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BI-AMALGAMATION OF RINGS DEFINED BY BÉZOUT-LIKE CONDITIONS

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Abstract. Let $f : A \longrightarrow B$ and $g : A \longrightarrow C$ be two ring homomorphisms and let J (resp.,J') be an ideal of B (resp.,C) such that $f^{-1}(J) = g^{-1}(J')$. In this paper, we investigate the transfer of notions of elementary divisor ring, almost Bézout domain (AB-domain) and almost valuation domain (AV-domain) to the bi-amalgamation of A with (B, C) along (J, J') with respect to (f, g) (denoted by $A \bowtie^{f,g} (J, J')$, introduced and studied by Kabbaj, Louartiti and Tamekkante in 2013.

1 Introduction

Throughout this paper, all rings are commutative with identity elements, and all modules are unitary.

A ring R is called an elementary divisor ring (respectively, Hermite ring) if for every matrix M over R there exist nonsingular matrices P, Q such that PMQ (respectively, MQ) is a diagonal matrix (respectively, triangular matrix). It proved in [15] that a ring R is an Hermite ring if and only if for all $a, b \in R$, there exist $a_1, b_1, d \in R$ such that $a = a_1d$, $b = b_1d$, and $Ra_1 + Rb_1 = R$. A ring is called Bézout ring if every finitely generated ideal is principal. It is clear that every elementary divisor ring is an Hermite ring, and that every Hermite ring is a Bézout ring.

Following [20] a ring R is said to be a valuation ring if for any two elements in R, one divides the other. Kaplansky proved that any valuation ring is an elementary divisor ring.

In [2], D. D. Anderson and M. Zaffrullah introduced the notion of almost valuation domain (AV-domain for short) as a ring R such that for any two elements a and b in R, there exists a positive integer n such that a^n divides b^n or b^n divides a^n. Also they introduced the notion of almost Bézout domain (AB-domain) as a ring R such that for any two elements a and b in R, there exists a positive integer n such that the ideal (a^n, b^n) is principal. Among others, they proved that the integral closure of an almost valuation (resp., almost Bézout) domain is a valuation domain (resp., a Prüfer domain with torsion class group). Moreover, the notion of almost Bézout domains runs along lines somewhat similar to those of Bézout domain (i.e, every two generated, equivalently, every finitely generated, ideal is principal). In [1], D. D. Anderson, K. R. Knopp, and R. L. Lewin continued the study of almost Bézout domains, and after observing that each Bézout domain is nearly Bézout, they used the construction $K + XL[X]$ to disprove the converse. the same example shows that a Noetherian almost Bézout domain need not be an almost principal ideal domain (API-domain), even though each Noetherian Bézout domain is a principal ideal domain (PID). In [3], Anderson and Zaffrullah continued their study of almost Bézout domains and gave a new characterisation of Cohen-Kaplansky domains. They also showed that a finite intersection of almost valuation domains with the same quotient field is an almost Bézout domain. This result generalizes the classical case that a finite intersection of valuation domains with the same quotient field is a Bézout domain. In [5], A. Badawi introduced a new class of integral domains closely related to AVD's, that is the class of pseudo-almost valuation domains (PAVD's). He showed that the class of almost valuation domains is properly contained in the class pseudo-almost valuation domains, and that PAVD's are precisely the pullbacks of AVD's. In [25], A. Mimouni studied the transfer of the notions of almost valuation, almost Prüfer and almost Bézout domains to pullbacks.

Let $f : A \longrightarrow B$ and $g : A \longrightarrow C$ be two ring homomorphisms and let J (resp.,J') be an ideal of B (resp.,C) such that $f^{-1}(J) = g^{-1}(J')$. In this setting, we can consider the following subring of $B \times C$:

$$A \bowtie^{f,g} (J, J') := \{(f(a) + j, g(a) + j')|a \in A, j \in J, j' \in J'\}$$

called the bi-amalgamation of A with (B, C) along (J, J') with respect to (f, g) (introduced and studied by Kabbaj, Louartiti and Tamekkante in 2013 in [21]).

