

Some remarks on $\mathcal{I}(\Delta^m)$ -summability in gradual normed linear spaces

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MSC 2010 Classifications: Primary 97I30, 40A35; Secondary 26E50, 46S40.

Keywords and phrases: $\mathcal{I}$ -convergence, difference operator, statistical convergence, Cauchy sequences and gradual normed spaces.

The authors appreciate the reviewers and editor for their helpful feedback and insightful suggestions that enhanced the clarity of our paper.

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Abstract Using the difference operator Δ^m , the present work aims to define $\mathcal{I}(\Delta^m)$ -summability and $\mathcal{I}^*(\Delta^m)$ -summability in a more general framework, gradual normed linear spaces. We also demonstrate $\mathcal{I}(\Delta^m)$ and $\mathcal{I}^*(\Delta^m)$ -Cauchy sequences and acquire some intriguing connections.

1 Introduction

The concept of statistical convergence was introduced by Fast [1] in relation to the notion of natural density of subsets of $\mathbb{N}$, the set of positive integers.

A sequence $w = (w_k)$ is said to be statistical convergent to w_0 if for each $\epsilon > 0$,

$$\lim_{n \rightarrow \infty} \frac{1}{n} |\{k \leq n : |w_k - w_0| \geq \epsilon\}| = 0,$$

i.e. $\delta = \{k \in \mathbb{N} : |w_k - w_0| \geq \epsilon\}$ [2]. Subsequently, statistical convergence and related notions have been developed by many authors, including Schoenberg [3], Salat [4], Connor [5], Fridy and Orhan [6], Mursaleen [7] and many others.

Using the concept of ideals, Kostyrko et al. [8] proposed the concept of ideal convergence, which is an extension of statistical convergence. Dems [9] defined $\mathcal{I}$ -Cauchy sequence in metric spaces and studied its connection with ideal convergence. Recently Choudhury and Debnath [10] introduced $\mathcal{I}$ -convergence, $\mathcal{I}^*$ -convergence, $\mathcal{I}$ -Cauchy and $\mathcal{I}^*$ -Cauchy in gradual normed linear spaces. They additionally examined certain properties within these spaces. For further development on ideal convergence, we refer to [11, 12, 13, 14, 15, 16, 17, 18, 19].

In past years, many sequence spaces have been studied using the difference operator Δ (see [20, 21]). For any sequence $w = (w_k)$, the m^{th} difference operator of w_k is defined as follows.

$$\begin{aligned} \Delta^0 w_k &= w_k; \\ \Delta^1 w_k &= w_k - w_{k+1}; \\ &\vdots \\ \Delta^m w_k &= \sum_{j=0}^m (-1)^j \binom{m}{j} w_{k+j} \quad (k \in \mathbb{N}). \end{aligned}$$

Many problems of engineering and science cannot be handled via crisp sets due to the presence of huge uncertainty. To remove this difficulty, Zadeh [22] proposed the idea of fuzzy sets, which turned out to be a very effective tool to deal with these situations. As fuzzy numbers are fuzzy intervals, to avoid confusion between fuzzy numbers and fuzzy intervals, Fortin et al. [23] introduced the notion of gradual real numbers using an assignment function with domain $(0,1]$ and co-domain $\mathbb{R}$. Recently, Sadeqi and Azari [24] introduced the concept of gradual normed linear spaces and defined the convergence structure in these spaces. After their pioneer work, some summability methods have also been discussed, as can be seen in [25, 26, 27, 28, 29, 30,

31, 32, 33, 34, 35, 36]. In the present paper, we aim to introduce a new summability method using ideals and the difference operator Δ^m in gradual normed linear spaces and examine certain characteristics within these areas.

2 Preliminaries

This section emphasizes a few essential definitions and outcomes necessary for the results.

By a gradual real number $\tilde{n}$, we mean an assignment function $A_{\tilde{n}}$ with domain $(0,1]$ and co-domain $\mathbb{R}$, i.e. $A_{\tilde{n}} : (0, 1] \rightarrow \mathbb{R}$. Let,

$$\mathcal{G}(\mathbb{R}) = \{\tilde{n} : \tilde{n} \text{ is a gradual real numbers}\}.$$

Further, any $\tilde{n} \in \mathcal{G}(\mathbb{R})$ is said to be positive if $A_{\tilde{n}}(\eta) \geq 0$ for $\eta \in (0, 1]$. Fortin et al. [23] introduced a new representation of fuzzy intervals. They have considered fuzzy intervals as crisp sets of entities instead of fuzzy sets and called them gradual real numbers. They also defined gradual operations on $\mathcal{G}(\mathbb{R})$ as follows:

[23] Let $\tilde{n}_1, \tilde{n}_2 \in \mathcal{G}(\mathbb{R})$ with corresponding assignment functions $A_{\tilde{n}_1}, A_{\tilde{n}_2}$. then $\tilde{n}_1 \circ \tilde{n}_2 \in \mathcal{G}(\mathbb{R})$ is defined by

$$A_{\tilde{n}_1 \circ \tilde{n}_2}(\eta) = A_{\tilde{n}_1}(\eta) \circ A_{\tilde{n}_2}(\eta) \text{ for } \eta \in (0, 1];$$

where $A_{\tilde{n}_1 \circ \tilde{n}_2}$ denotes the assignment function of $\tilde{n}_1 \circ \tilde{n}_2 \in \mathcal{G}(\mathbb{R})$.

