

Some remarks on S_σ -summability via generalized neutrosophic norm

A. Sharma, S. Murtaza and V. Kumar

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Corresponding Author: Archana Sharma

Abstract In this paper, we introduce S_σ -summability using σ -mean in neutrosophic- n -normed linear spaces ($N-n-NLS$) and investigate some of its properties within these spaces. We also define S_σ -Cauchy sequence and S_σ -completeness in $N-n-NLS$ and give examples to show the strongness of this method in these spaces.

1 Introduction

H. Fast[1] introduced an extension of the usual concept of sequential limit which he called Statistical convergence. Schoenberg[2] gave some basic properties of Statistical convergence and also studied the concept as a summability method. "A sequence $x = (x_k)$ is said to be statistical convergent to x_0 if for each $\eta > 0$,

$$\lim_{n \rightarrow \infty} \frac{1}{n} |\{k \leq n : |x_k - x_0| \geq \eta\}| = 0,$$

i.e., $\delta(\{k \in \mathbb{N} : |x_k - x_0| \geq \eta\}) = 0$ where δ denotes the asymptotic density of subsets of $\mathbb{N}$ ".

Many authors including Connor[3], Fridy [4], Maddox [5], Salat [6], Savas and Nuray[7], Mursaleen[8] and Tripathy[9, 10] have made significant contribution to develop statistical convergence in different directions.

For a map $\sigma : \mathbb{N} \times \mathbb{N}$, the invariant mean or σ -mean is a linear functional h on l_∞ , the space of real bounded sequences s.t

- (i) $h(p) \geq 0$ when the sequence $p = (p_n)$ has $p_n \geq 0$ for all n ,
- (ii) $h(\theta) = 1$, where $\theta = (1, 1, \dots)$ and
- (iii) $h(p_{\sigma(n)}) = h(p)$ for all $p \in l_\infty$.

Let $\sigma^m(n)$ be the m -th iterate of σ at n , where $m, n \in \mathbb{N}$. We observe that σ is 1-1 with $\sigma^m(n) \neq n$ for $m, n \in \mathbb{N}$. If we denote c , the space of convergent sequences, then h extends the limit functional on c with $h(p) = \lim p_n$ for all $p = (p_n) \in c$. Further, $c \subset \mathbb{V}_\sigma$ where

$$\mathbb{V}_\sigma = \left\{ \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n (p_{\sigma^k(m)} - p_0) = 0 \text{ uniformly in } m \right\}.$$

For the choice $\sigma(n) = n + 1$, the σ -means are the Banach limits on l_∞ and $\mathbb{V}_\sigma$ is the set of almost convergent sequences. Several researchers including M. A. Mursaleen[11], Mursaleen[8], Savas[12], Schaefer[13] etc., used σ -mean to study many problems of Analysis in different aspects. However, Savas and Nuray[7] gave one of the remarkable extension of statistical convergence with the help of σ -mean, called σ -statistical convergence. For development on σ -statistical convergence we refer to the reader [7, 8, 12] etc.

Further, many real-life problems cannot be effectively modeled using crisp sets due to complex uncertainty in the data. To address these problems, Zadeh[14] introduced the concept of fuzzy sets as a generalization of crisp sets. Fuzzy sets have been widely used in various fields of engineering such as chaos control [15], fuzzy physics [16], population dynamics [17], non-linear dynamical systems [18] and computer programming [19]. As the time grown up, many generalization of fuzzy sets have been appeared in literature as per our requirement. The most

recent among these is neutrosophic sets, initially introduced by samarandache [20] with the help of membership, non-membership and indeterminacy function. For further development on these sets, we refer [21, 22, 23, 24]. In recent years, these sets are naturally used to define neutrosophic metric spaces [25], neutrosophic topological spaces [26] and neutrosophic normed linear spaces [27] to study distance related problems and their topological structures. One of the important area required in this study is the development of various convergence structures in these spaces. Consequently, nowadays the study of summability methods in neutrosophic normed linear spaces is a point of attraction for many researchers. Some latest works can be found in [28, 29, 30] etc. In addition, we also recommend to the reader [31, 32, 33, 34, 35, 36, 37, 38, 39, 40]. In present paper, we define S_σ -summability, S_σ -Cauchy sequence, S_σ -completeness in $N - n - NLS$ and develop some of their properties. We also give example that shows the strongness of this method in these spaces.