This construction is a generalisation of the amalgamated algebra along an ideal (introduced and studied by D'Anna and Fontana in [13, 14].) Moreover, other classical constructions (such as the $A + XB[X]$, $A + XB[[X]]$, and the $D + M$ constructions)can be studied as particular cases of the amalgamation [13, Exemples 2.5 and 2.6] and other classical constructions , such as the Nagata's idealization ([26, page 2]), and the CPI extensions are strictly related to it ([13, Exemple 2.7 and Remark 2.8]). In [21], the authors studied the basic properties of this construction (e.g., characterized for $A \bowtie^f J$ to be a Noetherian ring, an integral domain, a reduced ring) and they characterized those distinguished pullbacks that can be expressed as a bi-amalgamation. Moreover, they pursued the investigation on the structure of the rings of the form $A \bowtie^{f,g} (J, J')$, with particular attention to the prime spectrum.

This paper aims at studying the transfer of the notions of elementary divisor ring, almost Bézout domain (AB-domain) and almost valuation domain (AV-domain) to the bi-amalgamation of algebra. It contains in addition to the introduction three sections, the first one deals with the transfer of the notions of elementary divisor ring, and investigate the relationship between this notion and Hermite ring, and Bézout ring in the context of bi-amalgamation of algebra. The second and third sections investigates the transfer of the notions of almost valuation domain (AV-domain), and the almost Bézout domain (AB-domain) to the pre-mentioned construction.

2 ON ELEMENTARY DIVISOR PROPERTY

The main result of this section examines necessary and sufficient conditions for bi-amalgamation $A \bowtie^{f,g} (J, J')$ to inherit the notions of elementary divisor ring, and establish the relationship between this notion and Hermite ring, and Bézout ring.

The set of all $n \times n$ matrices with entries from a ring R will be denoted by $M_n(R)$. We will let $GL_n(R)$ denote the units in $M_n(R)$. Let B and C be rings, for every matrix $M = ((b_{i,j}, c_{i,j}))_{1\leq i,j\leq n} \in M_n(B\times C)$, we shall use the notation $M_b = (b_{i,j})_{1\leq i,j\leq n}$, $M_c = (c_{i,j})_{1\leq i,j\leq n}$ and $M = M_b \times M_c$. Let $M, N \in M_n(B \times C)$, it is easy to see that the product MN of M and N is given by $MN = (M_bN_b) \times (M_cN_c)$.

Theorem 2.1. *Let A, B and C be integral domains, $f : A \longrightarrow B$ and $g : A \longrightarrow C$ be two ring homomorphisms and let J (resp.,J') be an ideal of B (resp.,C) such that $f^{-1}(J) = g^{-1}(J')$. Then, $A \bowtie^{f,g} (J, J')$ is an elementary divisor ring if and only if the following statements hold:*

(i) $f(A) + J$ *and* $g(A) + J'$ *are elementary divisor rings.*

(ii) $J = 0$ *or* $J' = 0$

The proof of this theorem requires the following lemmas.

Lemma 2.2. *Let A, B and C be integral domains, $f : A \longrightarrow B$ and $g : A \longrightarrow C$ be two ring homomorphisms and let J (resp.,J') be a proper ideal of B (resp.,C) such that $f^{-1}(J) = g^{-1}(J')$. If $A \bowtie^{f,g} (J, J')$ is a Bézout ring then $J = 0$ or $J' = 0$.*

Proof. Assume that $A \bowtie^{f,g} (J, J')$ is a Bézout ring. We claim that $J = 0$ or $J' = 0$. Deny. There are some $0 \neq j \in J$ and $0 \neq j' \in J'$. It is clear that $(j, 0)$ and $(0, j') \in A \bowtie^{f,g} (J, J')$. Since $A \bowtie^{f,g} (J, J')$ is a Bézout ring the ideal generated by $(j, 0)$ and $(0, j')$ is principal. Hence, there exists $(f(d) + t, g(d) + t') \in A \bowtie^{f,g} (J, J')$ such that

$$(j, 0)A \bowtie^{f,g} (J, J') + (0, j')A \bowtie^{f,g} (J, J') = (f(d) + t, g(d) + t')A \bowtie^{f,g} (J, J').$$

So, there exist $(f(b)+l, g(b)+l')$, $(f(c)+k, g(c)+k')$, $(f(\alpha)+r, g(\alpha)+r')$, $(f(\beta)+h, g(\beta)+h')$ in $A \bowtie^{f,g} (J, J')$ such that

$$\begin{array}{rcl} (j,0) & = & (f(d)+t, g(d)+t')(f(b)+l, g(b)+l') \\ (0,j') & = & (f(d)+t, g(d)+t')(f(c)+k, g(c)+k') \\ (f(d)+t, g(d)+t') & = & (j,0)(f(\alpha)+r, g(\alpha)+r') + (0,j')(f(\beta)+h, g(\beta)+h') \end{array}$$