Similarly, we define gradual addition $\tilde{n}_1 + \tilde{n}_2$ and scalar multiplication $c\tilde{n}_1$ for $\tilde{n}_1, \tilde{n}_2 \in \mathcal{G}(\mathbb{R})$ and $c \in \mathbb{R}$

$$\begin{aligned} A_{\tilde{n}_1 + \tilde{n}_2}(\eta) &= A_{\tilde{n}_1}(\eta) + A_{\tilde{n}_2}(\eta) \text{ for } \eta \in (0, 1] \text{ and} \\ A_{c\tilde{n}_1}(\eta) &= cA_{\tilde{n}_1}(\eta) \text{ for } \eta \in (0, 1]. \end{aligned}$$

Moreover, the constant gradual real number $\tilde{l}$ defined by the constant assignment function $A_{\tilde{l}}(\eta) = l$ for $\eta \in (0, 1]$.

Using the above operations on $\mathcal{G}(\mathbb{R})$ Sadeqi and Azari [24] defined gradual normed linear space as follows :

Definition 2.1. Let $\mathcal{W}$ be a real vector space. The function $\|\cdot\|_{\mathcal{G}} : \mathcal{W} \rightarrow \mathcal{G}^*(\mathbb{R})$ is called a gradual norm, for every $\eta \in (0, 1]$, it satisfies the following properties for any $w, t \in \mathcal{W}$,

$$(\mathcal{G}_1) A_{\|w\|_{\mathcal{G}}}(\eta) = A_{\tilde{0}}(\eta) \text{ iff } w = 0;$$

$$(\mathcal{G}_2) A_{\|\lambda w\|_{\mathcal{G}}}(\eta) = |\lambda| A_{\|w\|_{\mathcal{G}}}(\eta) \text{ for any } \lambda \in \mathbb{R};$$

$$(\mathcal{G}_3) A_{\|w+t\|_{\mathcal{G}}}(\eta) \leq A_{\|w\|_{\mathcal{G}}}(\eta) + A_{\|t\|_{\mathcal{G}}}(\eta) \text{ [24].}$$

The pair $(\mathcal{W}, \|\cdot\|_{\mathcal{G}})$ is called a gradual normed linear space (GNLS).[24]

Let $\mathcal{W} = \mathbb{R}^n$ and $w = (w_1, w_2, \dots, w_n) \in \mathbb{R}^n$ be any point and $0 < \eta \leq 1$. If the function $\|\cdot\|_{\mathcal{G}}$ is defined by

$$A_{\|w\|_{\mathcal{G}}}(\eta) = e^{\eta} \sum_{j=1}^n |w_j|$$

then Sadeqi and Azari [24] have shown that $(\mathcal{W}, \|\cdot\|_{\mathcal{G}})$ becomes a GNLS.

We now give the convergence structure and the Cauchy sequence in GNLS as given in [24]. A sequence $w = (w_k)$ in the $(\mathcal{W}, \|\cdot\|_{\mathcal{G}})$ is said to be convergent to w_0 with respect to the $\|\cdot\|_{\mathcal{G}}$, if for every $\eta \in (0, 1]$ and $\epsilon > 0$, there exist $m(= m_{\epsilon}(\eta)) \in \mathbb{N}$ such that $A_{\|w_k - w_0\|_{\mathcal{G}}}(\eta) < \epsilon, \forall k \geq m$. In this case we will write, $\mathcal{G} - \lim_{k \rightarrow \infty} w_k = w_0$ [24].

A sequence $w = (w_k)$ in the $(\mathcal{W}, \|\cdot\|_{\mathcal{G}})$ is said to be Cauchy with respect to $\|\cdot\|_{\mathcal{G}}$, if for every $\eta \in (0, 1]$ and $\epsilon > 0$, there exists $m_0(= m_{0_{\epsilon}}(\eta)) \in \mathbb{N}$ such that $A_{\|w_k - w_l\|_{\mathcal{G}}}(\eta) < \epsilon, \forall k, l \geq m_0$ [24].

We now recall the terminology of ideal convergence as given in [8]. A family of subsets $\mathcal{I} \subset \mathcal{P}(X)$ is said to be an ideal on X if and only if (i) $\emptyset \in \mathcal{I}$; (ii) for each $C, D \in \mathcal{I}$ implies $C \cup D \in \mathcal{I}$; (iii) for each $C \in \mathcal{I}$ and $C \subset D$ implies $D \in \mathcal{I}$ [8]. A family of subsets $\mathbb{F} \subset \mathcal{P}(X)$