2 Preliminaries

Definition 2.1. [41] A binary operation $\bullet : [0, 1] \times [0, 1] \rightarrow [0, 1]$ is said to be a t -norm if $\bullet$ satisfies the following conditions:

- (i) $d \bullet e = e \bullet d$ and $d \bullet (e \bullet f) = (d \bullet e) \bullet f$;
- (ii) $\bullet$ is continuous;
- (iii) $d \bullet 1 = 1 \bullet d = d$ for all $d \in [0, 1]$ and
- (iv) $d \bullet e \leq f \bullet g$ if $d \leq f, e \leq g$ with $d, e, f, g \in [0, 1]$.

Definition 2.2. [41] A binary operation $\diamond : [0, 1] \times [0, 1] \rightarrow [0, 1]$ is said to be a t -conorm if $\diamond$ satisfies the following conditions:

- (i) $d \diamond e = e \diamond d$ and $d \diamond (e \diamond f) = (d \diamond e) \diamond f$;
- (ii) $\diamond$ is continuous;
- (iii) $d \diamond 0 = 0 \diamond d = d$ for all $d \in [0, 1]$ and
- (iv) $d \diamond e \leq f \diamond g$ if $d \leq f, e \leq g$ with $d, e, f, g \in [0, 1]$.

Kirisci and Simsek [25] utilized above definitions to define neutrosophic normed linear spaces as follows.

Definition 2.3. [27] A six tuple $(U, G, B, Y, \bullet, \diamond)$ consisting of linear space U over some field F , $\bullet$ is a t -norm, $\diamond$ is a t -conorm, G, B, Y are fuzzy sets on $U \times (0, \infty)$, G is the degree of belongingness, B is the degree of indeterminacy and Y is the degree of non-belongingness of $(u, \lambda_1) \in U(0, \infty)$ satisfying the following conditions:

For every $u, v \in U, \lambda_1, \lambda_2 > 0$ and scalar $\alpha \neq 0$, we have

- (i) $0 \leq G(u, \lambda_1) \leq 1, 0 \leq B(u, \lambda_1) \leq 1, 0 \leq Y(u, \lambda_1) \leq 1$;
- (ii) $G(u, \lambda_1) + B(u, \lambda_1) + Y(u, \lambda_1) \leq 3$;
- (iii) $G(u, \lambda_1) = 1$, if and only if, $u = 0$;
- (iv) $G(\alpha u, \lambda_1) = G(u, \frac{\lambda_1}{|\alpha|})$;
- (v) $G(u, \lambda_1) \bullet G(v, \lambda_2) \leq G(u + v, \lambda_1 + \lambda_2)$;
- (vi) $G(u, \cdot)$ is continuous non-decreasing function;
- (vii) $\lim_{\lambda_1 \rightarrow \infty} G(u, \lambda_1) = 1$;
- (viii) $B(u, \lambda_1) = 0$, if and only if, $u = 0$;
- (ix) $B(\alpha u, \lambda_1) = B(u, \frac{\lambda_1}{|\alpha|})$;
- (x) $B(u, \lambda_1) \diamond G(v, \lambda_2) \geq G(u + v, \lambda_1 + \lambda_2)$;
- (xi) $B(u, \cdot)$ is continuous non-decreasing function;
- (xii) $\lim_{\lambda_1 \rightarrow \infty} B(u, \lambda_1) = 0$;
- (xiii) $Y(u, \lambda_1) = 0$, if and only if, $u = 0$;
- (xiv) $Y(\alpha u, \lambda_1) = Y(u, \frac{\lambda_1}{|\alpha|})$;
- (xv) $Y(u, \lambda_1) \diamond G(v, \lambda_2) \geq G(u + v, \lambda_1 + \lambda_2)$;
- (xvi) $Y(u, \cdot)$ is continuous non-decreasing function;
- (xvii) $\lim_{\lambda_1 \rightarrow \infty} Y(u, \lambda_1) = 0$ and
- (xviii) If $\lambda_1 \leq 0$, then $G(u, \lambda_1) = 0, B(u, \lambda_1) = 1 = Y(u, \lambda_1)$.