It follows that $g(d)+t' \neq 0$ since $j' = (g(d)+t')(g(c)+k') \neq 0$. Also $g(b)+l' = 0$ since $(g(b)+l')(g(d)+t') = 0$ and C is an integral domain. Remark that $g(b)+l' = 0$ imply that $f(b)+l \in J$.
From the previous equalities we deduce that

$$f(d)+t = j(f(\alpha)+r) = (f(d)+t)(f(b)+l)(f(\alpha)+r).$$

Hence $1 = (f(b)+l)(f(\alpha)+r)$ since B is an integral domain. Therefore $1 \in J$ since $f(b)+l \in J$ which is a contradiction.

□

Lemma 2.3. *The following assertions holds:*

(i) Let A and B be two rings. Then $A \times B$ is an elementary divisor ring if and only if so are both A and B.

(ii) Let $f : A \longrightarrow B$ and $g : A \longrightarrow C$ be two ring homomorphisms and let J (resp.,J') be an ideal of B (resp.,C) such that $f^{-1}(J) = g^{-1}(J')$. Then: If $A \bowtie^{f,g} (J, J')$ is an elementary divisor ring then so are both $f(A)+J$ and $g(A)+J'$.

Proof. (i) See for instance [23, Lemma 2.2].

(ii) Let $U = (f(a_{i,j}) + t_{i,j})_{1 \leq i,j \leq n} \in M_n(f(A)+J)$ and let M be the matrix defined by $M = ((f(a_{i,j}) + t_{i,j}, g(a_{i,j}))_{1 \leq i,j \leq n}$ with entries from $A \bowtie^{f,g} (J, J')$. We have the equality $U = M_b$. Since $A \bowtie^{f,g} (J, J')$ is an elementary divisor ring M is equivalent to a diagonal matrix. From the previous part of the proof we deduce that there exist P and Q in $GL_n(f(A)+J)$ such that PUQ is a diagonal matrix. Therefore $f(A)+J$ is an elementary divisor ring. With a similar argument as in above, we get that $g(A)+J'$ is an elementary divisor ring.

□

Proof. Assume that $A \bowtie^{f,g} (J, J')$ is an elementary divisor ring.
(1) By lemma 2.2, $f(A)+J$ and $g(A)+J'$ are elementary divisor rings.
(2) By lemma 2.1, since every elementary divisor ring is a Bézout ring.
Conversely, assume that (1) and (2) hold. If $J = 0$ or $J' = 0$, then by [21, Proposition 4.1 (b)], $A \bowtie^{f,g} (J, J') \simeq f(A)+J$ or $A \bowtie^{f,g} (J, J') \simeq g(A)+J'$ which are elementary divisor rings. Hence, $A \bowtie^{f,g} (J, J')$ is an elementary divisor ring, as desired.

□

Recall that the amalgamation of A with B along J with respect to f is given by

$$A \bowtie^f J := \{(a, f(a)+j) \mid a \in A, j \in J\}$$

Clearly, every amalgamation can be viewed as a special bi-amalgamation, since $A \bowtie^f J = A \bowtie^{Id,f} (f^{-1}(J), J)$. Accordingly, Theorem 2.3 covers the special case of amalgamation [23], as recorded below.

Corollary 2.4. *Let A and B a pair of integral domains, let $f : A \longrightarrow B$ be a ring homomorphism and let J be an ideal of B.*

(i) Assume that f is injective.

- *If $J = B$ then $A \bowtie^f J$ is an elementary divisor ring if and only if so are both A and B.*
- *If $J \neq B$ then $A \bowtie^f J$ is an elementary divisor ring if and only if so are both $f(A)+J$ and $f(A) \cap J = 0$.*

(ii) Assume that f is not injective. Then $A \bowtie^f J$ is an elementary divisor ring if and only one of the following conditions holds:

- *$J = 0$ and A is an elementary divisor ring.*
- *$J = B$ and (A, B) is a pair of elementary divisor rings.*

Now, we establish the relationship between elementary divisor ring, Hermite ring and Bézout ring in the context of bi-amalgamation of algebra.