is called a filter on X if and only if (i) $\emptyset \neq \mathbb{F}$; (ii) for each $C, D \in \mathbb{F}$ implies $C \cap D \in \mathbb{F}$; (iii) for each $C \in \mathbb{F}$ and $C \subset D$ implies $D \in \mathbb{F}$. An ideal $\mathcal{I}$ is called a non-trivial if $\mathcal{I} \neq \emptyset$ and $X \notin \mathcal{I}$ [8]. The filter $\mathbb{F} = \mathbb{F}(\mathcal{I}) = \{X - A : A \in \mathcal{I}\}$ is called the filter associated with the ideal $\mathcal{I}$ [8]. A non-trivial ideal $\mathcal{I} \subset \mathcal{P}(X)$ is called an admissible ideal in X if and only if $\{x : x \in X\} \subset \mathcal{I}$ [8]. Further, an admissible ideal $\mathcal{I}$ is said to satisfy the condition (AP), if for every countable family of mutually disjoint sets $\{L_j\}_{j \in \mathbb{N}}$ from $\mathcal{I}$, there exists a countable family of sets $\{K_j\}_{j \in \mathbb{N}}$ such that the symmetric difference $L_j \Delta K_j$ is finite for every $j \in \mathbb{N}$ and $\bigcup_{j=1}^{\infty} K_j \in \mathcal{I}$ [8].

Kostiryko et al. [8] defined ideal convergence of number sequences as follows. Let $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$ be a non-trivial ideal, a real sequence (w_k) is said to be $\mathcal{I}$ -convergent to w_0 if for each $\epsilon > 0$, the set $T(\epsilon) = \{k \in \mathbb{N} : |w_k - w_0| \leq \epsilon\} \in \mathcal{I}$, in this case, we'll write $\mathcal{I} - \lim_{k \rightarrow \infty} w_k = w_0$ [8]. Further, a sequence $w = (w_k)$ is said to be $\mathcal{I}^*$ -summable to w_0 if there exists an infinite set $K = \{k_1 < k_2 < \dots < k_j < \dots\} \in \mathbb{F}(\mathcal{I})$, such that $\lim_{k \in K, k \rightarrow \infty} w_k = w_0$. In this case we write, $\mathcal{I}^* - \lim_{j \rightarrow \infty} w_{k_j} = w_0$ [8].

Dems [9] define $\mathcal{I}$ -Cauchy sequence as follows.

A sequence $w = (w_k)$ of real numbers is said to be $\mathcal{I}$ -Cauchy, if for every $\epsilon > 0$, there exists a $m \in \mathbb{N}$ such that $\{k \in \mathbb{N} : |w_k - w_m| \geq \epsilon\} \in \mathcal{I}$ [9].

A sequence $w = (w_k)$ of real numbers is said to be $\mathcal{I}^*$ -Cauchy, if there exists a set $K = \{k_1 < k_2 < \dots < k_j < \dots\} \in \mathbb{F}(\mathcal{I})$ such that the subsequence (w_{k_j}) is a Cauchy sequence i.e. $\lim_{i, j \rightarrow \infty} |w_{k_i} - w_{k_j}| = 0$ [9].

3 $\mathcal{I}(\Delta^m)$ -summability in $(\mathcal{W}, ||\cdot||_{\mathcal{G}})$

This section presents our primary findings. We start with the definition is motivated by the notion of ideal convergence and difference operators introduced in earlier works [8, 9].

Definition 3.1. Let $(\mathcal{W}, ||\cdot||_{\mathcal{G}})$ be a GNLS and Δ^m is the m^{th} order difference operator. A sequence $w = (w_k)$ in $\mathcal{W}$ is said to be gradually $\mathcal{I}(\Delta^m)$ -summable to $w_0 \in \mathcal{W}$, for every $0 < \eta \leq 1$ and for every positive real number ϵ then,

$$\mathcal{S}(\eta, \epsilon) = \left\{ k \in \mathbb{N} : A_{||\Delta^m w_k - w_0||_{\mathcal{G}}}(\eta) \geq \epsilon \right\} \in \mathcal{I}.$$

Here, we write $\mathcal{I}[\Delta^m : \mathcal{G}] - \lim_{k \rightarrow \infty} w_k = w_0$; or simply $w_k \xrightarrow{\mathcal{I}(\Delta^m : \mathcal{G})} w_0$. Let,

$$\mathcal{I}[\Delta^m : \mathcal{G}] = \left\{ w = (w_k) \in \mathcal{W} \text{ such that for some } w_0 \right\}.$$

Theorem 3.2. If a sequence $w = (w_k)$ is gradual summable to $w_0 \in \mathcal{W}$ then (w_k) is gradually $\mathcal{I}(\Delta^m)$ -summable to $w_0 \in \mathcal{W}$.

Proof. Let $w = (w_k)$ is gradually summable to $w_0 \in \mathcal{I}$ and consider $\mathcal{I}$ to be an admissible ideal. For a given $\epsilon > 0$ and $0 < \eta \leq 1$, then a natural number exists m thus,

$$A_{||\Delta^m w_k - w_0||_{\mathcal{G}}}(\eta) < \epsilon \quad \forall k \geq m,$$

and therefore, $\{k \in \mathbb{N} : A_{||\Delta^m w_k - w_0||_{\mathcal{G}}}(\eta) \geq \epsilon\} \in \mathcal{I}$ since it is a finite set [10]. This shows that (w_k) is gradually $\mathcal{I}(\Delta^m)$ -summable to w_0 . $\square$

Theorem 3.3. If a sequence $w = (w_k)$ is gradually $\mathcal{I}(\Delta^m)$ -summable to w_0 i.e. $w_k \xrightarrow{\mathcal{I}(\Delta^m : \mathcal{G})} w_0 \in \mathcal{W}$. Then w_0 is uniquely determined.

