

ON THE POINTWISE AND UNIFORM DEFERRED STATISTICAL SUMMABILITY OF FUNCTION SEQUENCE VIA INTUITIONISTIC FUZZY NORM

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Abstract This paper explores the concept of deferred statistical summability for functions sequences within the framework of intuitionistic fuzzy normed linear spaces (briefly termed as IFNLS). We introduce and formalize two key variants: pointwise deferred statistical summability and uniform deferred statistical summability. Through illustrative examples, we clarify these new summability concepts and examine their distinctive properties. Additionally, we examine related notions of Cauchy sequences and equicontinuity in this setting. Intuitionistic fuzzy normed spaces offer a valuable framework for modeling situations with uncertainty or partial truth.

1 Introduction

The concept of summability has long been a cornerstone of mathematical analysis, offering ways to assign finite values to divergent sequences and analyze the convergence behavior of sequences. Various summability methods have been developed and studied extensively, providing powerful tools for theoretical and practical applications. Among these methods, statistical convergence stands out as a significant extension. Independent ideas for statistical convergence, which extends the concept of ordinary convergence, were made for a sequence of integers by Steinhaus[28] and Fast[9]. This approach differs from matrix summability approaches, and it is based on the idea of natural density (represented by δ) for sets of positive integers. Subsequently, many authors have generalized the concept of statistical convergence; one of these generalizations is deferred statistical convergence. For any pair of sequences $\mu = (\mu_n)$ and $\nu = (\nu_n)$, where $n \in \mathbb{N}$, satisfying the following properties: $\mu_n < \nu_n$, and $\nu_n \rightarrow \infty$ as $n \rightarrow \infty$, Agnew [1] introduced a generalization of the Cesàro mean known as the deferred Cesàro mean, which offers stronger properties. By using this concept to statistical convergence, Küçükaslan and Yilmaztürk [19] presented deferred statistical convergence. They studied this generalized summability method for number sequences with the help of deferred density $\delta_{\beta, \omega}$ of subsets of $\mathbb{N}$. For $\mathcal{K} \subseteq \mathbb{N}$, it is defined by $\delta_{\beta, \omega}(\mathcal{K}) = \lim_{n \rightarrow \infty} \frac{1}{\omega_n - \beta_n} |\{\beta_n < k \leq \omega_n : k \in \mathcal{K}\}|$.

In parallel with these developments, the fuzzy set theory given by Zadeh [29] and its later extensions, intuitionistic fuzzy sets defined by Atanassov [2]. Compared to standard fuzzy sets, intuitionistic fuzzy sets provide a more advanced approach, since they are defined by using both membership function $\mathcal{M}$ and non-membership function $\mathcal{N}$. Katsaras [18] defines fuzzy norms through the use of fuzzy sets. This idea of a fuzzy norm is valuable for addressing problems where a crisp norm cannot be precisely determined due to huge uncertainty or vagueness. The field of fuzzy normed spaces has seen significant growth, with valuable contributions from many authors (see [3], [5], [10]). Saadati and Park [26] proposed an intuitionistic fuzzy norm, an extension of the fuzzy norm. This extension is useful in handling the uncertainty and imprecision in mathematical modeling and has applications in multiple areas such as decision-making, control theory, and image processing, etc.

The combination of these two ideas—summability theory and intuitionistic fuzzy set theory—has led to the exploration of summability methods within the framework of intuitionistic fuzzy normed linear spaces. This combination not only enriches the theory of summability but also enhances the modeling capabilities in environments with inherent uncertainty and fuzziness. For some additional research in this area, we refer to [13], [14], [15], [16], [20], [21], [24], [22], [23].

Additionally, a fundamental concept in the study of sequences of functions is convergence. There exist two primary types of convergence: Pointwise convergence and uniform convergence. Many authors have investigated statistical convergence for sequences of functions (ϵ_κ) from various angles (see [11], [12], [6], [4], [7]).

In this context, the present research aims to introduce deferred statistical summability for function sequences in intuitionistic fuzzy normed linear spaces. This concept is a more generalized version of all variants of statistical convergence. This extension is motivated by the need to handle more complex forms of uncertainty and to address the convergence of function sequences in a fuzzy environment more effectively. We establish some notions and properties of this concept. Additionally, we provide illustrative examples to demonstrate the applicability of deferred statistical summability for function sequences in the context of IFNLS.

2 Preliminaries

In this section, we recall some fundamental definitions crucial for our paper. Saadati and Park [26] used the binary operations continuous t -norm (denoted by $\otimes$) and continuous t -conorm (denoted by $\oplus$) [27] to establish intuitionistic fuzzy normed linear spaces. Their definition of IFNLS is provided below:

Definition 2.1. The structure $(\mathcal{L}_1, \mathcal{M}, \mathcal{N}, \otimes, \oplus)$ be an IFNLS, where $\mathcal{L}_1$ is a linear space over a field $\mathcal{F}$. $\mathcal{M}, \mathcal{N}$ are fuzzy sets on $\mathcal{L}_1 \times (0, \infty)$ which fulfills the below axioms $\forall \ell, s \in \mathcal{L}_1$ and $\varsigma_1, \varsigma_2 > 0$

- (i) $\mathcal{M}(\ell, \varsigma) + \mathcal{N}(\ell, \varsigma) \leq 1$,
- (ii) $\mathcal{M}(\ell, \varsigma) > 0$; $\mathcal{N}(\ell, \varsigma) < 1$,
- (iii) $\mathcal{M}(\ell, \varsigma) = 1 \iff \ell = 0$; $\mathcal{N}(\ell, \varsigma) = 0 \iff \ell = 0$,
- (iv) $\mathcal{M}(c\ell, \varsigma) = \mathcal{M}\left(\ell, \frac{\varsigma}{|c|}\right)$; $\mathcal{N}(c\ell, \varsigma) = \mathcal{N}\left(\ell, \frac{\varsigma}{|c|}\right)$ for $c \neq 0$ and $c \in F$,
- (v) $\mathcal{M}(\ell, \varsigma_1) \otimes \mathcal{M}(s, \varsigma_2) \leq \mathcal{M}(\ell + s, \varsigma_1 + \varsigma_2)$; $\mathcal{N}(\ell, \varsigma_1) \oplus \mathcal{N}(s, \varsigma_2) \geq \mathcal{N}(\ell + s, \varsigma_1 + \varsigma_2)$,
- (vi) $\mathcal{M}(\ell, \cdot)$ and $\mathcal{N}(\ell, \cdot) : (0, \infty) \rightarrow [0, 1]$ are continuous,
- (vii) $\lim_{\varsigma \rightarrow \infty} \mathcal{M}(\ell, \varsigma) = 1$ and $\lim_{\varsigma \rightarrow 0} \mathcal{M}(\ell, \varsigma) = 0$; $\lim_{\varsigma \rightarrow \infty} \mathcal{N}(\ell, \varsigma) = 0$, $\lim_{\varsigma \rightarrow 0} \mathcal{N}(\ell, \varsigma) = 1$.

Throughout this work, $(\mathcal{M}, \mathcal{N})$ represents an intuitionistic fuzzy norm.

Example 2.2. Consider $(\mathcal{L}_1, \|\bullet\|)$ as a normed linear space. Define $\varrho_1 \otimes \varrho_2 = \varrho_1 \varrho_2$ and $\varrho_1 \oplus \varrho_2 = \min\{\varrho_1 + \varrho_2, 1\}$ for $\varrho_1, \varrho_2 \in [0, 1]$. Consider $\mathcal{M}(\ell, \varsigma) = \frac{\varsigma}{\varsigma + \|\ell\|}$ and $\mathcal{N}(\ell, \varsigma) = \frac{\|\ell\|}{\varsigma + \|\ell\|} \forall \varsigma > 0$ and $\ell \in \mathcal{L}_1$. Then $(\mathcal{L}_1, \mathcal{M}, \mathcal{N}, \otimes, \oplus)$ be an IFNLS.

Definition 2.3. [26] Consider $(\mathcal{L}_1, \mathcal{M}, \mathcal{N}, \otimes, \oplus)$ as an IFNLS. A number sequence $\ell = (\ell_k)$ in $\mathcal{L}_1$ is convergent to ℓ_0 with respect to IF norm- $(\mathcal{M}, \mathcal{N})$ if for every $\Upsilon \in (0, 1)$ and $\varsigma > 0$, $\exists k_0 \in \mathbb{N}$ such that $\mathcal{M}(\ell_k - \ell_0, \varsigma) > 1 - \Upsilon$ and $\mathcal{N}(\ell_k - \ell_0, \varsigma) < \Upsilon \forall k \geq k_0$. We write it as $(\mathcal{M}, \mathcal{N}) - \ell_k = \ell_0$.

Definition 2.4. [8] Let $(\mathcal{L}_1, \mathcal{M}, \mathcal{N}, \otimes, \oplus)$ and $(\mathcal{L}_2, \mathcal{M}', \mathcal{N}', \otimes, \oplus)$ be two IFNLS over same field $\mathcal{F}$ and $\epsilon_\kappa : (\mathcal{L}_1, \mathcal{M}, \mathcal{N}, \otimes, \oplus) \rightarrow (\mathcal{L}_2, \mathcal{M}', \mathcal{N}', \otimes, \oplus)$ is a function sequence. Then,

- (i) The sequence $(\epsilon_\kappa)_{k \in \mathbb{N}}$ is referred to as pointwise intuitionistic fuzzy convergent on $\mathcal{L}_1$ with respect to $(\mathcal{M}, \mathcal{N})$. If $\ell \in \mathcal{L}_1$, then the sequence $(\epsilon_\kappa(\ell))_{k \in \mathbb{N}}$ is convergent with respect to $(\mathcal{M}', \mathcal{N}')$.

