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A NOTE ON TWO RATIONAL INVARIANTS FOR A PARTICULAR 2×2 MATRIX

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Abstract We state and prove the invariance, with respect to matrix power, of both the diagonals and anti-diagonals ratio of a special case 2×2 matrix. The proof methodology is new, contrasting with those deployed previously in establishing anti-diagonals matrix invariants.

1 Introduction

1.1 Background

Let

$$\mathbf{N} = \mathbf{N}(A, B, C, D) = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{1.1}$$

be a general 2×2 matrix, assuming $A, B, C, D \neq 0$. A somewhat surprising result holds—namely, that (unless otherwise indeterminate) the ratio of the two anti-diagonal terms in any exponentiated instance of $\mathbf{N}$ is the quantity B/C—which is apparently little known in the literature. In a previous publication [2] the observation was formulated in four different ways. Our short note offers a new proof of this occurrence and a similar one which, by necessity, both apply in the case when $D = A$ and so to a slightly less general matrix than $\mathbf{N}$ above.

1.2 The Result

We begin by decomposing $\mathbf{N}(A, B, C, D)$ as $\mathbf{N}(A, B, C, D) = \mathbf{F}(A, D) + \mathbf{S}(B, C)$, where

$$\mathbf{F}(A, D) = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \qquad \text{and} \qquad \mathbf{S}(B, C) = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}. \tag{1.2}$$

Then $\mathbf{N}^n(A, B, C, D) = [\mathbf{F}(A, D) + \mathbf{S}(B, C)]^n = \sum_{k=0}^{n} \binom{n}{k} \mathbf{F}^{n-k}(A, D)\mathbf{S}^k(B, C)$ only if the matrices $\mathbf{F}(A, D)$ and $\mathbf{S}(B, C)$ commute. It is elementary to see that this occurs when $D = A$, which is the imposition mentioned above. Thus, $\mathbf{F}(A, D)$ simplifies to $\mathbf{F} = \mathbf{F}(A) = A\mathbf{I}_2$ (denoting the 2×2 identity matrix by $\mathbf{I}_2$), with $\mathbf{S}(B, C)$ as in (1.2) and

$$\mathbf{F}(A) + \mathbf{S}(B, C) = \mathbf{M}(A, B, C), \tag{1.3}$$

say, where

$$\mathbf{M}(A, B, C) = \mathbf{N}(A, B, C, A) = \begin{pmatrix} A & B \\ C & A \end{pmatrix}, \tag{1.4}$$

and for which we will prove these invariance results:

Theorem 1.1. *Unless otherwise indeterminate, the ratio of the two anti-diagonal terms in $\mathbf{M}^n$ is the quantity B/C while the ratio of the two diagonal terms therein is unity, both being invariant with respect to integer power $n \geq 1$.*

2 The Proof

Proof. Consider, from (1.3),

$$\begin{aligned}
\mathbf{M}^n(A,B,C) &= [\mathbf{F}(A)+\mathbf{S}(B,C)]^n \\
&= \sum_{k=0}^{n}\binom{n}{k}\mathbf{F}^{n-k}(A)\mathbf{S}^k(B,C) \\
&= \sum_{k(\text{even})\geq 0}\binom{n}{k}\mathbf{F}^{n-k}(A)\mathbf{S}^k(B,C)+\sum_{k(\text{odd})\geq 1}\binom{n}{k}\mathbf{F}^{n-k}(A)\mathbf{S}^k(B,C) \\
&= \sum_{m\geq 0}\binom{n}{2m}\mathbf{F}^{n-2m}(A)\mathbf{S}^{2m}(B,C) \\
&\qquad +\sum_{m\geq 0}\binom{n}{2m+1}\mathbf{F}^{n-(2m+1)}(A)\mathbf{S}^{2m+1}(B,C); \qquad \text{(P.1)}
\end{aligned}$$

the index m takes values according to the parity of n, upper limits for which being a detail not required in the proof. Noting that $\mathbf{F}^p(A)=A^p\mathbf{I}_2$ (integer power $p\geq 0$) and, for $m=0,1,2,\ldots$, $\mathbf{S}^{2m}(B,C)=(BC)^m\mathbf{I}_2$ with $\mathbf{S}^{2m+1}(B,C)=(BC)^m\mathbf{S}(B,C)$, we continue as