Proof. To show uniqueness, suppose on the contrary that there exist two distinct limits w_0 and w'_0 satisfying $w_k \xrightarrow{\mathcal{I}(\Delta^m : \mathcal{G})} w_0$ and $w_k \xrightarrow{\mathcal{I}(\Delta^m : \mathcal{G})} w'_0$ for any $\epsilon > 0$ and $0 < \eta \leq 1$, representing sets

as

$$\mathcal{S}_1(\eta, \epsilon) = \left\{ k \in \mathbb{N} : A_{\|\Delta^m w_k - w_0\|_{\mathcal{G}}}(\eta) < \frac{\epsilon}{2} \right\} \text{ and}$$

$$\mathcal{S}_2(\eta, \epsilon) = \left\{ k \in \mathbb{N} : A_{\|\Delta^m w_k - w'_0\|_{\mathcal{G}}}(\eta) < \frac{\epsilon}{2} \right\}$$

then by assumptions, both sets $\mathcal{S}_1(\eta, \epsilon)$ and $\mathcal{S}_2(\eta, \epsilon)$ belongs to $\mathbb{F}(\mathcal{I})$. Here, via definition of $\mathbb{F}(\mathcal{I})$, $\emptyset \neq \mathcal{S}_1(\eta, \epsilon) \cap \mathcal{S}_2(\eta, \epsilon) \in \mathbb{F}(\mathcal{I})$.

So, Let $k_0 \in \mathcal{S}_1(\eta, \epsilon) \cap \mathcal{S}_2(\eta, \epsilon)$ then

$$A_{\|\Delta^m w_{k_0} - w_0\|_{\mathcal{G}}}(\eta) < \frac{\epsilon}{2} \text{ and } A_{\|\Delta^m w_{k_0} - w'_0\|_{\mathcal{G}}}(\eta) < \frac{\epsilon}{2}$$

Hence, by triangle inequality

$$\begin{aligned} A_{\|w_0 - w'_0\|_{\mathcal{G}}}(\eta) &\leq A_{\|\Delta^m w_{k_0} - w_0\|_{\mathcal{G}}}(\eta) + A_{\|\Delta^m w_{k_0} - w'_0\|_{\mathcal{G}}}(\eta) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &< \epsilon \end{aligned}$$

Since ϵ is arbitrary, hence $A_{\|w_0 - w'_0\|_{\mathcal{G}}}(\eta) = A_{\bar{0}}(\eta)$ and therefore, $w_0 = w'_0$. Hence w_0 is uniquely determined. $\square$

Theorem 3.4. Let (w_k) and (z_k) be two sequences in $(\mathcal{W}, \|\cdot\|_{\mathcal{G}})$ and Δ^m is the m^{th} order difference operator, such that $w_k \xrightarrow{\mathcal{I}(\Delta^m; \mathcal{G})} w_0$ and $z_k \xrightarrow{\mathcal{I}(\Delta^m; \mathcal{G})} z_0$. Then

- (i) $(w_k + z_k) \xrightarrow{\mathcal{I}(\Delta^m; \mathcal{G})} (w_0 + z_0)$,
- (ii) $cw_k \xrightarrow{\mathcal{I}(\Delta^m; \mathcal{G})} cw_0$ where c is any scalar.

Proof. (i) Suppose $w_k \xrightarrow{\mathcal{I}(\Delta^m; \mathcal{G})} w_0$ and $z_k \xrightarrow{\mathcal{I}(\Delta^m; \mathcal{G})} z_0$, for given $\epsilon > 0$, $\mathcal{S}_1(\eta, \epsilon)$, $\mathcal{S}_2(\eta, \epsilon) \in \mathcal{I}$,

$$\mathcal{S}_1(\eta, \epsilon) = \{k \in \mathbb{N} : A_{\|\Delta^m w_k - w_0\|_{\mathcal{G}}}(\eta) \geq \frac{\epsilon}{2}\} \text{ and}$$

$$\mathcal{S}_2(\eta, \epsilon) = \{k \in \mathbb{N} : A_{\|\Delta^m z_k - z_0\|_{\mathcal{G}}}(\eta) \geq \frac{\epsilon}{2}\}.$$

Since $\mathcal{S}_1(\eta, \epsilon)$, $\mathcal{S}_2(\eta, \epsilon) \in \mathcal{I}$ and $\mathcal{I}$ is closed with respect to union, now by using the concept as given in [10] we have,

$$\begin{aligned} \mathcal{S}_1(\eta, \epsilon) \cup \mathcal{S}_2(\eta, \epsilon) &\in \mathcal{I}; \\ (\mathcal{S}_1(\eta, \epsilon) \cup \mathcal{S}_2(\eta, \epsilon))^{\mathbb{G}} &\in \mathbb{F}(\mathcal{I}); \\ \text{i.e. } \mathcal{S}_1^{\mathbb{G}}(\eta, \epsilon) \cap \mathcal{S}_2^{\mathbb{G}}(\eta, \epsilon) &\in \mathbb{F}(\mathcal{I}); \\ \mathcal{S}_1^{\mathbb{G}}(\eta, \epsilon) \cap \mathcal{S}_2^{\mathbb{G}}(\eta, \epsilon) &\neq \emptyset \text{ as } \emptyset \notin \mathbb{F}(\mathcal{I}). \end{aligned}$$