We call $N(G, B, Y)$ as the neutrosophic norm and $(U, G, B, Y, \bullet, \diamond)$, the neutrosophic normed space.

Let $(U, |||)$ be a normed linear space. Define $f \bullet g = fg$ and $f \diamond g = \min\{f + g, 1\}$ for all $f, g \in [0, 1]$. For $u \in U$ and $\zeta > 0$, if we define

$$G(u, \zeta) = \frac{\zeta}{\zeta + |||u|||}, B(u, \zeta) = \frac{|||u|||}{\zeta + |||u|||} \quad \text{and} \quad Y(x, \zeta) = \frac{|||u|||}{\zeta},$$

then it is easy to see that $(U, G, B, Y, \bullet, \diamond)$ is an *NNLS*.

We now recall the convergence structure and Cauchy sequence in a *NNLS* as given in [27].

A sequence $v = (v_k)$ in U is said to be convergent to $v_0 \in U$ w.r.t the neutrosophic norm (G, B, Y) if for $0 < \eta < 1, \zeta > 0, \exists n_0 \in \mathbb{N}$ s.t $G(v_k - v_0 : \zeta) > 1 - \eta, B(v_k - v_0 : \zeta) < \eta$ and $Y(v_k - v_0 : \zeta) < \eta$ for all $k \geq n_0$. In this case, we write $(G, B, Y) - \lim_{k \rightarrow \infty} v_k = V_0$.

A sequence $v = (v_k)$ in U is said to be Cauchy w.r.t the neutrosophic norm (G, B, Y) if for $0 < \eta < 1, \zeta > 0, \exists n_0 \in \mathbb{N}$ s.t for all $k, p \geq n_0$ $G(v_k - v_p : \zeta) > 1 - \eta, B(v_k - v_p : \zeta) < \eta$ and $Y(v_k - v_p : \zeta) < \eta$.

Kumar et al [42] extended the above concept of neutrosophic normed linear spaces to neutrosophic $-n$ -normed linear space and defined convergence structure and Cauchy sequences in these spaces as follows.

Definition 2.4. [42] A six tuple $(U, G, B, Y, \bullet, \diamond)$ consisting of linear space U over some field F , $\bullet$ is a t -norm, $\diamond$ is a t -conorm, G, B, Y are fuzzy sets on $U^n \times (0, \infty)$, G is the degree of belongingness, B is the degree of indeterminacy and Y is the degree of non-belongingness of $(u_1, u_2 \dots u_{n-1}, u_n, \lambda_1) \in U^n \times (0, \infty)$ satisfying the following conditions:

$\lambda_1, \lambda_2 > 0$ and scalar $\alpha \neq 0$, we have

- (i) $G(u_1, u_2 \dots u_{n-1}, u_n, \lambda_1) + B(u_1, u_2 \dots u_{n-1}, u_n, \lambda_1) + Y(u_1, u_2 \dots u_{n-1}, u_n, \lambda_1) \leq 3$;
 - (ii) $G(u_1, u_2 \dots u_{n-1}, u_n, \lambda_1) > 0$;
 - (iii) $G(u_1, u_2 \dots u_{n-1}, u_n, \lambda_1) = 1$, if and only if, u_i are dependent for $1 \leq i \leq n$;
 - (iv) $G(u_1, u_2 \dots u_{n-1}, u_n, \lambda_1)$ remains invariant for $u_i, 1 \leq i \leq n$;
 - (v) $G(\alpha u_1, u_2 \dots u_{n-1}, \alpha u_n, \lambda_1) = G(u_1, u_2 \dots u_{n-1}, u_n, \frac{\lambda_1}{|\alpha|})$;
 - (vi) $G(u_1, u_2 \dots u_{n-1}, u_n, \lambda_1) \bullet G(u_1, u_2 \dots u_{n-1}, u'_n, \lambda_2)$
 $\leq G(u_1, u_2 \dots u_{n-1}, u_n + u'_n, \lambda_1 + \lambda_2)$;
 - (vii) $G(u_1, u_2 \dots u_{n-1}, u_n, \cdot)$ is continuous non-decreasing function;
 - (viii) $\lim_{\lambda_1 \rightarrow \infty} G(u_1, u_2 \dots u_{n-1}, u_n, \lambda_1) = 1$ and $\lim_{\lambda_1 \rightarrow 0} G(u_1, u_2 \dots u_{n-1}, u_n, \lambda_1) = 0$
 - (ix) $B(u_1, u_2 \dots u_{n-1}, u_n, \lambda_1) > 0$;
 - (x) $B(u_1, u_2 \dots u_{n-1}, u_n, \lambda_1) = 1$, if and only if, u_i are dependent for $1 \leq i \leq n$;
 - (xi) $B(u_1, u_2 \dots u_{n-1}, u_n, \lambda_1)$ remains invariant for $u_i, 1 \leq i \leq n$;
 - (xii) $B(\alpha u_1, u_2 \dots u_{n-1}, \alpha u_n, \lambda_1) = B(u_1, u_2 \dots u_{n-1}, u_n, \frac{\lambda_1}{|\alpha|})$;
 - (xiii) $B(u_1, u_2 \dots u_{n-1}, u_n, \lambda_1) \diamond B(u_1, u_2 \dots u_{n-1}, u'_n, \lambda_2)$
 $\geq B(u_1, u_2 \dots u_{n-1}, u_n + u'_n, \lambda_1 + \lambda_2)$;
 - (xiv) $B(u_1, u_2 \dots u_{n-1}, u_n, \cdot)$ is continuous non-increasing function;
 - (xv) $\lim_{\lambda_1 \rightarrow \infty} B(u_1, u_2 \dots u_{n-1}, u_n, \lambda_1) = 0$ and $\lim_{\lambda_1 \rightarrow 0} B(u_1, u_2 \dots u_{n-1}, u_n, \lambda_1) = 1$
 - (xvi) $Y(u_1, u_2 \dots u_{n-1}, u_n, \lambda_1) > 0$;
 - (xvii) $Y(u_1, u_2 \dots u_{n-1}, u_n, \lambda_1) = 1$, if and only if, u_i are dependent for $1 \leq i \leq n$;
 - (xviii) $Y(u_1, u_2 \dots u_{n-1}, u_n, \lambda_1)$ remains invariant for $u_i, 1 \leq i \leq n$;
 - (xix) $Y(\alpha u_1, u_2 \dots u_{n-1}, \alpha u_n, \lambda_1) = Y(u_1, u_2 \dots u_{n-1}, u_n, \frac{\lambda_1}{|\alpha|})$;
 - (xx) $Y(u_1, u_2 \dots u_{n-1}, u_n, \lambda_1) \diamond Y(u_1, u_2 \dots u_{n-1}, u'_n, \lambda_2)$
 $\geq Y(u_1, u_2 \dots u_{n-1}, u_n + u'_n, \lambda_1 + \lambda_2)$;
 - (xxi) $Y(u_1, u_2 \dots u_{n-1}, u_n, \cdot)$ is continuous non-increasing function;
 - (xxii) $\lim_{\lambda_1 \rightarrow \infty} Y(u_1, u_2 \dots u_{n-1}, u_n, \lambda_1) = 0$ and $\lim_{\lambda_1 \rightarrow 0} Y(u_1, u_2 \dots u_{n-1}, u_n, \lambda_1) = 1$
- We call $N_n = (G, B, Y)_n$ as the neutrosophic $-n$ -norm and $U_n = (U, G, B, Y, \bullet, \diamond)$, the neutrosophic $-n$ -normed space.

Throughout this work unless otherwise stated $U_n = (U, G, B, Y, \bullet, \diamond)$ will denote an neutrosophic $-n$ -normed space.