$$\begin{aligned}
\mathbf{M}^n(A,B,C) &= \sum_{m\geq 0}\binom{n}{2m}A^{n-2m}\mathbf{I}_2\cdot(BC)^m\mathbf{I}_2 \\
&\qquad +\sum_{m\geq 0}\binom{n}{2m+1}A^{n-(2m+1)}\mathbf{I}_2\cdot(BC)^m\mathbf{S}(B,C) \\
&= \sum_{m\geq 0}\binom{n}{2m}A^{n-2m}(BC)^m\mathbf{I}_2 \\
&\qquad +\sum_{m\geq 0}\binom{n}{2m+1}A^{n-(2m+1)}(BC)^m\mathbf{S}(B,C) \\
&= \mathbf{\Omega}_e(A,B,C;n)+\mathbf{\Omega}_o(A,B,C;n), \qquad \text{(P.2)}
\end{aligned}$$

where, writing $\binom{n}{2m}A^{n-2m}(BC)^m=g_m(n)=g_m(A,B,C;n)$ and $\binom{n}{2m+1}A^{n-(2m+1)}(BC)^m=f_m(n)=f_m(A,B,C;n)$,

$$\mathbf{\Omega}_o(A,B,C;n)=\sum_{m\geq 0}f_m(n)\mathbf{S}(B,C)=\begin{pmatrix}0 & B\sum_{m\geq 0}f_m(n)\\ C\sum_{m\geq 0}f_m(n) & 0\end{pmatrix} \qquad \text{(P.3)}$$

is anti-diagonal, with

$$\mathbf{\Omega}_e(A,B,C;n)=\sum_{m\geq 0}g_m(n)\mathbf{I}_2=\begin{pmatrix}\sum_{m\geq 0}g_m(n) & 0\\ 0 & \sum_{m\geq 0}g_m(n)\end{pmatrix} \qquad \text{(P.4)}$$

diagonal. It is now immediate from (P.2)-(P.4) that the anti-diagonals ratio of $\mathbf{M}^n$ is precisely that of $\mathbf{\Omega}_o$ and is B/C, and that the unity diagonals ratio of $\mathbf{\Omega}_e$ is that of $\mathbf{M}^n$ also, both independent of matrix power n. □

Combinations of A,B,C that induce ratio indeterminacy are not discussed here, although some are immediate by inspection of the first few powers of $\mathbf{M}$ which are provided in Appendix A and also illustrate Theorem 1.1.

Remark 2.1. For the purpose of completeness we note that the authors' attention has been drawn to a 2005 paper by Cisneros-Molina where [1, Lemma 3.2, p. 148] the entries of a fully general exponentiated 2×2 matrix are listed in closed form and from which Theorem 1.1 is an immediate corollary (see Appendix B)—this, though, does not diminish our proof here in any way.

3 Summary

This note presents proofs of the diagonals and anti-diagonals ratio invariance for a particular 2×2 matrix, complementing previous work on the fully general case in which the anti-diagonals ratio invariance was stated [2, Theorem 1.1, p. 360] and proven, and the diagonals ratio invariance was merely noted (p. 362 therein). The line of argument used here does not, however, seem applicable to an examination of the invariance of those anti-diagonals ratios within an arbitrary dimension tri-diagonal matrix addressed in [3], where background information on the topic of matrix anti-diagonals ratio invariance may be found along with references to closed form entries for exponentiated matrices (see also [2]).

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Appendix A

The initial powers of the matrix $\mathbf{M}$ (1.4) are calculated to be the following:

$$\begin{aligned}
\mathbf{M}^2 &= \begin{pmatrix} A^2+BC & 2AB \\ 2AC & A^2+BC \end{pmatrix}, \\
\mathbf{M}^3 &= \begin{pmatrix} A(A^2+3BC) & B(3A^2+BC) \\ C(3A^2+BC) & A(A^2+3BC) \end{pmatrix}, \\
\mathbf{M}^4 &= \begin{pmatrix} A^4+6A^2BC+B^2C^2 & 4AB(A^2+BC) \\ 4AC(A^2+BC) & A^4+6A^2BC+B^2C^2 \end{pmatrix}, \\
\mathbf{M}^5 &= \begin{pmatrix} A(A^4+10A^2BC+5B^2C^2) & B(5A^4+10A^2BC+B^2C^2) \\ C(5A^4+10A^2BC+B^2C^2) & A(A^4+10A^2BC+5B^2C^2) \end{pmatrix}, \\
\mathbf{M}^6 &= \begin{pmatrix} (A^2+BC)(A^4+14A^2BC+B^2C^2) & 2AB(A^2+3BC)(3A^2+BC) \\ 2AC(A^2+3BC)(3A^2+BC) & (A^2+BC)(A^4+14A^2BC+B^2C^2) \end{pmatrix}, \\
&\vdots
\end{aligned} \tag{A.1}$$