It is enough to demonstrate (i) that,

$$\mathcal{S}_1^{\mathbb{G}}(\eta, \epsilon) \cap \mathcal{S}_2^{\mathbb{G}}(\eta, \epsilon) \subseteq \{k \in \mathbb{N} : A_{\|\Delta^m w_k + \Delta^m z_k - w_0 - z_0\|_{\mathcal{G}}}(\eta) < \epsilon\} = S(\text{say}).$$

Let $k_0 \in \mathcal{S}_1^{\mathbb{G}}(\eta, \epsilon) \cap \mathcal{S}_2^{\mathbb{G}}(\eta, \epsilon)$ then $k_0 \in \mathcal{S}_1^{\mathbb{G}}(\eta, \epsilon)$, $k_0 \in \mathcal{S}_2^{\mathbb{G}}(\eta, \epsilon)$ and therefore,

$$A_{\|\Delta^m w_{k_0} - w_0\|_{\mathcal{G}}}(\eta) < \frac{\epsilon}{2} \text{ and } A_{\|\Delta^m z_{k_0} - z_0\|_{\mathcal{G}}}(\eta) < \frac{\epsilon}{2} \quad (3.1)$$

Now,

$$\begin{aligned} A_{\|\Delta^m w_{k_0} + \Delta^m z_{k_0} - w_0 - z_0\|_{\mathcal{G}}}(\eta) &= A_{\|(\Delta^m w_{k_0} - w_0) + (\Delta^m z_{k_0} - z_0)\|_{\mathcal{G}}}(\eta) \\ &\leq A_{\|\Delta^m w_{k_0} - w_0\|_{\mathcal{G}}}(\eta) + A_{\|\Delta^m z_{k_0} - z_0\|_{\mathcal{G}}}(\eta) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \end{aligned}$$

so $k_0 \in \{k \in \mathbb{N} : A_{\|\Delta^m w_k + \Delta^m z_k - w_0 - z_0\|_{\mathcal{G}}}(\eta) < \epsilon\}$; therefore, $\mathcal{S}_1^{\mathcal{G}}(\eta, \epsilon) \cap \mathcal{S}_2^{\mathcal{G}}(\eta, \epsilon) \subseteq S$. This implies that $\mathcal{S}^{\mathcal{G}} \subseteq S_1 \cup S_2$. Since $S_1 \cup S_2 \in \mathcal{I}$, so $\mathcal{S}^{\mathcal{G}} = \{k \in \mathbb{N} : A_{\|\Delta^m w_k + \Delta^m z_k - w_0 - z_0\|_{\mathcal{G}}}(\eta) \geq \epsilon\} \in \mathcal{I}$. Therefore, $(w_k + z_k) \xrightarrow{\mathcal{I}(\Delta^m; \mathcal{G})} (w_0 + z_0)$.

(ii) Case I: When $c = 0$, It is not necessary for us to prove anything.

Case II: When $c \neq 0$, since $w_k \xrightarrow{\mathcal{I}(\Delta^m; \mathcal{G})} w_0$, for given $\epsilon > 0$, $\mathcal{S}_1(\eta, \epsilon_1) = \{k \in \mathbb{N} : A_{\|\Delta^m w_k - w_0\|_{\mathcal{G}}}(\eta) \geq \epsilon_1\} \in \mathcal{I}$ where, $\epsilon_1 = \frac{\epsilon}{|c|}$. Let $\mathcal{S}_2(\eta, \epsilon) = \{k \in \mathbb{N} : A_{\|\Delta^m c w_k - c w_0\|_{\mathcal{G}}}(\eta) \geq \epsilon\}$.

Now, let $k_0 \in \mathcal{S}_2(\eta, \epsilon_1)$, then $A_{\|\Delta^m c w_{k_0} - c w_0\|_{\mathcal{G}}}(\eta) \geq \epsilon$ and $|c| A_{\|\Delta^m w_{k_0} - w_0\|_{\mathcal{G}}}(\eta) \geq \epsilon$. This implies $A_{\|\Delta^m w_{k_0} - w_0\|_{\mathcal{G}}}(\eta) \geq \frac{\epsilon}{|c|} = \epsilon_1$. Thus $k_0 \in \{k \in \mathbb{N} : A_{\|\Delta^m w_k - w_0\|_{\mathcal{G}}}(\eta) \geq \epsilon_1\} = \mathcal{S}_1(\eta, \epsilon_1)$, therefore, $\mathcal{S}_2(\eta, \epsilon) \subseteq \mathcal{S}_1(\eta, \epsilon_1)$. Hence $\mathcal{S}_2(\eta, \epsilon) \in \mathcal{I}$ as $\mathcal{S}_1(\eta, \epsilon_1) \in \mathcal{I}$. $\square$