A sequence $v = (v_k)$ in U_n is said to be convergent to $v_0 \in U_n$ w.r.t the neutrosophic $-n$ -norm $(G, B, Y)_n$ if for $0 < \eta < 1, \zeta > 0$ and $u_1, u_2 \dots u_{n-1} \in U, \exists n_0 \in \mathbb{N}$ s.t $G(u_1, u_2 \dots u_{n-1}, v_k - v_0 : \zeta) > 1 - \eta, B(u_1, u_2 \dots u_{n-1}, v_k - v_0 : \zeta) < \eta$ and $Y(u_1, u_2 \dots u_{n-1}, v_k - v_0 : \zeta) < \eta$ for all $k \geq n_0$. In this case, we write $(G, B, Y)_n - \lim_{k \rightarrow \infty} v_k = v_0$.

A sequence $v = (v_k)$ in U_n is said to be Cauchy w.r.t the neutrosophic- n -norm $(G, B, Y)_n$ if for $0 < \eta < 1, \zeta > 0, \exists n_0 \in \mathbb{N}$ s.t for all $k, p \geq n_0$ and $u_1, u_2 \dots u_{n-1} \in U, G(u_1, u_2 \dots u_{n-1}, v_k - v_p : \zeta) > 1 - \eta, B(u_1, u_2 \dots u_{n-1}, v_k - v_p : \zeta) < \eta$ and $Y(u_1, u_2 \dots u_{n-1}, v_k - v_p : \zeta) < \eta$.

A sequence $v = (v_k)$ in U_n is said to be statistical convergent to $v_0 \in U_n$ w.r.t the neutrosophic- n -norm $(G, B, Y)_n$ if for $0 < \eta < 1, \zeta > 0$ and $u_1, u_2 \dots u_{n-1} \in U,$

$$\begin{aligned} \delta(\{k \in \mathbb{N} : G(u_1, u_2 \dots u_{n-1}, v_k - v_0 : \zeta) \leq 1 - \eta \text{ or} \\ B(u_1, u_2 \dots u_{n-1}, v_k - v_0 : \zeta) \geq \eta, \\ Y(u_1, u_2 \dots u_{n-1}, v_k - v_0 : \zeta) \geq \eta\}) = 0 \end{aligned}$$

In this case, we write $S(G, B, Y)_n - \lim_{k \rightarrow \infty} v_k = v_0$.

A sequence $v = (v_k)$ in U_n is said to be statistical Cauchy w.r.t the neutrosophic- n -norm $(G, B, Y)_n$ if for $0 < \eta < 1, \zeta > 0$ and $u_1, u_2 \dots u_{n-1} \in U, \exists p \in \mathbb{N}$ s.t for all $k \geq n_0,$

$$\begin{aligned} \delta(\{k \in \mathbb{N} : G(u_1, u_2 \dots u_{n-1}, v_k - v_p : \zeta) \leq 1 - \eta \text{ or} \\ B(u_1, u_2 \dots u_{n-1}, v_k - v_p : \zeta) \geq \eta, \\ Y(u_1, u_2 \dots u_{n-1}, v_k - v_p : \zeta) \geq \eta\}) = 0. \end{aligned}$$

3 S_σ -Summability in $N - n - NLS$

In this section, we introduce S_σ -convergence in $N - n - NLS$ and develop some of its properties.

Definition 3.1. A sequence $v = (v_k)$ in U_n is said to be S_σ -convergent to v_0 with respect to $(G, B, Y)_n$ if for $0 < \eta < 1, \zeta > 0$ and $u_1, u_2 \dots u_{n-1} \in U,$

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{n} |\{k \leq n : G(u_1, u_2 \dots u_{n-1}, v_{\sigma^k(m)} - v_0 : \zeta) \leq 1 - \eta \text{ or} \\ B(u_1, u_2 \dots u_{n-1}, v_{\sigma^k(m)} - v_0 : \zeta) \geq \eta, \\ Y(u_1, u_2 \dots u_{n-1}, v_{\sigma^k(m)} - v_0 : \zeta) \geq \eta\}| = 0 \end{aligned}$$

uniformly in $m = 1, 2, \dots$, i.e., $\delta(A) = 0$ uniformly in $m = 1, 2, \dots$ where, A is given by

$$\begin{aligned} A = \{k \leq n : G(u_1, u_2 \dots u_{n-1}, v_{\sigma^k(m)} - v_0 : \zeta) \leq 1 - \eta \text{ or} \\ B(u_1, u_2 \dots u_{n-1}, v_{\sigma^k(m)} - v_0 : \zeta) \geq \eta, \\ Y(u_1, u_2 \dots u_{n-1}, v_{\sigma^k(m)} - v_0 : \zeta) \geq \eta\}. \end{aligned}$$

In this case, we write $S_\sigma(G, B, Y)_n - \lim_{k \rightarrow \infty} v_k = v_0$ or $v_k \rightarrow v_0(S_\sigma)$ with respect to $(G, B, Y)_n$ -norm.

