 0$. By Lemma 2.3(ii), we arrive at $A(p, k, g)cgkp - pkgcA(p, k, g) - A(p, k, g)cpkg + gkpcA(p, k, g) = 0$. Hence, $A(p, k, g)cB(p, k, g) + B(p, k, g)cA(p, k, g) = 0 \quad \forall c, p, g, k \in S$. $\square$

Theorem 2.6. *Let R be a prime ring with characteristic not two and $S \not\subseteq Z$ square-closed Lie ideal of R . If $f(pep) = f(p)ep + pf(e)p + pf(p) + \beta(p, e)p + \beta(pe, p) \quad \forall p, e \in S$ where β is a Hochschild 2-cocycle, then $f(pe) = f(p)e + pf(e) + \beta(p, e) \quad \forall p, e \in S$.*

Proof. By Lemma 2.5, we have $A(p, e, g)cB(p, e, g) + B(p, e, g)cA(p, e, g) = 0 \quad \forall p, e, g, c \in S$. So, $4A(p, e, g)cB(p, e, g) + 4B(p, e, g)cA(p, e, g) = 0$. Since S square-closed, then $4B(p, e, g) \in S$ and [10, Lemma 2.6] gives $4A(p, e, g)cB(p, e, g) = 0$. Hence, we obtain $A(p, e, g)cB(p, e, g) = 0 \quad \forall c, p, e, g \in S$. It is obvious that A and B are tri-additive maps. Thus, [10, Lemma 2.7] arrives at $A(p, e, g)cB(d, h, l) = 0$ holds $\forall p, e, g, c, d, h, l \in S$. Since $S \not\subseteq Z$, then [2, Lemma 4] leads to either $B(d, h, l) = 0 \quad \forall d, h, l \in S$ or $A(p, e, g) = 0 \quad \forall p, e, g \in S$. If $B(d, h, l) = 0 \quad \forall d, h, l \in S$, then by Lemma 2.2 $S \subseteq Z$, a contradiction. So, we have $A(p, e, g) = 0$ holds $\forall p, e, g \in S$. Lemma 2.4 gives $f(pe) = f(p)e + pf(e) + \beta(p, e) \quad \forall p, e \in S$. $\square$

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