Theorem 3.5. *If every subsequence of $w = (w_k) \in \mathcal{I}$ is gradual $\mathcal{I}(\Delta^m)$ -summable to w_0 , then (w_k) is gradual $\mathcal{I}(\Delta^m)$ -summable to w_0 .*

Proof. If possible, let (w_k) is not gradually $\mathcal{I}(\Delta^m)$ -summable to w_0 . Then $\exists \epsilon > 0$ and $\eta \in (0, 1]$ such that $\mathcal{S}(\eta, \epsilon) \notin \mathcal{I}$; where $\mathcal{S}(\eta, \epsilon) = \{k \in \mathbb{N} : A_{\|\Delta^m w_k - w_0\|_{\mathcal{G}}}(\eta) \geq \epsilon\}$, so $\mathcal{S}(\eta, \epsilon)$ must be an infinite set, suppose $\mathcal{S}(\eta, \epsilon) = \{k_1 < k_2 < \dots < k_j < \dots\}$; we define another sequence (z_j) as $(z_j) = (w_{k_j})$ for $j \in \mathbb{N}$ [10]. Then (z_j) is a subsequence of (w_k) that is not gradually $\mathcal{I}$ -summable to w_0 , which is a contradiction. Therefore, (w_k) is gradually $\mathcal{I}(\Delta^m)$ -summable to w_0 . $\square$

Theorem 3.6. *Let (w_k) and (z_k) be two sequences in $(\mathcal{W}, \|\cdot\|_{\mathcal{G}})$ such that (z_k) is Δ^m gradual summable and $\{k \in \mathbb{N} : \Delta^m w_k \neq \Delta^m z_k\} \in \mathcal{I}$. Then (w_k) is gradually $\mathcal{I}(\Delta^m)$ -summable.*

Proof. Suppose $\{k \in \mathbb{N} : \Delta^m w_k \neq \Delta^m z_k\} \in \mathcal{I}$ and $z_k \xrightarrow{\Delta^m \|\cdot\|_{\mathcal{G}}} z_0$, for every $\epsilon > 0$ and $0 < \eta \leq 1$, $\{k \in \mathbb{N} : A_{\|\Delta^m z_k - z_0\|_{\mathcal{G}}}(\eta) \geq \epsilon\}$ is a finite set, and therefore,

$$\{k \in \mathbb{N} : A_{\|\Delta^m z_k - z_0\|_{\mathcal{G}}}(\eta) \geq \epsilon\} \in \mathcal{I}.$$

Having been included,

$\{k \in \mathbb{N} : A_{\|\Delta^m w_k - z_0\|_{\mathcal{G}}}(\eta) \geq \epsilon\} \subseteq \{k \in \mathbb{N} : A_{\|\Delta^m z_k - z_0\|_{\mathcal{G}}}(\eta) \geq \epsilon\} \cap \{k \in \mathbb{N} : \Delta^m w_k \neq \Delta^m z_k\}$ holds, by applying the above equation along with the hypothesis, we conclude,

$$w_k \xrightarrow{\mathcal{I}(\Delta^m; \mathcal{G})} z_0. \quad \square$$

Definition 3.7. A sequence $(w_k) \in (\mathcal{W}, \|\cdot\|_{\mathcal{G}})$ is said to be gradually $\mathcal{I}(\Delta^m)$ -Cauchy if for each $\epsilon > 0$ and $0 < \eta \leq 1$, $\exists$ a natural number $j(= j_{\epsilon}(\eta))$ such that the set $\mathcal{S}(\eta, \epsilon) = \{k \in \mathbb{N} : A_{\|\Delta^m w_k - \Delta^m w_j\|_{\mathcal{G}}}(\eta) \geq \epsilon\} \in \mathcal{I}$.

Theorem 3.8. *Any sequence in $\mathcal{W}$ that is gradually $\mathcal{I}(\Delta^m)$ -summable necessarily satisfies the gradually $\mathcal{I}(\Delta^m)$ -Cauchy condition.*

Proof. Let $w = (w_k) \in \mathcal{W}$ i.e. $w_k \xrightarrow{\mathcal{I}(\Delta^m; \mathcal{G})} w_0$, for every $\epsilon > 0$ and $0 < \eta \leq 1$,

$$\mathcal{S}(\eta, \epsilon) \in \mathcal{I}, \text{ where } \mathcal{S}(\eta, \epsilon) = \{k \in \mathbb{N} : A_{\|\Delta^m w_k - w_0\|_{\mathcal{G}}}(\eta) \geq \epsilon\}$$