- (ii) The sequence $(\epsilon_\kappa)_{\kappa \in \mathbb{N}}$ is termed uniformly intuitionistic fuzzy convergent on $\mathcal{L}_1$ to a function ϵ with respect to $(\mathcal{M}, \mathcal{N})$. If $\Upsilon \in (0, 1)$, there exists a positive integer $k_0 = k_0(\Upsilon, \varsigma)$ such that $\forall \ell \in \mathcal{L}_1$ and $\forall \kappa \geq k_0$,

$$\mathcal{M}(\epsilon_\kappa(\ell) - \epsilon(\ell), \varsigma) > 1 - \Upsilon \quad \text{and} \quad \mathcal{N}(\epsilon_\kappa(\ell) - \epsilon(\ell), \varsigma) < \Upsilon.$$

Definition 2.5. [25] Consider $(\mathcal{L}_1, \mathcal{M}, \mathcal{N}, \otimes, \oplus)$ as an IFNLS. A number sequence $\ell = (\ell_k)$ in $\mathcal{L}_1$ is called deferred statistical summable to ℓ_0 w.r.t IF norm- $(\mathcal{M}, \mathcal{N})$ if for every $\Upsilon \in (0, 1)$ and $\varsigma > 0$ we have

$$\lim_{n \rightarrow \infty} \frac{1}{\nu_n - \mu_n} \left| \left\{ \mu_n < k \leq \nu_n : \mathcal{M}(\ell_k - \ell_0, \varsigma) \leq 1 - \Upsilon \right. \right. \\ \left. \left. \text{or } \mathcal{N}(\ell_k - \ell_0, \varsigma) \geq \Upsilon \right\} \right| = 0,$$

equivalently,

$$\delta_\mu^\nu(\{\mu_n < k \leq \nu_n : \mathcal{M}(\ell_k - \ell_0, \varsigma) \leq 1 - \Upsilon \text{ or } \mathcal{N}(\ell_k - \ell_0, \varsigma) \geq \Upsilon\}) = 0.$$

We write it as $\ell_k \rightarrow \ell_0 \left(\mathcal{DS}_\mu^\nu(\mathcal{M}, \mathcal{N}) \right)$.

3 Pointwise and Uniform deferred statistical convergence

We begin this section by introducing the concept of pointwise deferred statistical convergence (PDSC) for sequences of functions. PDSC builds upon the concepts of pointwise convergence and deferred statistical convergence, which utilizes two auxiliary sequences of non-negative integers. These sequences denoted by $\mu = (\mu_n)$ and $\nu = (\nu_n)$, where $n \in \mathbb{N}$, satisfy the following properties:

- (i) $\mu_n < \nu_n$,
- (ii) $\nu_n \rightarrow \infty$ as $n \rightarrow \infty$.

Through this work, $(\mathcal{L}_1, \mathcal{M}, \mathcal{N}, \otimes, \oplus)$ and $(\mathcal{L}_2, \mathcal{M}', \mathcal{N}', \otimes, \oplus)$ represents two IFNLS over same field $\mathcal{F}$.

Definition 3.1. Consider $\epsilon_\kappa : (\mathcal{L}_1, \mathcal{M}, \mathcal{N}, \otimes, \oplus) \rightarrow (\mathcal{L}_2, \mathcal{M}', \mathcal{N}', \otimes, \oplus)$ is a functions sequence. Then, (ϵ_κ) is pointwise deferred statistical convergent to limit function ϵ with respect to $(\mathcal{M}, \mathcal{N})$ if for each $\ell \in \mathcal{L}_1$, the sequence $(\epsilon_\kappa(\ell))$ is deferred statistical convergent to $\epsilon(\ell)$ with respect to $(\mathcal{M}', \mathcal{N}')$ i.e., if for each $\ell \in \mathcal{L}_1$, and for all $\Upsilon \in (0, 1)$, $\varsigma > 0$,

$$\delta_\mu^\nu \left(\left\{ \mu_n < \kappa \leq \nu_n : \mathcal{M}'(\epsilon_\kappa(\ell) - \epsilon(\ell), \varsigma) \leq 1 - \Upsilon \right. \right. \\ \left. \left. \text{or } \mathcal{N}'(\epsilon_\kappa(\ell) - \epsilon(\ell), \varsigma) \geq \Upsilon \right\} \right) = 0.$$

We denote it as $\mathcal{DS}_\mu^\nu(\mathcal{M}, \mathcal{N}) - \epsilon_\kappa \rightarrow \epsilon$.

Example 3.2. Consider $(\mathbb{R}, \bullet)$ represents the space of real numbers equipped with the standard norm, and let $\varrho_1 \otimes \varrho_2 = \varrho_1 \varrho_2$ and $\varrho_1 \oplus \varrho_2 = \min\{\varrho_1 + \varrho_2, 1\}$ for $\varrho_1, \varrho_2 \in [0, 1]$. For all $\ell \in \mathbb{R}$ and every $\varsigma > 0$, let

$$\mathcal{M}(\ell, \varsigma) = \frac{\varsigma}{\varsigma + |\ell|} \quad \text{and} \quad \mathcal{N}(\ell, \varsigma) = \frac{|\ell|}{\varsigma + |\ell|}$$

with these definitions, $(\mathbb{R}, \mathcal{M}, \mathcal{N}, \otimes, \oplus)$ is an IFNLS (Also, $([0, 1], \mathcal{M}, \mathcal{N}, \otimes, \oplus)$ is an IFNLS).

Consider the function sequence $\epsilon_\kappa : [0, 1] \rightarrow \mathbb{R}$ defined as

$$\epsilon_\kappa(\ell) = \begin{cases} \ell^\kappa + 1 & ; [\sqrt{\nu_n}] - c_0 < \kappa \leq [\sqrt{\nu_n}] \text{ and } \ell \in [0, \frac{1}{2}) \\ 0 & ; \kappa \notin [[\sqrt{\nu_n}] - c_0, [\sqrt{\nu_n}]] \text{ and } \ell \in [0, \frac{1}{2}) \\ \ell^\kappa + \frac{1}{2} & ; [\sqrt{\nu_n}] - c_0 < \kappa \leq [\sqrt{\nu_n}] \text{ and } \ell \in [\frac{1}{2}, 1) \\ 1 & ; \kappa \notin [[\sqrt{\nu_n}] - c_0, [\sqrt{\nu_n}]] \text{ and } \ell \in [\frac{1}{2}, 1) \\ 2 & ; \ell = 1 \end{cases}$$

where ν_n is a monotonically increasing sequence of positive integers and $c_0 \in \mathbb{N}$ (fixed number). Also, $0 < \mu_n \leq [\sqrt{\nu_n}] - c_0$.

Now we will prove that $(\mathbf{e}_\kappa)$ is pointwise deferred statistical convergent on $[0, 1]$ with respect to $(\mathcal{M}, \mathcal{N})$. For that we will prove that $(\mathbf{e}_\kappa(\ell))$ is deferred statistical convergent on $[0, 1]$ for each $\ell \in [0, 1]$.

(i) If we take $\ell \in [0, \frac{1}{2})$.

$$\begin{aligned}
 \mathcal{W}(\Upsilon, \varsigma) &= \left\{ \kappa \in \mathbb{N} : \mathcal{M}(\mathbf{e}_\kappa(\ell) - \mathbf{e}(\ell), \varsigma) \leq 1 - \Upsilon \text{ or } \right. \\
 &\quad \left. \mathcal{N}(\mathbf{e}_\kappa(\ell) - \mathbf{e}(\ell), \varsigma) \geq \Upsilon \right\} \\
 &= \left\{ \kappa \in \mathbb{N} : \frac{\varsigma}{\varsigma + |\mathbf{e}_\kappa(\ell) - \mathbf{e}(\ell)|} \leq 1 - \Upsilon \text{ or } \right. \\
 &\quad \left. \frac{|\mathbf{e}_\kappa(\ell) - \mathbf{e}(\ell)|}{\varsigma + |\mathbf{e}_\kappa(\ell) - \mathbf{e}(\ell)|} \geq \Upsilon \right\} \\
 &= \left\{ \kappa \in \mathbb{N} : \frac{\varsigma}{\varsigma + |\mathbf{e}_\kappa(\ell) - 0|} \leq 1 - \Upsilon \text{ or } \frac{|\mathbf{e}_\kappa(\ell) - 0|}{\varsigma + |\mathbf{e}_\kappa(\ell) - 0|} \geq \Upsilon \right\} \\
 &= \left\{ \kappa \in \mathbb{N} : |\mathbf{e}_\kappa(\ell)| \geq \frac{\Upsilon \varsigma}{1 - \Upsilon} \right\} \\
 &= \{ \kappa \in \mathbb{N} : [\sqrt{\nu_n}] - c_0 < \kappa \leq [\sqrt{\nu_n}] \} \\
 |\mathcal{W}(\Upsilon, \varsigma)| &\leq ([\sqrt{\nu_n}]) - ([\sqrt{\nu_n}] - c_0) \\
 \frac{|\mathcal{W}(\Upsilon, \varsigma)|}{(\nu_n - \mu_n)} &\leq \frac{c_0}{(\nu_n - \mu_n)} \\
 \delta_\mu^\nu(\mathcal{W}(\Upsilon, \varsigma)) &\leq 0 \quad \text{as } n \rightarrow \infty.
 \end{aligned}$$

As deferred density cannot be negative therefore $\delta_\mu^\nu(\mathcal{W}(\Upsilon, \varsigma)) = 0$. Hence, in this case $(\mathbf{e}_\kappa)$ is deferred statistical convergent to 0 with respect to $(\mathcal{M}, \mathcal{N})$.

(ii) If we take $\ell \in [\frac{1}{2}, 1)$.

$$\begin{aligned}
 \mathcal{W}'(\Upsilon, \varsigma) &= \left\{ \kappa \in \mathbb{N} : \mathcal{M}(\mathbf{e}_\kappa(\ell) - \mathbf{e}(\ell), \varsigma) \leq 1 - \Upsilon \text{ or } \right. \\
 &\quad \left. \mathcal{N}(\mathbf{e}_\kappa(\ell) - \mathbf{e}(\ell), \varsigma) \geq \Upsilon \right\} \\
 &= \left\{ \kappa \in \mathbb{N} : \frac{\varsigma}{\varsigma + |\mathbf{e}_\kappa(\ell) - 1|} \leq 1 - \Upsilon \text{ or } \right. \\
 &\quad \left. \frac{|\mathbf{e}_\kappa(\ell) - 1|}{\varsigma + |\mathbf{e}_\kappa(\ell) - 1|} \geq \Upsilon \right\} \\
 &= \left\{ \kappa \in \mathbb{N} : |\mathbf{e}_\kappa(\ell) - 1| \geq \frac{\Upsilon \varsigma}{1 - \Upsilon} \right\} \\
 &= \{ \kappa \in \mathbb{N} : [\sqrt{\nu_n}] - c_0 < \kappa \leq [\sqrt{\nu_n}] \} \\
 |\mathcal{W}'(\Upsilon, \varsigma)| &\leq ([\sqrt{\nu_n}]) - ([\sqrt{\nu_n}] - c_0) \\
 \frac{|\mathcal{W}'(\Upsilon, \varsigma)|}{(\nu_n - \mu_n)} &\leq \frac{c_0}{(\nu_n - \mu_n)} \\
 \delta_\mu^\nu(\mathcal{W}'(\Upsilon, \varsigma)) &\leq 0 \quad \text{as } n \rightarrow \infty.
 \end{aligned}$$

As deferred density cannot be negative therefore $\delta_\mu^\nu(\mathcal{W}'(\Upsilon, \varsigma)) = 0$. Hence, in this case $(\mathbf{e}_\kappa)$ is deferred statistical convergent to 1 with respect to $(\mathcal{M}, \mathcal{N})$.