With reference to the proof of Section 2, we here check the entries above in a couple of selected instances. We have, for example, that $\sum_{m\geq 0} g_m(5) = \sum_{m\geq 0}\binom{5}{2m}A^{5-2m}(BC)^m = \binom{5}{0}A^5(BC)^0+\binom{5}{2}A^3(BC)^1+\binom{5}{4}A^1(BC)^2 = 1\cdot A^5\cdot 1+10\cdot A^3\cdot(BC)+5\cdot A\cdot(BC)^2 = A(A^4+10A^2BC+5B^2C^2)$, seen as the expected diagonal entries of $\mathbf{M}^5$. The common element to the anti-diagonal entries of $\mathbf{M}^6$ is $\sum_{m\geq 0} f_m(6) = \sum_{m\geq 0}\binom{6}{2m+1}A^{6-(2m+1)}(BC)^m = \binom{6}{1}A^5(BC)^0+\binom{6}{3}A^3(BC)^1+\binom{6}{5}A^1(BC)^2 = 6A^5+20A^3BC+6AB^2C^2 = 2A(3A^4+10A^2BC+3B^2C^2) = 2A(A^2+3BC)(3A^2+BC)$; the interested reader is invited to verify further ones similarly.

Appendix B

Here we verify Theorem 1.1 solely by appeal to the results of [1] (note that these formulas are not derived as such in the article, but we have checked them for matrix powers up to and beyond 500 using computer software).

Writing the exponentiated general matrix $\mathbf{N}$ as

$$\mathbf{N}^n(A,B,C,D) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}^n = \begin{pmatrix} \alpha_n & \beta_n \\ \gamma_n & \delta_n \end{pmatrix}, \tag{B.1}$$

closed forms for $\alpha_n = \alpha_n(A,B,C,D), \ldots, \delta_n = \delta_n(A,B,C,D)$ are listed by Cisneros-Molina as

$$\begin{aligned}
\alpha_n &= A^n + \sum_{s=1}^{\lfloor n/2 \rfloor} \sum_{m=0}^{n-2s} \binom{n-s-m}{s} \binom{m+s-1}{m} A^{n-2s-m} B^s C^s D^m, \\
\beta_n &= \sum_{s=0}^{\lfloor (n-1)/2 \rfloor} \sum_{m=0}^{n-2s-1} \binom{n-s-m-1}{s} \binom{m+s}{m} A^{n-2s-m-1} B^{s+1} C^s D^m, \\
\gamma_n &= \sum_{s=0}^{\lfloor (n-1)/2 \rfloor} \sum_{m=0}^{n-2s-1} \binom{n-s-m-1}{s} \binom{m+s}{m} A^{n-2s-m-1} B^{s} C^{s+1} D^m, \\
\delta_n &= D^n + \sum_{s=1}^{\lfloor n/2 \rfloor} \sum_{m=0}^{n-2s} \binom{n-s-m-1}{s-1} \binom{m+s}{m} A^{n-2s-m} B^s C^s D^m. \qquad \text{(B.2)}
\end{aligned}$$

By inspection, the anti-diagonals ratio $\beta_n/\gamma_n = B/C$ (regardless of the constraint $D = A$, and equally so when it applies). In the case when $D = A$ then the diagonals ratio α_n/δ_n is unity since now $\alpha_n = \delta_n$ because the inner sums of α_n, δ_n in (B.2), namely

$$S_1(n;s) = \sum_{m=0}^{n-2s} \binom{n-s-m-1}{s-1} \binom{m+s}{m} \qquad \text{(B.3)}$$

and

$$S_2(n;s) = \sum_{m=0}^{n-2s} \binom{n-s-m}{s} \binom{m+s-1}{m}, \qquad \text{(B.4)}$$

are identical; this is readily seen by a reversal in the order of summation in $S_1(n;s)$ according to $S_1(n;s) = \sum_{m=0}^{n-2s} \binom{n-s-m-1}{s-1}\binom{m+s}{m} = \sum_{m'=0}^{n-2s} \binom{n-s-(n-2s-m')-1}{s-1}\binom{(n-2s-m')+s}{n-2s-m'}$, which reduces to $\sum_{m'=0}^{n-2s} \binom{m'+s-1}{s-1}\binom{n-s-m'}{n-2s-m'} = \sum_{m'=0}^{n-2s} \binom{m'+s-1}{m'}\binom{n-s-m'}{s} = S_2(n;s)$.

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