Clearly, $\mathcal{S}^{\mathcal{G}}(\eta, \epsilon) \in \mathbb{F}(\mathcal{I})$ and therefore, $\mathcal{S}^{\mathcal{G}}(\eta, \epsilon) \neq \phi$ as $\phi \notin \mathbb{F}(\mathcal{I})$. Choose $k_0 \in \mathcal{S}^{\mathcal{G}}(\eta, \epsilon)$. Then,

$$A_{\|\Delta^m w_{k_0} - w_0\|_{\mathcal{G}}}(\eta) < \epsilon. \quad (3.2)$$

Suppose, $\mathcal{U}(\eta, \epsilon) = \{k \in \mathbb{N} : A_{\|\Delta^m w_k - \Delta^m w_{k_0}\|_{\mathcal{G}}}(\eta) \geq 2\epsilon\}$. We will now demonstrate that the subsequent inclusion holds true,

$$\mathcal{U}(\eta, \epsilon) \subseteq \mathcal{S}(\eta, \epsilon).$$

If $p \in \mathcal{U}(\eta, \epsilon)$ then,

$$A_{\|\Delta^m w_p - \Delta^m w_{k_0}\|_{\mathcal{G}}}(\eta) \geq 2\epsilon$$

$$\text{Now, } 2\epsilon \leq A_{\|\Delta^m w_p - \Delta^m w_{k_0}\|_{\mathcal{G}}}(\eta) \leq A_{\|\Delta^m w_p - w_0\|_{\mathcal{G}}}(\eta) + A_{\|w_0 - \Delta^m w_{k_0}\|_{\mathcal{G}}}(\eta)$$

$$2 \leq A_{\|\Delta^m w_p - w_0\|_{\mathcal{G}}}(\eta) + \epsilon \quad (\text{by 3.2})$$

$$A_{\|\Delta^m w_p - w_0\|_{\mathcal{G}}}(\eta) \geq \epsilon.$$

i.e. $\{p \in \mathbb{N} : A_{\|\Delta^m w_p - w_0\|_{\mathcal{G}}}(\eta) \geq \epsilon\} \in \mathcal{I}$ therefore, $p \in \mathcal{S}(\eta, \epsilon)$. This implies that $\mathcal{U}(\eta, \epsilon) \in I$; as a result, the sequence (w_k) is gradually $\mathcal{I}(\Delta^m)$ -Cauchy. $\square$

4 $\mathcal{I}^*(\Delta^m)$ -summability in $(\mathcal{W}, \|\cdot\|_{\mathcal{G}})$

Definition 4.1. A sequence $w = (w_k)$ in $(\mathcal{W}, \|\cdot\|_{\mathcal{G}})$ is said to be gradually $\mathcal{I}^*(\Delta^m)$ -summable to an element $w_0 \in \mathcal{W}$ if there exists an infinite set $K = \{k_1 < k_2 < \dots < k_j < \dots\} \in \mathbb{F}(\mathcal{I})$ such that the subsequence (w_{k_j}) is gradually convergent to w_0 . In this case, we write

$$\mathcal{I}^*(\Delta^m : \mathcal{G})\text{-}\lim_{k \rightarrow \infty} w_k = w_0.$$

Theorem 4.2. If $w = (w_k) \in (\mathcal{W}, \|\cdot\|_{\mathcal{G}})$ such that $w_k \xrightarrow{\mathcal{I}^*(\Delta^m : \mathcal{G})} w_0$. Then $w_k \xrightarrow{\mathcal{I}(\Delta^m : \mathcal{G})} w_0$.

Proof. Let $w_k \xrightarrow{\mathcal{I}^*(\Delta^m : \mathcal{G})} w_0$. Then, $\exists K = \{k_1 < k_2 < \dots < k_j < \dots\} \in \mathbb{F}(\mathcal{I})$ such that the subsequence (w_{k_j}) is gradual convergent to w_0 so, for each $\epsilon > 0$ and $0 < \eta \leq 1$, $\exists 0 < k_0 \in \mathbb{N}$ such that $A_{\|\Delta^m w_{k_j} - w_0\|_{\mathcal{G}}}(\eta) < \epsilon \forall k \geq k_0$. Therefore,

$$\{k \in \mathbb{N} : A_{\|w_k - w_0\|_{\mathcal{G}}}(\eta) \geq \epsilon\} \subseteq \{1, 2, \dots, k_0 - 1\} \cup K^{\mathbb{C}} \in \mathcal{I},$$

as $\{1, 2, \dots, k_0 - 1\} \in \mathcal{I}$ since it is finite set and $K^{\mathbb{C}} \in \mathcal{I}$. This shows that $\mathcal{I}(\Delta^m : \mathcal{G})\text{-}\lim_{k \rightarrow \infty} w_k = w_0$. $\square$

The proof of following theorem follows the classical method used in ideal convergence theory (see, e.g., [8, 10]).

Theorem 4.3. If $\mathcal{I}$ satisfies the condition (AP) and $w = (w_k) \in (\mathcal{W}, \|\cdot\|_{\mathcal{G}})$ such that $w_k \xrightarrow{\mathcal{I}(\Delta^m : \mathcal{G})} w_0$, then $w_k \xrightarrow{\mathcal{I}^*(\Delta^m : \mathcal{G})} w_0$.