(iii) If we take $\ell = 1$.

$$\begin{aligned}
 \mathcal{U}''(\Upsilon, \varsigma) &= \left\{ \kappa \in \mathbb{N} : \mathcal{M}(\mathbf{e}_\kappa(\ell) - \mathbf{e}(\ell), \varsigma) \leq 1 - \Upsilon \text{ or } \right. \\
 &\quad \left. \mathcal{N}(\mathbf{e}_\kappa(\ell) - \mathbf{e}(\ell), \varsigma) \geq \Upsilon \right\} \\
 &= \left\{ \kappa \in \mathbb{N} : \frac{\varsigma}{\varsigma + |\mathbf{e}_\kappa(\ell) - 2|} \leq 1 - \Upsilon \text{ or } \frac{|\mathbf{e}_\kappa(\ell) - 2|}{\varsigma + |\mathbf{e}_\kappa(\ell) - 2|} \geq \Upsilon \right\} \\
 &= \left\{ \kappa \in \mathbb{N} : |\mathbf{e}_\kappa(\ell) - 2| \geq \frac{\Upsilon \varsigma}{1 - \Upsilon} \right\} \\
 &= \left\{ \kappa \in \mathbb{N} : 0 \geq \frac{\Upsilon \varsigma}{1 - \Upsilon} \right\} = \phi \\
 |\mathcal{U}''(\Upsilon, \varsigma)| &= 0 \\
 \frac{|\mathcal{U}''(\Upsilon, \varsigma)|}{(\nu_n - \mu_n)} &= \frac{0}{(\nu_n - \mu_n)} \\
 \delta_\mu^\nu(\mathcal{U}''(\Upsilon, \varsigma)) &= 0 \quad \text{as } n \rightarrow \infty.
 \end{aligned}$$

This implies that $(\mathbf{e}_\kappa(\ell))$ is deferred statistical convergent to 2 with respect to $(\mathcal{M}, \mathcal{N})$ for $\ell = 1$.

By considering all three cases we conclude that

$$\mathcal{DS}_\mu^\nu(\mathcal{M}, \mathcal{N}) - \mathbf{e}_\kappa \rightarrow \begin{cases} 2 & ; \ell = 1 \\ 0 & ; \ell \in [0, \frac{1}{2}) \\ 1 & ; \ell \in [\frac{1}{2}, 1) \end{cases}$$

Hence, $(\mathbf{e}_\kappa(\ell))$ is pointwise deferred statistical convergent on $[0, 1]$ with respect to $(\mathcal{M}, \mathcal{N})$.

Lemma 3.3. Consider function sequence $\mathbf{e}_\kappa : (\mathcal{L}_1, \mathcal{M}, \mathcal{N}, \otimes, \oplus) \longrightarrow (\mathcal{L}_2, \mathcal{M}', \mathcal{N}', \otimes, \oplus)$. Then for each $\ell \in \mathcal{L}_1$, and for all $\Upsilon \in (0, 1)$, $\varsigma > 0$, the following assertions are identical

- (i) $\mathcal{DS}_\mu^\nu(\mathcal{M}, \mathcal{N}) - \mathbf{e}_\kappa \rightarrow \mathbf{e}$ (i.e., pointwise deferred statistical convergence),
- (ii) $\delta_\mu^\nu(\{\mu_n < k \leq \nu_n : \mathcal{M}'(\mathbf{e}_\kappa(\ell) - \mathbf{e}(\ell), \varsigma) \leq 1 - \Upsilon\}) =$
 $\delta_\mu^\nu(\{\mu_n < k \leq \nu_n : \mathcal{N}'(\mathbf{e}_\kappa(\ell) - \mathbf{e}(\ell), \varsigma) \geq \Upsilon\}) = 0$ for each $\ell \in \mathcal{L}_1$,
- (iii) $\delta_\mu^\nu(\{\mu_n < \kappa \leq \nu_n : \mathcal{M}'(\mathbf{e}_\kappa(\ell) - \mathbf{e}(\ell), \varsigma) > 1 - \Upsilon$
and $\mathcal{N}'(\mathbf{e}_\kappa(\ell) - \mathbf{e}(\ell), \varsigma) < \Upsilon\}) = 1$ for each $\ell \in \mathcal{L}_1$,
- (iv) $\delta_\mu^\nu(\{\mu_n < k \leq \nu_n : \mathcal{M}'(\mathbf{e}_\kappa(\ell) - \mathbf{e}(\ell), \varsigma) > 1 - \Upsilon\}) =$
 $\delta_\mu^\nu(\{\mu_n < k \leq \nu_n : \mathcal{N}'(\mathbf{e}_\kappa(\ell) - \mathbf{e}(\ell), \varsigma) < \Upsilon\}) = 1$ for each $\ell \in \mathcal{L}_1$,
- (v) For each $\ell \in \mathcal{L}_1$, $\mathcal{DS}_\mu^\nu\text{-}\lim \mathcal{M}'(\mathbf{e}_\kappa(\ell) - \mathbf{e}(\ell), \varsigma) = 1$ and $\mathcal{DS}_\mu^\nu\text{-}\lim \mathcal{N}'(\mathbf{e}_\kappa(\ell) - \mathbf{e}(\ell), \varsigma) = 0$.

In next Theorem (3.4), we prove a relation between pointwise intuitionistic fuzzy convergent sequences and pointwise deferred statistical convergent sequences in IFNLS.

Theorem 3.4. Let $\mathbf{e}_\kappa : (\mathcal{L}_1, \mathcal{M}, \mathcal{N}, \otimes, \oplus) \longrightarrow (\mathcal{L}_2, \mathcal{M}', \mathcal{N}', \otimes, \oplus)$ is a functions sequence. If $(\mathbf{e}_\kappa)$ is pointwise intuitionistic fuzzy convergent to a function $\mathbf{e}$ on $\mathcal{L}_1$ with respect to $(\mathcal{M}, \mathcal{N})$, then $\mathcal{DS}_\mu^\nu(\mathcal{M}, \mathcal{N}) - \mathbf{e}_\kappa \rightarrow \mathbf{e}$.

Proof. Since $(\mathbf{e}_\kappa)_{\kappa \in \mathbb{N}}$ is pointwise intuitionistic fuzzy convergent therefore by definition (2.4) for given $\Upsilon \in (0, 1)$, $\varsigma > 0$, there exists a positive integer $k_0(\Upsilon) \in \mathbb{N}$ such that for each $\ell \in \mathcal{L}_1$ we have

$$\mathcal{M}'(\mathbf{e}_\kappa(\ell) - \mathbf{e}(\ell), \varsigma) > 1 - \Upsilon \quad \text{and} \quad \mathcal{N}'(\mathbf{e}_\kappa(\ell) - \mathbf{e}(\ell), \varsigma) < \Upsilon$$

for all $k \geq k_0$. Thus, the following set has finite number of terms for each $\ell \in \mathcal{L}_1$,

$$\mathcal{U}(\Upsilon, \varsigma) = \{\mu_n < \kappa \leq \nu_n : \mathcal{M}'(\mathfrak{e}_\kappa(\ell) - \mathfrak{e}(\ell), \varsigma) \leq 1 - \Upsilon \text{ or} \\ \mathcal{N}'(\mathfrak{e}_\kappa(\ell) - \mathfrak{e}(\ell), \varsigma) \geq \Upsilon\}.$$

Since finite set has deferred density zero. So, $\delta_\mu^\nu(\mathcal{U}(\Upsilon, \varsigma)) = 0$.

Hence, $\mathcal{DS}_\mu^\nu(\mathcal{M}, \mathcal{N}) - \mathfrak{e}_\kappa \rightarrow \mathfrak{e}$. □

Theorem 3.5. Let $\mathfrak{e}_\kappa$ and $\mathfrak{e}'_\kappa$ from $(\mathcal{L}_1, \mathcal{M}, \mathcal{N}, \otimes, \oplus)$ to $(\mathcal{L}_2, \mathcal{M}', \mathcal{N}', \otimes, \oplus)$ be two function sequence. If $\mathcal{DS}_\mu^\nu(\mathcal{M}, \mathcal{N}) - \mathfrak{e}_\kappa \rightarrow \mathfrak{e}$ and $\mathcal{DS}_\mu^\nu(\mathcal{M}, \mathcal{N}) - \mathfrak{e}'_\kappa \rightarrow \mathfrak{e}'$ then $\mathcal{DS}_\mu^\nu(\mathcal{M}, \mathcal{N}) - (c_1\mathfrak{e}_\kappa + c_2\mathfrak{e}'_\kappa) \rightarrow (c_1\mathfrak{e} + c_2\mathfrak{e}')$.

Proof. Case I: Argument is simple if both $c_1 = 0$ and $c_2 = 0$.