We now give the following Lemma.

Lemma 3.2. For a sequence $v = (v_k)$ in U_n , the following statements are equivalent:

- (i) $S_\sigma(G, B, Y)_n - \lim_{k \rightarrow \infty} v_k = v_0$;
- (ii) $\delta(\{k \in \mathbb{N} : G(u_1, u_2 \dots u_{n-1}, v_{\sigma^k(m)} - v_0 : \zeta) \leq 1 - \eta\})$
 $= \delta(\{k \in \mathbb{N} : B(u_1, u_2 \dots u_{n-1}, v_{\sigma^k(m)} - v_0 : \zeta) \geq \eta\})$
 $= \delta(\{k \in \mathbb{N} : Y(u_1, u_2 \dots u_{n-1}, v_{\sigma^k(m)} - v_0 : \zeta) \geq \eta\}) = 0$;
- (iii) $\delta(\{k \in \mathbb{N} : G(u_1, u_2 \dots u_{n-1}, v_{\sigma^k(m)} - v_p : \zeta) > 1 - \eta \text{ and}$
 $B(u_1, u_2 \dots u_{n-1}, v_{\sigma^k(m)} - v_p : \zeta) < \eta,$
 $Y(u_1, u_2 \dots u_{n-1}, v_{\sigma^k(m)} - v_p : \zeta) < \eta\}) = 1$.
- (iv) $\delta(\{k \in \mathbb{N} : G(u_1, u_2 \dots u_{n-1}, v_{\sigma^k(m)} - v_0 : \zeta) > 1 - \eta\})$
 $= \delta(\{k \in \mathbb{N} : B(u_1, u_2 \dots u_{n-1}, v_{\sigma^k(m)} - v_0 : \zeta) < \eta\})$
 $= \delta(\{k \in \mathbb{N} : Y(u_1, u_2 \dots u_{n-1}, v_{\sigma^k(m)} - v_0 : \zeta) < \eta\}) = 1$;
- (v) $S_\sigma - \lim_{k \rightarrow \infty} G(u_1, u_2 \dots u_{n-1}, u_n, v_k - v_0, \zeta) = 1,$
 $S_\sigma - \lim_{k \rightarrow \infty} B(u_1, u_2 \dots u_{n-1}, u_n, v_k - v_0, \zeta) = 0$ and
 $S_\sigma - \lim_{k \rightarrow \infty} G(u_1, u_2 \dots u_{n-1}, u_n, v_k - v_0, \zeta) = 0$.

Proof. The proof is omitted as can be easily obtained by Definition 3.1. $\square$

Theorem 3.3. For a sequence $v = (v_k)$ in U_n , if $S_\sigma(G, B, Y)_n - \lim_{k \rightarrow \infty} v_k$ exists, then it is unique.

Proof. Omitted. $\square$

Theorem 3.4. Let $u = (u_k)$ and $v = (v_k)$ be in U_n s.t $S_\sigma(G, B, Y)_n - \lim_{k \rightarrow \infty} u_k = u_0$ and $S_\sigma(G, B, Y)_n - \lim_{k \rightarrow \infty} v_k = v_0$, then

- (i) $S_\sigma(G, B, Y)_n - \lim_{k \rightarrow \infty} (u_k + v_k) = u_0 + v_0$;
- (ii) $S_\sigma(G, B, Y)_n - \lim_{k \rightarrow \infty} (\alpha u_k) = \alpha u_0$ where $0 \neq \alpha \in F$.

Proof. Omitted, as it can be obtained analogously to the proof of Theorem 3.3. $\square$

Theorem 3.5. Every $(G, B, Y)_n$ -convergent sequence in a $N-n-NLS$ U_n is $S_\sigma(G, B, Y)_n$ -convergent.