Proof. Suppose, $w_k \xrightarrow{\mathcal{I}(\Delta^m : \mathcal{G})} w_0$, then for each $\epsilon > 0$ and $0 < \eta \leq 1$, the set

$$\mathcal{S}(\eta, \epsilon) = \{k \in \mathbb{N} : A_{\|\Delta^m w_k - w_0\|_{\mathcal{G}}}(\eta) \geq \epsilon\} \in \mathcal{I}.$$

Now, define a set $\{\mathcal{S}_j(\eta, \epsilon)\}_{j \in \mathbb{N}} \in \mathcal{I}$ by considering

$$\begin{aligned} \mathcal{S}_1(\eta, 1) &= \{k \in \mathbb{N} : A_{\|\Delta^m w_k - w_0\|_{\mathcal{G}}}(\eta) \geq 1\} \text{ and} \\ \mathcal{S}_j(\eta, j) &= \{k \in \mathbb{N} : \frac{1}{j} \leq A_{\|\Delta^m w_k - w_0\|_{\mathcal{G}}}(\eta) \leq \frac{1}{j-1}\} \text{ for } j \geq 2. \end{aligned}$$

Now, applying the method which is used in [8, 10]. Here, clearly $\{\mathcal{S}_j(\eta, \epsilon)\}_{j \in \mathbb{N}}$ is a countable family of mutually disjoint sets in $\mathcal{I}$ and $\mathcal{I}$ satisfies the condition (AP), so there exists an another countable family of sets $\{\mathcal{T}_j(\eta, \epsilon)\}_{j \in \mathbb{N}} \in \mathcal{I}$ such that $\mathcal{S}_j \Delta \mathcal{T}_j$ is a finite set for each $j \in \mathbb{N}$ and $\mathcal{T} = \bigcup_{j=1}^{\infty} \mathcal{T}_j \in \mathcal{I}$. Since $\mathcal{T} \in \mathcal{I}$ so $\mathcal{T}^{\mathbb{C}} = \mathbb{N} - \mathcal{T} \in \mathbb{F}(\mathcal{I})$. It is enough to demonstrate that to prove the result, $\lim_{k \in \mathcal{T}, k \rightarrow \infty} w_k = w_0$. Let $\epsilon > 0$ be given and $\eta \in (0, 1]$. Choose a positive integer r such that $\epsilon > \frac{1}{r-1}$. Then we have

$$\{k \in \mathbb{N} : A_{\|\Delta^m w_k - w_0\|_{\mathcal{G}}}(\eta) \geq \epsilon\} \subseteq \{k \in \mathbb{N} : A_{\|\Delta^m w_k - w_0\|_{\mathcal{G}}}(\eta) \geq \frac{1}{r+1}\} = \bigcup_{j=1}^{r+1} \mathcal{S}_j. \quad (4.1)$$

Since we have $\mathcal{S}_j \Delta \mathcal{T}_j$ is a finite set for every $j = \{1, 2, 3, \dots, r+1\}$, therefore there exists a positive integer n_0 such that

$$\left\{ \left\{ \bigcup_{j=1}^{r+1} \mathcal{T}_j \right\} \cap \{k \in \mathbb{N} : k \geq n_0\} \right\} = \left\{ \left\{ \bigcup_{j=1}^{r+1} \mathcal{S}_j \right\} \cap \{k \in \mathbb{N} : k \geq n_0\} \right\}.$$

If $k > n_0$ and $k \in \mathcal{T}^{\mathbb{C}}$, then $k \in \mathcal{T}$, this implies that $k \notin \bigcup_{j=1}^{r+1} \mathcal{T}_j$ and therefore $k \notin \bigcup_{j=1}^{r+1} \mathcal{S}_j$, hence for every $k > n_0$ and $k \in \mathcal{T}^{\mathbb{C}}$; then we have by (4.1) $A_{\|\Delta^m w_k - w_0\|_{\mathcal{G}}}(\eta) < \epsilon$. $\square$

Definition 4.4. A sequence $w = (w_k) \in (\mathcal{W}, \|\cdot\|_{\mathcal{G}})$ with Δ^m is said to be gradually $\mathcal{I}^*(\Delta^m)$ -Cauchy if there exists a set $K = \{k_1 < k_2 < \dots < k_j < \dots\} \in \mathbb{F}(\mathcal{I})$ and a subsequence (w_{k_j}) is gradual Cauchy sequence.

The proof of the following theorems 4.5 and 4.6 can be obtained analogously.

Theorem 4.5. Let $\mathcal{I}$ be an admissible ideal, $w = (w_k) \in (\mathcal{W}, \|\cdot\|_{\mathcal{G}})$. If (w_k) is gradually $\mathcal{I}^*(\Delta^m)$ -Cauchy, then it is gradually $\mathcal{I}(\Delta^m)$ -Cauchy.

Proof. Omitted. □

Theorem 4.6. If $\mathcal{I}$ satisfy the property (AP), $w = (w_k)$ be a sequence in $(\mathcal{W}, \|\cdot\|_{\mathcal{G}})$. Then the $\mathcal{I}(\Delta^m)$ -Cauchy and $\mathcal{I}^*(\Delta^m)$ -Cauchy coincides.

Proof. Omitted. □

5 Conclusion remarks

Statistical convergence is a non-matrix summable method and has wide application in many areas of mathematical analysis such as summability theory, approximation theory, Fourier series, functional analysis etc. The present study proposes a new summability approach, namely $\mathcal{I}^*(\Delta^m)$ -summability, formulated in the framework of gradual normed linear spaces. The established results offer an effective means to investigate convergence phenomena in these spaces involving high levels of uncertainty and ordinary norms fails to work in these situations.

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