Case II: Suppose $c_1 \neq 0$ and $c_2 \neq 0$. Since $\mathcal{DS}_\mu^\nu(\mathcal{M}, \mathcal{N}) - \mathfrak{e}_\kappa \rightarrow \mathfrak{e}$ and $\mathcal{DS}_\mu^\nu(\mathcal{M}, \mathcal{N}) - \mathfrak{e}'_\kappa \rightarrow \mathfrak{e}'$ then for each $\ell \in \mathcal{L}_1$ if we define

$$\mathcal{U}_1 = \left\{ \mu_n < \kappa \leq \nu_n : \mathcal{M}'\left(\mathfrak{e}_\kappa(\ell) - \mathfrak{e}(\ell), \frac{\varsigma}{2|c_1|}\right) \leq 1 - \Upsilon \text{ or} \right. \\ \left. \mathcal{N}'\left(\mathfrak{e}_\kappa(\ell) - \mathfrak{e}(\ell), \frac{\varsigma}{2|c_1|}\right) \geq \Upsilon \right\}, \\ \mathcal{U}_2 = \left\{ \mu_n < \kappa \leq \nu_n : \mathcal{M}'\left(\mathfrak{e}'_\kappa(\ell) - \mathfrak{e}'(\ell), \frac{\varsigma}{2|c_2|}\right) \leq 1 - \Upsilon \text{ or} \right. \\ \left. \mathcal{N}'\left(\mathfrak{e}'_\kappa(\ell) - \mathfrak{e}'(\ell), \frac{\varsigma}{2|c_2|}\right) \geq \Upsilon \right\},$$

then we have $\delta_\mu^\nu(\mathcal{U}_1) = 0$ and $\delta_\mu^\nu(\mathcal{U}_2) = 0$. If we define $\mathcal{U} = \mathcal{U}_1 \cup \mathcal{U}_2$ then $\delta_\mu^\nu(\mathcal{U}) = 0$. As $\mathcal{U} = \mathcal{U}_1 \cup \mathcal{U}_2 \neq \mathbb{N}$ and $\exists k_0 \in \mathbb{N}$ such that

$$\mathcal{M}'\left(\mathfrak{e}_{\kappa_0}(\ell) - \mathfrak{e}(\ell), \frac{\varsigma}{2|c_1|}\right) > 1 - \Upsilon \text{ and } \mathcal{N}'\left(\mathfrak{e}_{\kappa_0}(\ell) - \mathfrak{e}(\ell), \frac{\varsigma}{2|c_1|}\right) < \Upsilon, \\ \mathcal{M}'\left(\mathfrak{e}'_{\kappa_0}(\ell) - \mathfrak{e}'(\ell), \frac{\varsigma}{2|c_2|}\right) > 1 - \Upsilon \text{ and } \mathcal{N}'\left(\mathfrak{e}'_{\kappa_0}(\ell) - \mathfrak{e}'(\ell), \frac{\varsigma}{2|c_2|}\right) < \Upsilon.$$

Let

$$\mathcal{V} = \left\{ \mu_n < \kappa \leq \nu_n : \mathcal{M}'\left((c_1\mathfrak{e}_\kappa + c_2\mathfrak{e}'_\kappa)(\ell) - (c_1\mathfrak{e}(\ell) + c_2\mathfrak{e}'(\ell)), \varsigma\right) > 1 - \Upsilon \right. \\ \left. \text{and } \mathcal{N}'\left((c_1\mathfrak{e}_\kappa + c_2\mathfrak{e}'_\kappa)(\ell) - (c_1\mathfrak{e}(\ell) + c_2\mathfrak{e}'(\ell)), \varsigma\right) < \Upsilon \right\}$$

We will demonstrate that for every $\ell \in \mathcal{L}_1$, $\mathcal{U}^{\mathbb{G}} \subset \mathcal{V}$. Let $k_0 \in \mathcal{U}^{\mathbb{G}}$. In this case

$$\mathcal{M}'\left(\mathfrak{e}_{\kappa_0}(\ell) - \mathfrak{e}(\ell), \frac{\varsigma}{2|c_1|}\right) > 1 - \Upsilon \text{ and } \mathcal{N}'\left(\mathfrak{e}_{\kappa_0}(\ell) - \mathfrak{e}(\ell), \frac{\varsigma}{2|c_1|}\right) < \Upsilon, \\ \mathcal{M}'\left(\mathfrak{e}'_{\kappa_0}(\ell) - \mathfrak{e}'(\ell), \frac{\varsigma}{2|c_2|}\right) > 1 - \Upsilon \text{ and } \mathcal{N}'\left(\mathfrak{e}'_{\kappa_0}(\ell) - \mathfrak{e}'(\ell), \frac{\varsigma}{2|c_2|}\right) < \Upsilon.$$

Now

$$\mathcal{M}'\left((c_1\mathfrak{e}_{\kappa_0} + c_2\mathfrak{e}'_{\kappa_0})(\ell) - (c_1\mathfrak{e} + c_2\mathfrak{e}')(\ell), \varsigma\right) \\ \geq \mathcal{M}'\left(c_1\mathfrak{e}'_{\kappa_0}(\ell) - c_1\mathfrak{e}(\ell), \frac{\varsigma}{2}\right) \otimes \mathcal{M}'\left(c_2\mathfrak{e}'_{\kappa_0}(\ell) - c_2\mathfrak{e}'(\ell), \frac{\varsigma}{2}\right) \\ = \mathcal{M}'\left(\mathfrak{e}_{\kappa_0}(\ell) - \mathfrak{e}(\ell), \frac{\varsigma}{2|c_1|}\right) \otimes \mathcal{M}'\left(\mathfrak{e}'_{\kappa_0}(\ell) - \mathfrak{e}'(\ell), \frac{\varsigma}{2|c_2|}\right) \\ > (1 - \Upsilon) \otimes (1 - \Upsilon) \\ = (1 - \Upsilon)$$

and

$$\begin{aligned}
& \mathcal{N}' \left((c_1 \mathbf{e}_{\kappa_0} + c_2 \mathbf{e}'_{\kappa_0})(\ell) - (c_1 \mathbf{e} + c_2 \mathbf{e}')(\ell), \varsigma \right) \\
& \leq \mathcal{N}' \left(c_1 \mathbf{e}'_{\kappa_0}(\ell) - c_1 \mathbf{e}(\ell), \frac{\varsigma}{2} \right) \oplus \mathcal{N}' \left(c_2 \mathbf{e}'_{\kappa_0}(\ell) - c_2 \mathbf{e}'(\ell), \frac{\varsigma}{2} \right) \\
& = \mathcal{N}' \left(\mathbf{e}_{\kappa_0}(\ell) - \mathbf{e}(\ell), \frac{\varsigma}{2|c_1|} \right) \oplus \mathcal{N}' \left(\mathbf{e}'_{\kappa_0}(\ell) - \mathbf{e}'(\ell), \frac{\varsigma}{2|c_2|} \right) \\
& < \Upsilon \oplus \Upsilon \\
& = \Upsilon.
\end{aligned}$$

This implies $\mathcal{W}^{\mathbb{G}} \subset \mathcal{V}$. Since $\mathcal{V}^{\mathbb{G}} \subset \mathcal{W}$ and $\delta_{\mu}^{\nu}(\mathcal{W}) = 0$, thus $\delta_{\mu}^{\nu}(\mathcal{V}^{\mathbb{G}}) = 0$ i.e.

$$\begin{aligned}
& \delta_{\mu}^{\nu} \left(\left\{ \mu_n < \kappa \leq \nu_n : \mathcal{M}' \left((c_1 \mathbf{e}_{\kappa} + c_2 \mathbf{e}'_{\kappa})(\ell) - (c_1 \mathbf{e}(\ell) + c_2 \mathbf{e}'(\ell)), \varsigma \right) \leq 1 - \Upsilon \right. \right. \\
& \quad \left. \left. \text{or } \mathcal{N}' \left((c_1 \mathbf{e}_{\kappa} + c_2 \mathbf{e}'_{\kappa})(\ell) - (c_1 \mathbf{e}(\ell) + c_2 \mathbf{e}'(\ell)), \varsigma \right) \geq \Upsilon \right\} \right) = 0.
\end{aligned}$$

Hence,

$$\mathcal{DS}_{\mu}^{\nu}(\mathcal{M}, \mathcal{N}) - (c_1 \mathbf{e}_{\kappa} + c_2 \mathbf{e}'_{\kappa}) \rightarrow (c_1 \mathbf{e} + c_2 \mathbf{e}').$$

□

Definition 3.6. Let $\mathbf{e}_{\kappa} : (\mathcal{L}_1, \mathcal{M}, \mathcal{N}, \otimes, \oplus) \rightarrow (\mathcal{L}_2, \mathcal{M}', \mathcal{N}', \otimes, \oplus)$ is a functions sequence. Then, $(\mathbf{e}_{\kappa})$ is called pointwise deferred statistical Cauchy sequence with respect to $(\mathcal{M}, \mathcal{N})$ if for each $\ell \in \mathcal{L}_1$, the sequence $(\mathbf{e}_{\kappa}(\ell))$ is deferred statistical Cauchy sequence with respect to $(\mathcal{M}', \mathcal{N}')$ i.e., if for every $\Upsilon \in (0, 1)$ and $\varsigma > 0$, $\exists$ a number $r = r(\ell, \Upsilon, \varsigma)$ such that

$$\begin{aligned}
& \delta_{\mu}^{\nu} (\{ \mu_n < \kappa \leq \nu_n : \mathcal{M}' (\mathbf{e}_{\kappa}(\ell) - \mathbf{e}_r(\ell), \varsigma) \leq 1 - \Upsilon \text{ or } \\
& \quad \mathcal{N}' (\mathbf{e}_{\kappa}(\ell) - \mathbf{e}_r(\ell), \varsigma) \geq \Upsilon \text{ for each } \ell \in \mathcal{L}_1 \}) = 0.
\end{aligned}$$

Theorem 3.7. Let $\mathbf{e}_{\kappa} : (\mathcal{L}_1, \mathcal{M}, \mathcal{N}, \otimes, \oplus) \rightarrow (\mathcal{L}_2, \mathcal{M}', \mathcal{N}', \otimes, \oplus)$ is a functions sequence. If $(\mathbf{e}_{\kappa})$ is $\mathcal{DS}_{\mu}^{\nu}(\mathcal{M}, \mathcal{N}) - \mathbf{e}_{\kappa} \rightarrow \mathbf{e}$ with respect to $(\mathcal{M}, \mathcal{N})$, then $(\mathbf{e}_{\kappa})$ is pointwise deferred statistical Cauchy sequence with respect to $(\mathcal{M}, \mathcal{N})$.