Proof. Let $v = (v_k)$ be a sequence in U_n such that $(G, B, Y)_n - \lim_{k \rightarrow \infty} v_k = v_0$. Then for $0 < \eta < 1, \zeta > 0$ and $u_1, u_2 \dots u_{n-1} \in U$, $\exists$ a positive integer $k_0 \in \mathbb{N}$ s.t

$$\begin{aligned} G(u_1, u_2 \dots u_{n-1}, v_k - v_0 : \zeta) &> 1 - \eta \quad \text{and} \\ B(u_1, u_2 \dots u_{n-1}, v_k - v_0 : \zeta) &< \eta, \\ Y(u_1, u_2 \dots u_{n-1}, v_k - v_0 : \zeta) &< \eta \end{aligned}$$

for all $k \geq k_0$. This implies

$$\begin{aligned} A = \{k \leq n : G(u_1, u_2 \dots u_{n-1}, v_{\sigma^k(m)} - v_0 : \zeta) \leq 1 - \eta \quad \text{or} \\ B(u_1, u_2 \dots u_{n-1}, v_{\sigma^k(m)} - v_0 : \zeta) \geq \eta, \\ Y(u_1, u_2 \dots u_{n-1}, v_{\sigma^k(m)} - v_0 : \zeta) \geq \eta\}. \end{aligned}$$

is finite and therefore $\delta(A) = 0$. Hence, $S_\sigma(G, B, Y)_n - \lim_{k \rightarrow \infty} v_k = v_0$. $\square$

The converse of Theorem 3.5 is not valid in general as can be seen from the following example.

Example 3.6. Let $(\mathbb{R}^n, \|\dots\|)$ be a normed space and define $f \bullet g = fg$, $f \diamond g = \min\{f + g, 1\}$ for all $f, g \in [0, 1]$. For $v \in \mathbb{R}$ and every $\zeta > 0$, if we define

$$\begin{aligned} G(u_1, u_2 \dots u_{n-1}, v, \zeta) &= \frac{\zeta}{\zeta + \|u_1, u_2 \dots u_{n-1}, v\|}, \\ B(u_1, u_2 \dots u_{n-1}, v, \zeta) &= \frac{\|u_1, u_2 \dots u_{n-1}, v\|}{\zeta + \|u_1, u_2 \dots u_{n-1}, v\|} \quad \text{and} \\ Y(u_1, u_2 \dots u_{n-1}, v, \zeta) &= \frac{\|u_1, u_2 \dots u_{n-1}, v\|}{\zeta}, \end{aligned}$$

then $(\mathbb{R}^n, G, B, Y, \bullet, \diamond)$ is an $NN - n - LS$.

Define a sequence $v = (v_k)$ by

$$v_k = \begin{cases} (0, 0, \dots, 1) \in \mathbb{R}^n & \text{if } k \text{ is a prime} \\ (0, 0, \dots, 0) \in \mathbb{R}^n & \text{otherwise.} \end{cases}$$

For every $0 < \eta < 1$ and $\zeta > 0$, if we denote the set

$$\begin{aligned} T_n(\eta, \zeta) = \left\{ k \in \mathbb{N} : G(u_1, u_2 \dots u_{n-1}, v_{\sigma^k(m)} - v_0 : \zeta) \leq 1 - \eta \quad \text{or} \right. \\ B(u_1, u_2 \dots u_{n-1}, v_{\sigma^k(m)} - v_0 : \zeta) \geq \eta, \\ \left. Y(u_1, u_2 \dots u_{n-1}, v_{\sigma^k(m)} - v_0 : \zeta) \geq \eta \right\}, \end{aligned}$$