Theorem 2.5. *Let A, B and C be integral domains, $f : A \longrightarrow B$ and $g : A \longrightarrow C$ be two ring homomorphisms and let J (resp.,J') be an ideal of B (resp.,C) such that $f^{-1}(J) = g^{-1}(J')$. The following properties are equivalent:*

(i) $A \bowtie^{f,g} (J, J')$ is an Hermite ring.

(ii) $A \bowtie^{f,g} (J, J')$ is a Bézout ring.

(iii) One of the following conditions hold:

- *$f(A) + J$ is a Bézout ring, and $J' = 0$.*
- *$g(A) + J'$ is a Bézout ring, and $J = 0$.*

Proof. $(1) \Rightarrow (2)$ It is clear.
$(2) \Rightarrow (3)$ By lemma 2.1 $J = 0$ or $J' = 0$ and so $A \bowtie^{f,g} (J, J') \simeq f(A)+J$ or $A \bowtie^{f,g} (J, J') \simeq g(A) + J'$ which are a Bézout ring.
$(3) \Rightarrow (1)$ In this case, we get that $A \bowtie^{f,g} (J, J') \simeq f(A) + J$ or $A \bowtie^{f,g} (J, J') \simeq g(A) + J'$ which are an Hermite ring, since every Bézout domain is an Hermite ring.

□

3 ON AV-RING PROPERTY

This section characterizes the bi-amalgamated algebra along ideals $A \bowtie^{f,g} (J, J')$ to be an almost valuation ring. The main result (Theorem 3.1) examines the property of almost valuation ring, that the bi-amalgamation $A \bowtie^{f,g} (J, J')$ might inherit from, $f(A) + J$, $g(A) + J'$ for some classes of ideals J, J' and homomorphisms f, g and hence generates new examples of almost valuation rings.

Theorem 3.1. *Let $f : A \longrightarrow B$ and $g : A \longrightarrow C$ be two ring homomorphisms and let J (resp.,J') be an ideal of B having no nontrivial nilpotent elements (resp.,C) such that $f^{-1}(J) = g^{-1}(J')$. Then $A \bowtie^{f,g} (J, J')$ is an AV-ring if and only if the following statements hold:*

(i) $f(A) + J$ and $g(A) + J'$ are AV-rings.

(ii) $J = 0$ or $J' = 0$.

The proof of this theorem involves the following lemmas.

Lemma 3.2. *Let $f : A \longrightarrow B$ and $g : A \longrightarrow C$ be two ring homomorphisms and let J (resp.,J') be an ideal of B (resp.,C) such that $f^{-1}(J) = g^{-1}(J')$. If $A \bowtie^{f,g} (J, J')$ is an AV-ring, then so are $f(A) + J$ and $g(A) + J'$.*

Proof. By [21, Proposition 4.1 (b)] and the fact that if A is an AV-ring and I is an ideal of A, then A/I is an AV-ring. □

Lemma 3.3. *Let $f : A \longrightarrow B$ and $g : A \longrightarrow C$ be two ring homomorphisms and let J (resp.,J') be ideal of B having no nontrivial nilpotent elements (resp.,C) such that $f^{-1}(J) = g^{-1}(J')$. If $A \bowtie^{f,g} (J, J')$ is an AV-ring then $J = 0$ or $J' = 0$.*

Proof. Assume that $A \bowtie^{f,g} (J, J')$ is an AV-ring. We claim that $J = 0$ or $J' = 0$. Deny. There are some $0 \neq j \in J$ and $0 \neq j' \in J'$. It is clear that $(j, 0)$ and $(0, j') \in A \bowtie^{f,g} (J, J')$. Since $A \bowtie^{f,g} (J, J')$ is a an AV-ring there exists a positive integer n such that
$(j, 0)^n \in A \bowtie^{f,g} (J, J')(0, j')^n$ or $(0, j')^n \in A \bowtie^{f,g} (J, J')(j, 0)^n$.
If $(j, 0)^n \in A \bowtie^{f,g} (J, J')(0, j')^n$, then $(j, 0)^n = (j^n, 0) = (f(d)+t, g(d)+t')(0, j'^n)$ for some $(f(d)+t, g(d)+t') \in A \bowtie^{f,g} (J, J')$. Thus $j^n = 0$ and since J is reduced, $j = 0$, which is absurd. Hence $(0, j')^n \in A \bowtie^{f,g} (J, J')(j, 0)^n$, and so $(0, j')^n = (0, j'^n) = (f(d) + t, g(d) + t')(j^n, 0)$ for some $(f(d) + t, g(d) + t') \in A \bowtie^{f,g} (J, J')$. Thus $j'^n = 0$ and since J' is reduced, $j' = 0$, which is again a contradiction. □