Proof. For $\Upsilon \in (0, 1)$, choose $\mathfrak{U} \in (0, 1)$ such that

$$(1 - \mathfrak{U}) \otimes (1 - \mathfrak{U}) > 1 - \Upsilon \text{ and } \mathfrak{U} \oplus \mathfrak{U} < \Upsilon. \quad (3.1)$$

Since $\mathcal{DS}_{\mu}^{\nu}(\mathcal{M}, \mathcal{N}) - \mathbf{e}_{\kappa} \rightarrow \mathbf{e}$, so for every $\varsigma > 0$ we have $\delta_{\mu, \nu}(\mathcal{Y}_{\ell}(\mathfrak{U}, \varsigma)) = 0$ and $\delta_{\mu, \nu}(\mathcal{Y}_{\ell}(\mathfrak{U}, \varsigma))^{\mathbb{G}} = 1$, where

$$\begin{aligned}
\mathcal{Y}_{\ell}(\mathfrak{U}, \varsigma) = & \left\{ \mu_n < \kappa \leq \nu_n : \mathcal{M}' \left(\mathbf{e}_{\kappa}(\ell) - \mathbf{e}(\ell), \frac{\varsigma}{2} \right) \leq 1 - \mathfrak{U} \text{ or } \right. \\
& \left. \mathcal{N}' \left(\mathbf{e}_{\kappa}(\ell) - \mathbf{e}(\ell), \frac{\varsigma}{2} \right) \geq \mathfrak{U} \right\}
\end{aligned}$$

for each $\ell \in \mathcal{L}_1$. Let $r \in (\mathcal{Y}_{\ell}(\mathfrak{U}, \varsigma))^{\mathbb{G}}$, then

$$\mathcal{M}' \left(\mathbf{e}_r(\ell) - \mathbf{e}(\ell), \frac{\varsigma}{2} \right) > 1 - \mathfrak{U} \text{ and } \mathcal{N}' \left(\mathbf{e}_r(\ell) - \mathbf{e}(\ell), \frac{\varsigma}{2} \right) < \mathfrak{U}, \quad (3.2)$$

Our aim is to demonstrate that there exists a number $r = r(\ell, \Upsilon, \varsigma)$ such that

$$\begin{aligned}
& \delta_{\mu}^{\nu} (\{ \mu_n < \kappa \leq \nu_n : \mathcal{M}' (\mathbf{e}_{\kappa}(\ell) - \mathbf{e}_r(\ell), \varsigma) \leq 1 - \Upsilon \text{ or } \\
& \quad \mathcal{N}' (\mathbf{e}_{\kappa}(\ell) - \mathbf{e}_r(\ell), \varsigma) \geq \Upsilon \text{ for each } \ell \in \mathcal{L}_1 \}) = 0.
\end{aligned}$$

However we define,

$$\mathcal{Z}_\ell(\Upsilon, \varsigma) = \{\mu_n < \kappa \leq \nu_n : \mathcal{M}'(\mathbf{e}_\kappa(\ell) - \mathbf{e}_r(\ell), \varsigma) \leq 1 - \Upsilon \text{ or } \mathcal{N}'(\mathbf{e}_\kappa(\ell) - \mathbf{e}_r(\ell), \varsigma) \geq \Upsilon \text{ for each } \ell \in \mathcal{L}_1\}.$$

To verify the result, it is necessary to demonstrate that $\mathcal{Z}_\ell(\Upsilon, \varsigma) \subseteq \mathcal{Y}_\ell(\mathfrak{U}, \varsigma)$. Let $\kappa \in \mathcal{Z}_\ell(\Upsilon, \varsigma)$ then

$$\mathcal{M}'(\mathbf{e}_\kappa(\ell) - \mathbf{e}_r(\ell), \varsigma) \leq 1 - \Upsilon \text{ or } \mathcal{N}'(\mathbf{e}_\kappa(\ell) - \mathbf{e}_r(\ell), \varsigma) \geq \Upsilon \quad (3.3)$$

Case (i): If $\mathcal{M}'(\mathbf{e}_\kappa(\ell) - \mathbf{e}_r(\ell), \varsigma) \leq 1 - \Upsilon$, then we have $\mathcal{M}'(\mathbf{e}_\kappa(\ell) - \mathbf{e}(\ell), \frac{\varsigma}{2}) \leq 1 - \mathfrak{U}$ therefore $\kappa \in \mathcal{Y}_\ell(\mathfrak{U}, \varsigma)$. As otherwise $\mathcal{M}'(\mathbf{e}_\kappa(\ell) - \mathbf{e}(\ell), \frac{\varsigma}{2}) > 1 - \mathfrak{U}$, then by (3.1), (3.2) and (3.3) we have

$$\begin{aligned} 1 - \Upsilon &\geq \mathcal{M}'(\mathbf{e}_\kappa(\ell) - \mathbf{e}_r(\ell), \varsigma) \\ &= \mathcal{M}'\left(\mathbf{e}_\kappa(\ell) - \mathbf{e}(\ell) + \mathbf{e}(\ell) - \mathbf{e}_r(\ell), \frac{\varsigma}{2} + \frac{\varsigma}{2}\right) \\ &\geq \mathcal{M}'\left(\mathbf{e}_\kappa(\ell) - \mathbf{e}(\ell), \frac{\varsigma}{2}\right) \otimes \mathcal{M}'\left(\mathbf{e}(\ell) - \mathbf{e}_r(\ell), \frac{\varsigma}{2}\right) \\ &> (1 - \mathfrak{U}) \otimes (1 - \mathfrak{U}) \\ &> 1 - \Upsilon \text{ (not possible).} \end{aligned}$$

Hence $\mathcal{Z}_\ell(\Upsilon, \varsigma) \subseteq \mathcal{Y}_\ell(\mathfrak{U}, \varsigma)$.

Case (ii): If $\mathcal{N}'(\mathbf{e}_\kappa(\ell) - \mathbf{e}_r(\ell), \varsigma) \geq \Upsilon$, then we have $\mathcal{M}'(\mathbf{e}_\kappa(\ell) - \mathbf{e}(\ell), \frac{\varsigma}{2}) \geq \mathfrak{U}$ therefore $\kappa \in \mathcal{Y}_\ell(\mathfrak{U}, \varsigma)$. As otherwise $\mathcal{M}'(\mathbf{e}_\kappa(\ell) - \mathbf{e}(\ell), \frac{\varsigma}{2}) < \mathfrak{U}$, then by (3.1), (3.2) and (3.3) we have

$$\begin{aligned} \Upsilon &\leq \mathcal{N}'(\mathbf{e}_\kappa(\ell) - \mathbf{e}_r(\ell), \varsigma) \\ &= \mathcal{N}'\left(\mathbf{e}_\kappa(\ell) - \mathbf{e}(\ell) + \mathbf{e}(\ell) - \mathbf{e}_r(\ell), \frac{\varsigma}{2} + \frac{\varsigma}{2}\right) \\ &\leq \mathcal{N}'\left(\mathbf{e}_\kappa(\ell) - \mathbf{e}(\ell), \frac{\varsigma}{2}\right) \oplus \mathcal{N}'\left(\mathbf{e}(\ell) - \mathbf{e}_r(\ell), \frac{\varsigma}{2}\right) \\ &< \mathfrak{U} \oplus \mathfrak{U} \\ &< \Upsilon \text{ (not possible).} \end{aligned}$$

Hence $\mathcal{Z}_\ell(\Upsilon, \varsigma) \subseteq \mathcal{Y}_\ell(\mathfrak{U}, \varsigma)$. Thus in all the cases, $\mathcal{Z}_\ell(\Upsilon, \varsigma) \subseteq \mathcal{Y}_\ell(\mathfrak{U}, \varsigma)$. Since $\delta_{\mu, \nu}(\mathcal{Y}_\ell(\mathfrak{U}, \varsigma)) = 0$ so $\delta_{\mu, \nu}(\mathcal{Z}_\ell(\Upsilon, \varsigma)) = 0$ and therefore $(\mathbf{e}_\kappa)$ is pointwise deferred statistical Cauchy sequence w.r.t. intuitionistic fuzzy norm $(\mathcal{M}, \mathcal{N})$.

Now in subsequent part we study uniform deferred statistical convergence and its properties.

Definition 3.8. Let $\mathbf{e}_\kappa : (\mathcal{L}_1, \mathcal{M}, \mathcal{N}, \otimes, \oplus) \rightarrow (\mathcal{L}_2, \mathcal{M}', \mathcal{N}', \otimes, \oplus)$ is a functions sequence. Then, $(\mathbf{e}_\kappa)$ is called uniform deferred statistical convergent to limit function $\mathbf{e}$ w.r.t. IF norm- $(\mathcal{M}, \mathcal{N})$ if for all $\ell \in \mathcal{L}_1$, the sequence $(\mathbf{e}_\kappa(\ell))$ is deferred statistical convergent to $\mathbf{e}(\ell)$ w.r.t. IF-norm $(\mathcal{M}', \mathcal{N}')$ i.e., if for all $\ell \in \mathcal{L}_1$ and for all $\Upsilon \in (0, 1)$, $\varsigma > 0$,

$$\begin{aligned} \delta_\mu^\nu \left(\left\{ \mu_n < \kappa \leq \nu_n : \mathcal{M}'(\mathbf{e}_\kappa(\ell) - \mathbf{e}(\ell), \varsigma) \leq 1 - \Upsilon \right. \right. \\ \left. \left. \text{or } \mathcal{N}'(\mathbf{e}_\kappa(\ell) - \mathbf{e}(\ell), \varsigma) \geq \Upsilon \right\} \right) = 0. \end{aligned}$$

We denote it as $\mathcal{DS}_\mu^\nu(\mathcal{M}, \mathcal{N}) - \mathbf{e}_\kappa \rightrightarrows \mathbf{e}$.

Example 3.9. Consider an IFNLS $(\mathbb{R}, \mathcal{M}, \mathcal{N}, \otimes, \oplus)$ as in Example(3.2).

Consider a function sequence $\mathbf{e}_\kappa : [0, 1] \rightarrow \mathbb{R}$ defined as

$$\mathbf{e}_\kappa(\ell) = \begin{cases} 1 & ; [\sqrt{\nu_n}] - c_0 < \kappa \leq [\sqrt{\nu_n}] \\ \ell^\kappa & ; \kappa \notin [[\sqrt{\nu_n}] - c_0, [\sqrt{\nu_n}]] \end{cases}$$

where ν_n is a monotonically increasing sequence of positive integers and $c_0 \in \mathbb{N}$ (fixed number). Also, $0 < \mu_n \leq [\sqrt{\nu_n}] - c_0$.

We shall now demonstrate that, $(\mathbf{e}_\kappa)$ is uniform deferred statistical convergent to 0 on $[0, 1)$ with respect to $(\mathcal{M}, \mathcal{N})$.