then

$$\begin{aligned}
 T_n(\eta, \zeta) &= \left\{ k \in \mathbb{N} : \frac{\zeta}{\zeta + \|u_1, u_2 \dots u_{n-1}, v_{\sigma^k(m)}\|} \leq 1 - \eta \text{ or} \right. \\
 &\quad \frac{\|u_1, u_2 \dots u_{n-1}, v_{\sigma^k(m)}\|}{\zeta + \|u_1, u_2 \dots u_{n-1}, v_{\sigma^k(m)}\|} \geq \eta, \\
 &\quad \left. \frac{\|u_1, u_2 \dots u_{n-1}, v_{\sigma^k(m)}\|}{\zeta} \geq \eta \right\} \\
 &= \left\{ k \in \mathbb{N} : \|u_1, u_2 \dots u_{n-1}, v_{\sigma^k(m)}\| \geq \frac{\eta\zeta}{1-\eta} > 0 \right\} \\
 &= \left\{ k \in \mathbb{N} : k \text{ is a prime} \right\}.
 \end{aligned}$$

So $\delta(T_n(\eta, \zeta)) = 0$ and therefore $v = (v_k)$ is σ -statistical convergent w.r.t the neutrosophic norm $(G, B, Y)_n$. Clearly $v = (v_k)$ is not convergent w.r.t $(G, B, Y)_n$.

Theorem 3.7. A sequence $v = (v_k)$ in a $N - n - NLS$ U_n is $S_\sigma(G, B, Y)_n$ -convergent to v_0 if and only if there exist a subset $K = \{k_j : j \in \mathbb{N}\} \subseteq \mathbb{N}$ s.t $\delta(K) = 1$ and $(G, B, Y)_n - \lim_{j \rightarrow \infty} v_{k_j} = v_0$.

Proof. Omitted. □

Definition 3.8. A sequence $v = (v_k)$ in a $N - n - NLS$ U_n is said to be S_σ -Cauchy if for $0 < \eta < 1, \zeta > 0$ and $u_1, u_2 \dots u_{n-1} \in U, \exists p \in \mathbb{N}$ s.t

$$\begin{aligned}
 \lim_{n \rightarrow \infty} \frac{1}{n} \left| \left\{ k \leq n : G(u_1, u_2 \dots u_{n-1}, v_{\sigma^k(m)} - v_{\sigma^p(m)} : \zeta) \leq 1 - \eta \text{ or} \right. \right. \\
 B(u_1, u_2 \dots u_{n-1}, v_{\sigma^k(m)} - v_{\sigma^p(m)} : \zeta) \geq \eta, \\
 \left. \left. Y(u_1, u_2 \dots u_{n-1}, v_{\sigma^k(m)} - v_{\sigma^p(m)} : \zeta) \geq \eta \right\} \right| = 0;
 \end{aligned}$$

uniformly in $m = 1, 2, \dots$

or equivalently, $\delta(A) = 0$ where A is given by

$$\begin{aligned}
 A = \left\{ k \in \mathbb{N} : G(u_1, u_2 \dots u_{n-1}, v_{\sigma^k(m)} - v_{\sigma^p(m)} : \zeta) \leq 1 - \eta \text{ or} \right. \\
 B(u_1, u_2 \dots u_{n-1}, v_{\sigma^k(m)} - v_{\sigma^p(m)} : \zeta) \geq \eta, \\
 \left. Y(u_1, u_2 \dots u_{n-1}, v_{\sigma^k(m)} - v_{\sigma^p(m)} : \zeta) \geq \eta \right\}.
 \end{aligned}$$

Theorem 3.9. Let $(U, G, B, Y, \bullet, \diamond)$ be a $N - n - NLS$. If a sequence $v = (v_k)$ is S_σ -convergent w.r.t $(G, B, Y)_n$ -norm, then it is S_σ -Cauchy w.r.t $(G, B, Y)_n$ -norm.

Proof. Omitted. □

Definition 3.10. A neutrosophic- n -normed linear space is said to be S_σ -complete if every S_σ -convergent sequence is S_σ -Cauchy.

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Author information

A. Sharma, Department of Mathematics, Chandigarh University, Mohali-140413, Punjab, India.
E-mail: dr.archanasharma1022@gmail.com

S. Murtaza, Department of Mathematics, Chandigarh University, Mohali-140413, Punjab, India.
E-mail: sajidsulimani8@gmail.com

V. Kumar, Department of Mathematics, Chandigarh University, Mohali-140413, Punjab, India.
E-mail: kaushikvjy@gmail.com