Proof. of theorem 3.1

Assume that $A \bowtie^{f,g} (J, J')$ is an AV-ring.
(1) By lemma 3.2, $f(A) + J$ and $g(A) + J'$ are AV-rings.
(2) By lemma 3.3
Conversely, assume that (1) and (2) hold. If $J = 0$ or $J' = 0$, then by [21, Proposition 4.1 (b)], $A \bowtie^{f,g} (J, J') \simeq f(A) + J$ or $A \bowtie^{f,g} (J, J') \simeq g(A) + J'$ which are AV-rings.
Hence, $A \bowtie^{f,g} (J, J')$ is an AV-ring, as desired. □

The Corollary below follows immediately from Theorem 3.1 which examines the case of amalgamation algebra.

Corollary 3.4. *Let A and B be a pair of rings and $f : A \longrightarrow B$ be a ring homomorphism. If J is a non-zero proper ideal of B having no nontrivial nilpotent elements and A is reduced. Then $A \bowtie^f J$ is an AV-ring if and only if f is injective, $f(A) + J$ is an AV-ring and $f(A) \cap J = (0)$.*

Theorem 3.1 enriches the literature with the new original examples of AV-rings.

Example 3.5. Let (A, m) be a local AV-domain, E an A-module such that $mE = 0$, and $B = A \propto E$ the trivial ring extension of A by E, and $C = A/m$. Consider the natural injective ring homomorphisms $f : A \longrightarrow B$ and the canonical surjective ring homomorphisms $g : A \longrightarrow A/m$ and let $J = m \propto \{0\}$. We claim that the bi-amalgamation $R = A \bowtie^{f,g} (J, 0)$ is an AV-ring. Indeed, notice first that $f^{-1}(J) = g^{-1}(0) = m$, $f(A) + J = B$ and $g(A) = A/m = C$. Further, B is an AV-ring by [27, Theorem 2.1 (3)] and C is an AV-ring. So, R is an AV-ring by theorem 3.1.

Example 3.6. Let A be an AV-ring, I, K be two ideals of A such that $I \subseteq K$ and I is a radical ideal, $B = A/I$, $C = A/K$. Consider the canonicals surjective ring homomorphisms $f : A \longrightarrow B$ and $g : A \longrightarrow C$ and let $J = K/I$. We claim that the bi-amalgamation $R = A \bowtie^{f,g} (J, 0)$ is an AV-ring. Indeed, notice first that $f^{-1}(J) = g^{-1}(0) = K$, $f(A) + J = B$, $g(A) = A/K = C$ and $J = K/I$ has no nonzero nilpotent element. Further, B and C are AV-rings. So, R is an AV-ring by theorem 3.1.

4 ON AB-RING PROPERTY

This section deals with the transfer of the notion of AB-ring to the bi-amalgamated algebras along ideals.

Theorem 4.1. *Let (A, B, C) be integral domains, $f : A \longrightarrow B$ and $g : A \longrightarrow C$ be two ring homomorphisms and let J (resp.,J') be a proper ideal of B (resp.,C) such that $f^{-1}(J) = g^{-1}(J')$. Then $A \bowtie^{f,g} (J, J')$ is an AB-ring if and only if the following statements hold:*

(i) $f(A) + J$ *and* $g(A) + J'$ *are AB-rings.*

(ii) $J = 0$ *or* $J' = 0$.

Before proving this theorem, we recall the following lemmas.

Lemma 4.2. *Let (A, B, C) be integral domains, $f : A \longrightarrow B$ and $g : A \longrightarrow C$ be two ring homomorphisms and let J (resp.,J') be an ideal of B (resp.,C) such that $f^{-1}(J) = g^{-1}(J')$. If $A \bowtie^{f,g} (J, J')$ is an AB-ring, then so are $f(A) + J$ and $g(A) + J'$.*

Proof. By [21, Proposition 4.1 (b)] and the fact that if A is an AB-ring and I is an ideal of A, then A/I is an AB-ring. □