For each $\ell \in [0, 1)$,

$$\begin{aligned}
 \mathcal{A}^*(\Upsilon, \varsigma) &= \left\{ \kappa \in \mathbb{N} : \mathcal{M}'(\mathbf{e}_\kappa(\ell) - \mathbf{e}(\ell), \varsigma) \leq 1 - \Upsilon \text{ or } \right. \\
 &\quad \left. \mathcal{N}'(\mathbf{e}_\kappa(\ell) - \mathbf{e}(\ell), \varsigma) \geq \Upsilon \right\} \\
 &= \left\{ \kappa \in \mathbb{N} : \frac{\varsigma}{\varsigma + |\mathbf{e}_\kappa(\ell) - \mathbf{e}(\ell)|} \leq 1 - \Upsilon \text{ or } \right. \\
 &\quad \left. \frac{|\mathbf{e}_\kappa(\ell) - \mathbf{e}(\ell)|}{\varsigma + |\mathbf{e}_\kappa(\ell) - \mathbf{e}(\ell)|} \geq \Upsilon \right\} \\
 &= \left\{ \kappa \in \mathbb{N} : \frac{\varsigma}{\varsigma + |\mathbf{e}_\kappa(\ell) - 0|} \leq 1 - \Upsilon \text{ or } \frac{|\mathbf{e}_\kappa(\ell) - 0|}{\varsigma + |\mathbf{e}_\kappa(\ell) - 0|} \geq \Upsilon \right\} \\
 &= \left\{ \kappa \in \mathbb{N} : |\mathbf{e}_\kappa(\ell)| \geq \frac{\Upsilon \varsigma}{1 - \Upsilon} \right\} \\
 &= \left\{ \kappa \in [[\sqrt{\nu_n}] - c_0, [\sqrt{\nu_n}]] : |\mathbf{e}_\kappa(\ell)| \geq \frac{\Upsilon \varsigma}{1 - \Upsilon} \right\} \\
 &\quad \cup \left\{ \kappa \notin [[\sqrt{\nu_n}] - c_0, [\sqrt{\nu_n}]] : |\mathbf{e}_\kappa(\ell)| \geq \frac{\Upsilon \varsigma}{1 - \Upsilon} \right\} \\
 &= \{ \kappa \in [[\sqrt{\nu_n}] - c_0, [\sqrt{\nu_n}]] : \mathbf{e}_\kappa(\ell) = 1 \} \\
 &\quad \cup \left\{ \kappa \notin [[\sqrt{\nu_n}] - c_0, [\sqrt{\nu_n}]] : |\ell^\kappa| \geq \frac{\Upsilon \varsigma}{1 - \Upsilon} \right\} \\
 &= \mathcal{A}_1 \cup \mathcal{A}_2.
 \end{aligned}$$

where,

$$\begin{aligned}
 \mathcal{A}_1 &= \{ \kappa \in [[\sqrt{\nu_n}] - c_0, [\sqrt{\nu_n}]] : \mathbf{e}_\kappa(\ell) = 1 \}, \\
 \mathcal{A}_2 &= \left\{ \kappa \notin [[\sqrt{\nu_n}] - c_0, [\sqrt{\nu_n}]] : |\ell^\kappa| \geq \frac{\Upsilon \varsigma}{1 - \Upsilon} \right\}.
 \end{aligned}$$

As $\mathcal{A}_2$ is either finite or empty set in relation to Υ and ς , its deferred density is consequently zero. Hence, we obtain

$$\begin{aligned}
 \delta_\mu^\nu(\mathcal{A}^*(\Upsilon, \varsigma)) &= \delta_\mu^\nu(\mathcal{A}_1) + \delta_\mu^\nu(\mathcal{A}_2) \\
 &\leq \frac{c_0}{(\nu_n - \mu_n)} + 0 \\
 &\leq 0 \quad \text{as } n \rightarrow \infty.
 \end{aligned}$$

As deferred density cannot be negative therefore $\delta_\mu^\nu(\mathcal{A}^*(\Upsilon, \varsigma)) = 0$. Hence, $(\mathbf{e}_\kappa(\ell))$ is deferred statistical convergent to 0 with respect to $(\mathcal{M}, \mathcal{N})$ for every $\ell \in [0, 1)$. Hence,

$$\mathcal{DS}_\mu^\nu(\mathcal{M}, \mathcal{N}) - \mathbf{e}_\kappa \Rightarrow 0.$$

Lemma 3.10. Consider function sequence $\mathbf{e}_\kappa : (\mathcal{L}_1, \mathcal{M}, \mathcal{N}, \otimes, \oplus) \rightarrow (\mathcal{L}_2, \mathcal{M}', \mathcal{N}', \otimes, \oplus)$. Then for all $\ell \in \mathcal{L}_1$, and for all $\Upsilon \in (0, 1)$, $\varsigma > 0$, the following assertions are identical

- (i) $\mathcal{DS}_\mu^\nu(\mathcal{M}, \mathcal{N}) - \mathbf{e}_\kappa \Rightarrow \mathbf{e}$ (i.e., uniform deferred statistical convergence),
- (ii) $\delta_\mu^\nu(\{\mu_n < k \leq \nu_n : \mathcal{M}'(\mathbf{e}_\kappa(\ell) - \mathbf{e}(\ell), \varsigma) \leq 1 - \Upsilon\}) =$
 $\delta_\mu^\nu(\{\mu_n < k \leq \nu_n : \mathcal{N}'(\mathbf{e}_\kappa(\ell) - \mathbf{e}(\ell), \varsigma) \geq \Upsilon\}) = 0$ for all $\ell \in \mathcal{L}_1$,
- (iii) $\delta_\mu^\nu(\{\mu_n < \kappa \leq \nu_n : \mathcal{M}'(\mathbf{e}_\kappa(\ell) - \mathbf{e}(\ell), \varsigma) > 1 - \Upsilon$
and $\mathcal{N}'(\mathbf{e}_\kappa(\ell) - \mathbf{e}(\ell), \varsigma) < \Upsilon\}) = 1$ for all $\ell \in \mathcal{L}_1$,

- (iv) $\delta_\mu^\nu(\{\mu_n < k \leq \nu_n : \mathcal{M}'(\mathbf{e}_\kappa(\ell) - \mathbf{e}(\ell), \varsigma) > 1 - \Upsilon\}) = \delta_\mu^\nu(\{\mu_n < k \leq \nu_n : \mathcal{N}'(\mathbf{e}_\kappa(\ell) - \mathbf{e}(\ell), \varsigma) < \Upsilon\}) = 1$ for all $\ell \in \mathcal{L}_1$,
- (v) For all $\ell \in \mathcal{L}_1$, $\mathcal{DS}_\mu^\nu - \lim \mathcal{M}'(\mathbf{e}_\kappa(\ell) - \mathbf{e}(\ell), \varsigma) = 1$ and $\mathcal{DS}_\mu^\nu - \lim \mathcal{N}'(\mathbf{e}_\kappa(\ell) - \mathbf{e}(\ell), \varsigma) = 0$.

Proposition 3.11. Let $\mathbf{e}_\kappa$ and $\mathbf{e}$ be bounded function from $(\mathcal{L}_1, \mathcal{M}, \mathcal{N}, \otimes, \oplus)$ to $(\mathcal{L}_2, \mathcal{M}', \mathcal{N}', \otimes, \oplus)$. Then $(\mathbf{e}_\kappa)$ is uniform deferred statistical intuitionistic fuzzy convergent to $\mathbf{e}$ iff

$$\begin{aligned} \mathcal{DS}_\mu^\nu - \liminf \mathcal{M}'(\mathbf{e}_\kappa(\ell) - \mathbf{e}(\ell), \varsigma) &= 1, \quad \text{and} \\ \mathcal{DS}_\mu^\nu - \limsup \mathcal{N}'(\mathbf{e}_\kappa(\ell) - \mathbf{e}(\ell), \varsigma) &= 0. \end{aligned}$$

where the sup and inf are taken over all $\ell \in \mathcal{L}_1$.

Remark 3.12. $\mathcal{DS}_\mu^\nu(\mathcal{M}, \mathcal{N}) - \mathbf{e}_\kappa \rightrightarrows \mathbf{e}$ (i.e., uniform deferred statistical convergence) implies $\mathcal{DS}_\mu^\nu(\mathcal{M}, \mathcal{N}) - \mathbf{e}_\kappa \rightarrow \mathbf{e}$ (i.e., pointwise deferred statistical convergence).

Theorem 3.13. Let $\mathbf{e}_\kappa : (\mathcal{L}_1, \mathcal{M}, \mathcal{N}, \otimes, \oplus) \rightarrow (\mathcal{L}_2, \mathcal{M}', \mathcal{N}', \otimes, \oplus)$ be a functions sequence. If $(\mathbf{e}_\kappa)$ is uniform intuitionistic fuzzy convergent to a function $\mathbf{e}$ on $\mathcal{L}_1$ with respect to $(\mathcal{M}, \mathcal{N})$, then $\mathcal{DS}_\mu^\nu(\mathcal{M}, \mathcal{N}) - \mathbf{e}_\kappa \rightrightarrows \mathbf{e}$.

Proof. Using a similar approach to Theorem (3.4), we can easily prove this result. Therefore, the proof is left out. $\square$

Theorem 3.14. Let $\mathbf{e}_\kappa$ and $\mathbf{e}'_\kappa$ from $(\mathcal{L}_1, \mathcal{M}, \mathcal{N}, \otimes, \oplus)$ to $(\mathcal{L}_2, \mathcal{M}', \mathcal{N}', \otimes, \oplus)$ be two function sequence. If $\mathcal{DS}_\mu^\nu(\mathcal{M}, \mathcal{N}) - \mathbf{e}_\kappa \rightrightarrows \mathbf{e}$ and $\mathcal{DS}_\mu^\nu(\mathcal{M}, \mathcal{N}) - \mathbf{e}'_\kappa \rightrightarrows \mathbf{e}'$ then $\mathcal{DS}_\mu^\nu(\mathcal{M}, \mathcal{N}) - (c_1\mathbf{e}_\kappa + c_2\mathbf{e}'_\kappa) \rightrightarrows (c_1\mathbf{e} + c_2\mathbf{e}')$.

Proof. Since the argument is similar to Theorem (3.5) therefore, it is omitted. $\square$

Let $\mathcal{B}$ represents the collection of all uniformly bounded functions sequences in $\mathcal{L}_1$ i.e.,

$$\mathcal{B} = \{\mathbf{e}_\kappa(\ell) \in \mathcal{L}_1 : |\mathbf{e}_\kappa(\ell)| < \mathbf{C} \text{ for all } \kappa \in \mathbb{N}, \ell \in \mathcal{L}_1 \text{ and for some } \mathbf{C} > 0\}. \quad (3.4)$$

Theorem 3.15. Let $\mathbf{e}_\kappa : (\mathcal{L}_1, \mathcal{M}, \mathcal{N}, \otimes, \oplus) \rightarrow (\mathcal{L}_2, \mathcal{M}', \mathcal{N}', \otimes, \oplus)$ be a uniformly bounded functions sequence. If $\mathcal{DS}_\mu^\nu(\mathcal{M}, \mathcal{N}) - \mathbf{e}_\kappa \rightrightarrows \mathbf{e}$ and $\mathbf{e}(\ell)$ is bounded for all $\ell \in \mathcal{L}_1$, then $\mathcal{DS}_\mu^\nu(\mathcal{M}, \mathcal{N}) - \mathbf{e}_\kappa^2 \rightrightarrows \mathbf{e}^2$.

Proof. Since $\mathbf{e}_\kappa(\ell)$ be a uniformly bounded functions sequence then by using 3.4 we have $|\mathbf{e}_\kappa(\ell)| < \mathbf{C}_1$ for all $\kappa \in \mathbb{N}$, $\ell \in \mathcal{L}_1$, and for some $\mathbf{C}_1 > 0$. Also $\mathcal{DS}_\mu^\nu(\mathcal{M}, \mathcal{N}) - \mathbf{e}_\kappa \rightrightarrows \mathbf{e}$ then for all $\ell \in \mathcal{L}_1$ and for all $\Upsilon \in (0, 1)$, $\varsigma > 0$,

$$\delta_\mu^\nu(\mathcal{P}(\Upsilon, \varsigma)) = 0 \text{ and } \delta_\mu^\nu((\mathcal{P}(\Upsilon, \varsigma))^{\mathbf{G}}) = 1$$

where

$$\begin{aligned} \mathcal{P}(\Upsilon, \varsigma) &= \{\mu_n < \kappa \leq \nu_n : \mathcal{M}'(\mathbf{e}_\kappa(\ell) - \mathbf{e}(\ell), \varsigma) \leq 1 - \Upsilon \text{ or} \\ &\quad \mathcal{N}'(\mathbf{e}_\kappa(\ell) - \mathbf{e}(\ell), \varsigma) \geq \Upsilon\} \\ (\mathcal{P}(\Upsilon, \varsigma))^{\mathbf{G}} &= \{\mu_n < \kappa \leq \nu_n : \mathcal{M}'(\mathbf{e}_\kappa(\ell) - \mathbf{e}(\ell), \varsigma) > 1 - \Upsilon \text{ and} \\ &\quad \mathcal{N}'(\mathbf{e}_\kappa(\ell) - \mathbf{e}(\ell), \varsigma) < \Upsilon\} \end{aligned}$$

As given that $\mathbf{e}(\ell)$ is also bounded for all $\ell \in \mathcal{L}_1$, then there exist $\mathbf{C}_2 > 0$ such that $|\mathbf{e}(\ell)| < \mathbf{C}_2$. Now for $\Upsilon \in (0, 1)$, and $\varsigma > 0$ consider the following sets

$$\begin{aligned} \mathcal{Q}(\Upsilon, \varsigma) &= \{\mu_n < \kappa \leq \nu_n : \mathcal{M}'(\mathbf{e}_\kappa^2(\ell) - \mathbf{e}^2(\ell), \varsigma) \leq 1 - \Upsilon \text{ or} \\ &\quad \mathcal{N}'(\mathbf{e}_\kappa^2(\ell) - \mathbf{e}^2(\ell), \varsigma) \geq \Upsilon\} \\ (\mathcal{Q}(\Upsilon, \varsigma))^{\mathbf{G}} &= \{\mu_n < \kappa \leq \nu_n : \mathcal{M}'(\mathbf{e}_\kappa^2(\ell) - \mathbf{e}^2(\ell), \varsigma) > 1 - \Upsilon \text{ and} \\ &\quad \mathcal{N}'(\mathbf{e}_\kappa^2(\ell) - \mathbf{e}^2(\ell), \varsigma) < \Upsilon\} \end{aligned}$$

Our aim is to prove that $\delta_\mu^\nu(\mathcal{Q}(\Upsilon, \varsigma)) = 0$ and $\delta_\mu^\nu((\mathcal{Q}(\Upsilon, \varsigma))^{\mathbb{G}}) = 1$. For this we have to show that $(\mathcal{P}(\Upsilon, \varsigma))^{\mathbb{G}} \subset (\mathcal{Q}(\Upsilon, \varsigma))^{\mathbb{G}}$. Let $\kappa_0 \in (\mathcal{P}(\Upsilon, \varsigma))^{\mathbb{G}}$, then we have

$$\mathcal{M}'(\mathbf{e}_{\kappa_0}(\ell) - \mathbf{e}(\ell), \varsigma) > 1 - \Upsilon \text{ and } \mathcal{N}'(\mathbf{e}_{\kappa_0}(\ell) - \mathbf{e}(\ell), \varsigma) < \Upsilon. \quad (3.5)$$

Now

$$\begin{aligned} \mathcal{M}'(\mathbf{e}_{\kappa_0}^2(\ell) - \mathbf{e}^2(\ell), \varsigma) &= \mathcal{M}'((\mathbf{e}_{\kappa_0}(\ell) - \mathbf{e}(\ell))(\mathbf{e}_{\kappa_0}(\ell) + \mathbf{e}(\ell)), \varsigma) \\ &= \mathcal{M}'((\mathbf{e}_{\kappa_0}(\ell) - \mathbf{e}(\ell)), \frac{\varsigma}{|\mathbf{e}_{\kappa_0}(\ell) + \mathbf{e}(\ell)|}) \\ &= \mathcal{M}'((\mathbf{e}_{\kappa_0}(\ell) - \mathbf{e}(\ell)), \frac{\varsigma}{\mathbb{C}_1 + \mathbb{C}_2}) \\ &= \mathcal{M}'((\mathbf{e}_{\kappa_0}(\ell) - \mathbf{e}(\ell)), \varsigma_1) \\ &> 1 - \Upsilon \quad (\text{by using 3.5}), \end{aligned}$$

and

$$\begin{aligned} \mathcal{N}'(\mathbf{e}_{\kappa_0}^2(\ell) - \mathbf{e}^2(\ell), \varsigma) &= \mathcal{N}'((\mathbf{e}_{\kappa_0}(\ell) - \mathbf{e}(\ell))(\mathbf{e}_{\kappa_0}(\ell) + \mathbf{e}(\ell)), \varsigma) \\ &= \mathcal{N}'((\mathbf{e}_{\kappa_0}(\ell) - \mathbf{e}(\ell)), \frac{\varsigma}{|\mathbf{e}_{\kappa_0}(\ell) + \mathbf{e}(\ell)|}) \\ &= \mathcal{N}'((\mathbf{e}_{\kappa_0}(\ell) - \mathbf{e}(\ell)), \frac{\varsigma}{\mathbb{C}_1 + \mathbb{C}_2}) \\ &= \mathcal{N}'((\mathbf{e}_{\kappa_0}(\ell) - \mathbf{e}(\ell)), \varsigma_1) \\ &< \Upsilon \quad (\text{by using 3.5}), \end{aligned}$$

where $\varsigma_1 = \frac{\varsigma}{\mathbb{C}_1 + \mathbb{C}_2} > 0$. This implies $\kappa_0 \in (\mathcal{Q}(\Upsilon, \varsigma))^{\mathbb{G}}$. Therefore, $(\mathcal{P}(\Upsilon, \varsigma))^{\mathbb{G}} \subset (\mathcal{Q}(\Upsilon, \varsigma))^{\mathbb{G}}$. Consequently, we get $\delta_\mu^\nu((\mathcal{Q}(\Upsilon, \varsigma))^{\mathbb{G}}) = 1$, i.e. $\delta_\mu^\nu(\mathcal{Q}(\Upsilon, \varsigma)) = 0$. Hence, the proof is verified. $\square$

Definition 3.16. Let $\mathbf{e}_\kappa : (\mathcal{L}_1, \mathcal{M}, \mathcal{N}, \otimes, \oplus) \longrightarrow (\mathcal{L}_2, \mathcal{M}', \mathcal{N}', \otimes, \oplus)$ is a functions sequence. The sequence $(\mathbf{e}_\kappa)$ is called uniform deferred statistical Cauchy sequence with respect to $(\mathcal{M}, \mathcal{N})$ if for all $\ell \in \mathcal{L}_1$, the sequence $(\mathbf{e}_\kappa(\ell))$ is deferred statistical Cauchy with respect to $(\mathcal{M}', \mathcal{N}')$ i.e., if for every $\Upsilon \in (0, 1)$ and $\varsigma > 0$, there exists a number $r = r(\Upsilon, \varsigma)$ such that

$$\begin{aligned} \delta_\mu^\nu(\{\mu_n < \kappa \leq \nu_n : \mathcal{M}'(\mathbf{e}_\kappa(\ell) - \mathbf{e}_r(\ell), \varsigma) \leq 1 - \Upsilon \text{ or} \\ \mathcal{N}'(\mathbf{e}_\kappa(\ell) - \mathbf{e}_r(\ell), \varsigma) \geq \Upsilon \text{ for all } \ell \in \mathcal{L}_1\}) = 0 \end{aligned}$$

Theorem 3.17. Let $\mathbf{e}_\kappa : (\mathcal{L}_1, \mathcal{M}, \mathcal{N}, \otimes, \oplus) \longrightarrow (\mathcal{L}_2, \mathcal{M}', \mathcal{N}', \otimes, \oplus)$ is a functions sequence. If $(\mathbf{e}_\kappa)$ is $\mathcal{DS}_\mu^\nu(\mathcal{M}, \mathcal{N}) - \mathbf{e}_\kappa \Rightarrow \mathbf{e}$ w.r.t. IF norm- $(\mathcal{M}, \mathcal{N})$, then $(\mathbf{e}_\kappa)$ is uniform deferred statistical Cauchy sequence w.r.t. IF norm- $(\mathcal{M}, \mathcal{N})$.