Lemma 4.3. *Let (A, B, C) be integral domains, $f : A \longrightarrow B$ and $g : A \longrightarrow C$ be two ring homomorphisms and let J (resp.,J') be a proper ideal of B (resp.,C) such that $f^{-1}(J) = g^{-1}(J')$. If $A \bowtie^{f,g} (J, J')$ is an AB-ring then $J = 0$ or $J' = 0$.*

Proof. Assume that $A \bowtie^{f,g} (J, J')$ is an AB-ring. We claim that $J = 0$ or $J' = 0$. Deny. There are some $0 \neq j \in J$ and $0 \neq j' \in J'$. It is clear that $(j, 0)$ and $(0, j') \in A \bowtie^{f,g} (J, J')$. Since $A \bowtie^{f,g} (J, J')$ is a an AB-ring there exists a positive integer n such that the ideal $((j, 0)^n, (0, j')^n)A \bowtie^{f,g} (J, J') = A \bowtie^{f,g} (J, J')(j^n, 0) + A \bowtie^{f,g} (J, J')(0, j'^n)$ is principal ideal of $A \bowtie^{f,g} (J, J')$.
Set $A \bowtie^{f,g} (J, J')(j^n, 0) + A \bowtie^{f,g} (J, J')(0, j'^n) = (f(d) + t, g(d) + t')A \bowtie^{f,g} (J, J')$ for some $(f(d) + t, g(d) + t') \in A \bowtie^{f,g} (J, J')$. Then there exist $(f(b) + l, g(b) + l')$, $(f(c) + k, g(c) + k')$, $(f(\alpha) + r, g(\alpha) + r')$, $(f(\beta) + h, g(\beta) + h')$ in $A \bowtie^{f,g} (J, J')$ such that

$$(j^n, 0) = (f(d) + t, g(d) + t')(f(b) + l, g(b) + l')$$

$$(0, j'^n) = (f(d) + t, g(d) + t')(f(c) + k, g(c) + k')$$

$$(f(d) + t, g(d) + t') = (j^n, 0)(f(\alpha) + r, g(\alpha) + r') + (0, j'^n)(f(\beta) + h, g(\beta) + h').$$

Hence $g(d) + t' \neq 0$ since $j'^n = (g(d) + t')(g(c) + k') \neq 0$. Also $g(b) + l' = 0$ since $(g(b) + l')(g(d) + t') = 0$ and C is an integral domain. Remark that $g(b) + l' = 0$ imply that $f(b) + l \in J$.
From the previous equalities we deduce that

$$f(d) + t = j^n((f(\alpha) + r) = (f(d) + t)(f(b) + l)(f(\alpha) + r)$$

Hence

$$1 = (f(b) + l)(f(\alpha) + r)$$

Since B is an integral domain. Therefore $1 \in J$ since $f(b) + l \in J$ which is a contradiction.

□

Proof. of theorem 4.1
Assume that $A \bowtie^{f,g} (J, J')$ is an AB-ring.
(1) By lemma 4.2, $f(A) + J$ and $g(A) + J'$ are AB-rings.
(2) By lemma 4.3
Conversely, assume that (1) and (2) hold. If $J = 0$ or $J' = 0$, then by [21, Proposition 4.1 (b)], $A \bowtie^{f,g} (J, J') \simeq f(A) + J$ or $A \bowtie^{f,g} (J, J') \simeq g(A) + J'$ which are AB-rings.
Hence, $A \bowtie^{f,g} (J, J')$ is an AB-ring, as desired.

□

The Corollary below recovers a known result for amalgamation algebra.

Corollary 4.4. *Let A, B be a pair of integral domains, $f : A \longrightarrow B$ be a ring homomorphism and J be a proper ideal of B. Then $A \bowtie^f J$ is an AB-ring if and only if f is injective, $f(A) + J$ is an AB-ring and $f(A) \cap J = (0)$.*

Example 4.5. Let A be an AB-domain, I, K be two prime ideals of A such that $I \subsetneq K \subsetneq A$, $B = A/I$, $C = A/K$. Consider the canonicals surjective ring homomorphisms $f : A \longrightarrow B$ and $g : A \longrightarrow C$ and let $J = K/I$. We claim that the bi-amalgamation $R = A \bowtie^{f,g} (J, 0)$ is an AB-ring. Indeed, notice first that $f^{-1}(J) = g^{-1}(0) = K$, $f(A) + J = B$, $g(A) = A/K = C$ and $J = K/I$. Further, B and C are AB-rings. So, R is an AB-ring by theorem 4.1.

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