Proof. The approach is analogous to Theorem 3.7, thus proof is omitted. $\square$

We use the following definition 3.18 of intuitionistic fuzzy continuity to prove Theorem 3.21.

Definition 3.18. [8] For a given $\tau > 0$ and $\Upsilon \in (0, 1)$, there exist $\varsigma = \varsigma(\Upsilon, \tau) > 0$, $\rho = \rho(\Upsilon, \tau) \in (0, 1)$ such that for all $\ell \in \mathcal{L}_1$, then the mapping $\mathbf{e} : (\mathcal{L}_1, \mathcal{M}, \mathcal{N}, \otimes, \oplus) \rightarrow (\mathcal{L}_2, \mathcal{M}', \mathcal{N}', \otimes, \oplus)$ is called intuitionistic fuzzy continuous at $\ell_0 \in \mathcal{L}_1$ if

$$\mathcal{M}(\ell - \ell_0, \varsigma) > 1 - \rho \text{ and } \mathcal{N}(\ell - \ell_0, \varsigma) < \rho.$$

implies

$$\mathcal{M}'(\mathbf{e}(\ell) - \mathbf{e}(\ell_0), \tau) > 1 - \Upsilon \text{ and } \mathcal{N}'(\mathbf{e}(\ell) - \mathbf{e}(\ell_0), \tau) < \Upsilon.$$

Definition 3.19. Consider $(\mathcal{L}_1, \mathcal{M}, \mathcal{N}, \otimes, \oplus)$ and $(\mathcal{L}_2, \mathcal{M}', \mathcal{N}', \otimes, \oplus)$ are two IFNLS. The family of functions $\mathcal{F}$ from $\mathcal{L}_1$ to $\mathcal{L}_2$ is called intuitionistic fuzzy equicontinuous at a point $\ell_0 \in \mathcal{L}_1$ if for every $\Upsilon \in (0, 1)$ and $\varsigma > 0$, there exists $\rho = \rho(\ell_0, \Upsilon, \varsigma)$ such that

$$\mathcal{M}'(\mathfrak{e}(\ell_0) - \mathfrak{e}(\ell), \varsigma) > 1 - \Upsilon \text{ and } \mathcal{N}'(\mathfrak{e}(\ell_0) - \mathfrak{e}(\ell), \varsigma) < \Upsilon$$

for all $\mathfrak{e} \in \mathcal{F}$ and all ℓ such that

$$\mathcal{M}(\ell_0 - \ell, \varsigma) > 1 - \rho \text{ and } \mathcal{N}(\ell_0 - \ell, \varsigma) < \rho.$$

If the family is equicontinuous at every point of $\mathcal{L}_1$, then it is intuitionistic fuzzy equicontinuous.

Remark 3.20. In the context of continuity, ρ can depend on three factors: Υ, ℓ_0 and $\mathfrak{e}$. But in case of equicontinuity, ρ must be independent of $\mathfrak{e}$.

Theorem 3.21. Let $\mathfrak{e}_\kappa : (\mathcal{L}_1, \mathcal{M}, \mathcal{N}, \otimes, \oplus) \rightarrow (\mathcal{L}_2, \mathcal{M}', \mathcal{N}', \otimes, \oplus)$ be a intuitionistic fuzzy equicontinuous functions sequence on $\mathcal{L}_1$ for each $\kappa \in \mathbb{N}$ and $(\mathfrak{e}_\kappa)$ is also $\mathcal{DS}_\mu^\nu(\mathcal{M}, \mathcal{N}) - \mathfrak{e}_\kappa \rightrightarrows \mathfrak{e}$. Then the function $\mathfrak{e} : \mathcal{L}_1 \rightarrow \mathcal{L}_2$ is intuitionistic fuzzy continuous on $\mathcal{L}_1$.

Proof. Assume that an arbitrary point $\ell_0 \in \mathcal{L}_1$. For any $\Upsilon \in (0, 1)$ and $\varsigma > 0$, there exists a $\rho = \rho(\ell_0, \Upsilon, \frac{\varsigma}{3}) > 0$ such that, by using the intuitionistic fuzzy equicontinuity of $\mathfrak{e}_\kappa$ we have

$$\mathcal{M}'(\mathfrak{e}_\kappa(\ell_0) - \mathfrak{e}_\kappa(\ell), \frac{\varsigma}{3}) > 1 - \Upsilon \text{ and } \mathcal{N}'(\mathfrak{e}_\kappa(\ell_0) - \mathfrak{e}_\kappa(\ell), \frac{\varsigma}{3}) < \Upsilon$$

for all ℓ such that $\mathcal{M}(\ell_0 - \ell, \varsigma) > 1 - \rho$ and $\mathcal{N}(\ell_0 - \ell, \varsigma) < \rho$. Let $\mathcal{B}(\ell_0, \rho, \varsigma)$ denote the open ball in $(\mathcal{L}_1, \mathcal{M}, \mathcal{N}, \otimes, \oplus)$ with radius ρ and center ℓ_0 . Consider a fixed point $\ell \in \mathcal{B}(\ell_0, \rho, \varsigma)$. Define the sets

$$\begin{aligned} \mathcal{P}(\Upsilon, \varsigma) &= \left\{ \mu_n < \kappa \leq \nu_n : \mathcal{M}'(\mathfrak{e}_\kappa(\ell) - \mathfrak{e}(\ell), \frac{\varsigma}{3}) \leq 1 - \Upsilon \text{ or } \right. \\ &\quad \left. \mathcal{N}'(\mathfrak{e}_\kappa(\ell) - \mathfrak{e}(\ell), \frac{\varsigma}{3}) \geq \Upsilon \text{ for all } \ell \in \mathcal{L}_1 \right\}, \\ \mathcal{Q}(\Upsilon, \varsigma) &= \left\{ \mu_n < \kappa \leq \nu_n : \mathcal{M}'(\mathfrak{e}_\kappa(\ell_0) - \mathfrak{e}(\ell_0), \frac{\varsigma}{3}) \leq 1 - \Upsilon \text{ or } \right. \\ &\quad \left. \mathcal{N}'(\mathfrak{e}_\kappa(\ell_0) - \mathfrak{e}(\ell_0), \frac{\varsigma}{3}) \geq \Upsilon \text{ for all } \ell \in \mathcal{L}_1 \right\}. \end{aligned}$$

Since $\mathcal{DS}_\mu^\nu(\mathcal{M}, \mathcal{N}) - \mathfrak{e}_\kappa \rightrightarrows \mathfrak{e}$, then $\delta_\mu^\nu(\mathcal{P}(\Upsilon, \varsigma)) = 0$ and $\delta_\mu^\nu(\mathcal{Q}(\Upsilon, \varsigma)) = 0$. Hence $\delta_\mu^\nu(\mathcal{P}(\Upsilon, \varsigma) \cup \mathcal{Q}(\Upsilon, \varsigma)) = 0$ and $\mathcal{P}(\Upsilon, \varsigma) \cup \mathcal{Q}(\Upsilon, \varsigma)$ is different from $\mathbb{N}$. Thus there exist $j \in \mathbb{N}$ such that

$$\begin{aligned} \mathcal{M}'(\mathfrak{e}_j(\ell) - \mathfrak{e}(\ell), \frac{\varsigma}{3}) &\leq 1 - \Upsilon \text{ or } \mathcal{N}'(\mathfrak{e}_j(\ell) - \mathfrak{e}(\ell), \frac{\varsigma}{3}) \geq \Upsilon, \\ \mathcal{M}'(\mathfrak{e}_j(\ell_0) - \mathfrak{e}(\ell_0), \frac{\varsigma}{3}) &\leq 1 - \Upsilon \text{ or } \mathcal{N}'(\mathfrak{e}_j(\ell_0) - \mathfrak{e}(\ell_0), \frac{\varsigma}{3}) \geq \Upsilon \end{aligned}$$

Now, we shall demonstrate that at ℓ_0 , limit function $\mathfrak{e}$ is an intuitionistic fuzzy continuous.

$$\begin{aligned} &\mathcal{M}'(\mathfrak{e}(\ell) - \mathfrak{e}(\ell_0), \varsigma) \\ &= \mathcal{M}'(\mathfrak{e}(\ell) - \mathfrak{e}_j(\ell) + \mathfrak{e}_j(\ell) - \mathfrak{e}_j(\ell_0) + \mathfrak{e}_j(\ell_0) - \mathfrak{e}(\ell_0), \varsigma) \\ &\geq \mathcal{M}'(\mathfrak{e}(\ell) - \mathfrak{e}_j(\ell), \frac{\varsigma}{3}) \otimes \mathcal{M}'(\mathfrak{e}_j(\ell) - \mathfrak{e}_j(\ell_0), \frac{\varsigma}{3}) \otimes \mathcal{M}'(\mathfrak{e}_j(\ell_0) - \mathfrak{e}(\ell_0), \frac{\varsigma}{3}) \\ &> (1 - \Upsilon) \otimes (1 - \Upsilon) \otimes (1 - \Upsilon) \\ &= (1 - \Upsilon) \end{aligned}$$

and

$$\begin{aligned}
 & \mathcal{N}'(\mathfrak{e}(\ell) - \mathfrak{e}(\ell_0), \varsigma) \\
 &= \mathcal{N}'(\mathfrak{e}(\ell) - \mathfrak{e}_j(\ell) + \mathfrak{e}_j(\ell) - \mathfrak{e}_j(\ell_0) + \mathfrak{e}_j(\ell_0) - \mathfrak{e}(\ell_0), \varsigma) \\
 &\leq \mathcal{N}'\left(\mathfrak{e}(\ell) - \mathfrak{e}_j(\ell), \frac{\varsigma}{3}\right) \oplus \mathcal{N}'\left(\mathfrak{e}_j(\ell) - \mathfrak{e}_j(\ell_0), \frac{\varsigma}{3}\right) \oplus \mathcal{N}'\left(\mathfrak{e}_j(\ell_0) - \mathfrak{e}(\ell_0), \frac{\varsigma}{3}\right) \\
 &< \Upsilon \oplus \Upsilon \oplus \Upsilon \\
 &= \Upsilon.
 \end{aligned}$$

Hence, the proof is completed. $\square$

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