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Continued Fraction Expansions of the quasi-arithmetic power means of positive matrices with parameter (p,α)

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Abstract The goal of this paper is to provide an efficient method for computing the quasi-arithmetic power means of two positive matrices with parameter (p, α) by using the continued fractions with matrix arguments. Furthermore, we give some numerical examples which illustrated the theoretical results.

1 Introduction

Over the last two centuries, the theory of continued fractions has been a topic of extensive study. The basic idea of this theory over real numbers is to give an approximation of various real numbers by the rational ones. A continued fraction is an expression obtained through an iterative process of representing a number as the sum of its integer part and the reciprocal of another number, then writing this other number as the sum of its integer part and another reciprocal, and so on. One of the main reasons why continued fractions are so useful in computation is that they often provide representation for transcendental functions that are much more generally valid than the classical representation by, say, the power series. Further; in the convergent case, the continued fractions expansions have the advantage that they converge more rapidly than other numerical algorithms.
Recently, the extension of continued fractions theory from real numbers to the matrix case has seen several developments and interesting applications. Since calculations involving matrix valued functions with matrix arguments are feasible with large computers, it will be an interesting attempt to develop such matrix theory.

In mathematics and statistics, the quasi-arithmetic mean or generalized f-mean is one generalization of the more familiar means such as the arithmetic mean and the geometric mean using a function f. It is also called Kolmogorov mean after Russian scientist Andrey Kolmogorov. The importance of quasi-arithmetic means has been well understood at least since the 1930s, and a number of writers have since contributed to their characterisation and to the study of their properties. Quasi-arithmetic means, in particular, have been applied in several disciplines. Their functional form has been used in the theory of copulas under the name of Archimedean copulas [3] and a rich literature can be found under this name. In the theory of aggregation operators and fuzzy measures, a growing literature related to the use of quasi-arithmetics includes the works of Frank (1979), Hajek (1998), Kolesarova (2001), Klement et al. (1999), Grabish (1995), Calvo and Mesiar [2].
The definition of the quasi-arithmetic power mean with parameter (p,α), defined for A >0 and B >0 is given as follows :

$$f_{p,\alpha}(A,B) = A^{1/2}((1-\alpha)I + \alpha(A^{-1/2}BA^{-1/2})^{p})^{1/p}A^{1/2}. \tag{1.1}$$

In a practical context, the computation of $f_{p,\alpha}(A,B)$ imposes many difficulties by virtue of the appearance of the rational exponents of matrices. One fundamental motivation and goal of this paper is to remove this difficulty and reveal a practical method involving matrix continued fractions, for the computation of $f_{p,\alpha}(A,B)$.
The class of quasi-arithmetic power means contain many kinds of means: the mean $f_{1,\alpha}(A,B)$ is

the α-weighed arithmetic mean. The case $f_{0,\alpha}(A,B)$ is the α-weighed geometric mean (this case is understood that we take limit as p⟶0). The case $f_{-1,\alpha}(A,B)$ is the α-weighed harmonic mean. The mean $f_{p,1/2}(A,B)$ is the power mean or binomial mean of order p. However, the rational exponents of matrices in $f_{p,\alpha}(A,B)$ imposes many difficulties. In this paper we gives a practical method, involving matrix continued fractions, for the computation of quasi-arithmetic power mean with parameter (p,α).

2 DEFINITIONS AND NOTATIONS

Throughout this paper, $\mathcal{M}_m$ will represent an algebra of real (or complex) matrices of sizes $m \times m$. Since the complex case can be stated similarly to the real case, then we limit our attention to the last case.

Let $A \in \mathcal{M}_m$, A is said to be positive semidefinite (resp. positive definite) if A is symmetric and

$$\forall x \in I\!R^m,\ (Ax,x) \geq 0 \quad (\text{ resp. } \forall x \in I\!R^m,\ x \neq 0\ (Ax,x) > 0)$$

where $(.,.)$ denotes the standard scalar product of $I\!R^m$.

We observe that positive semidefiniteness induces a partial ordering on the space of symmetric matrices: if A and B are two symmetric matrices, we write $A \leq B$ if $B - A$ is positive semidefinite. Henceforth, whenever we say that $A \in \mathcal{M}_m$ is positive semidefinite (or positive definite), it will be assumed that A is symmetric. It is easy to see that if $A \leq B$ then $CAC \leq CBC$ for any symmetric matrix C.
For any matrices A,$B \in \mathcal{M}_m$ with B invertible, we write $A/B := B^{-1}A$, in particular, if A=I, the identity matrix , then $I/B = B^{-1}$. It is easy to verify that for any invertible matrix X we have

$$\frac{A}{B} = \frac{XA}{XB}.$$

Now, we introduce some topological notions of continued fractions with matrix arguments.

We provide $\mathcal{M}_m$ with the standard induced norm(Definition 5.6.1 in [5]):

$$\forall A \in \mathcal{M}_m,\ \|A\| = Sup_{x\neq 0}\frac{\|Ax\|}{\|x\|} = Sup_{\|x\|=1}\|Ax\|.$$

Let $\{A_n\}$ be a sequence of matrices in $\mathcal{M}_m$. We say that $\{A_n\}$ converges in $\mathcal{M}_m$ if there exists a matrix $A \in \mathcal{M}_m$ such that $\|A_n - A\|$ tends to 0 when n tends to $+\infty$. In this case we write, $A_n \longrightarrow A$ or $\lim_{n\to+\infty} A_n = A$.

Definition 2.1. ([9, p.116]) Let $\{A_n\}_{n\geq 0}$ and $\{B_n\}_{n\geq 1}$ be two sequences of matrices in $\mathcal{M}_m$. We denote the continued fraction expansion by

$$A_0 + \cfrac{B_1}{A_1 + \cfrac{B_2}{A_2 + ...}} := \left[A_0; \frac{B_1}{A_1}, \frac{B_2}{A_2}, ...\right].$$

Sometimes, we denote this continued fraction by $\left[A_0; \frac{B_n}{A_n}\right]_{n=1}^{+\infty}$ or $K\left(B_n/A_n\right)$, where

$$\left[A_0; \frac{B_i}{A_i}\right]_{i=1}^{n} = \left[A_0; \frac{B_1}{A_1}, ..., \frac{B_n}{A_n}\right] = A_0 + \cfrac{B_1}{A_1 + \cfrac{B_2}{A_2 + ... + \cfrac{B_n}{A_n}}}$$

The fractions $\frac{B_n}{A_n}$ and $\frac{P_n}{Q_n} := \left[A_0; \frac{B_i}{A_i}\right]_{i=1}^{n}$ are called, respectively, the n^{th} partial quotient and

the n^{th} convergent
of the continued fraction $K\left(B_n/A_n\right)$.
The continued fraction $\left[A_0, \dfrac{B_k}{A_k}\right]_{k=1}^{+\infty}$ is said to be convergent in $\mathcal{M}_m$ if the sequence $\{P_n/Q_n\}_n = \{Q_n^{-1}P_n\}_n$
converges in $\mathcal{M}_m$ in the sense that there exists a matrix $F \in \mathcal{M}_m$ such that $\lim_{n\to+\infty}\|F_n - F\| = 0$.
In this case, we denote

$$F = \left[A_0; \frac{B_n}{A_n}\right]_{n=1}^{+\infty}.$$

If $A_k = A$ and $B_k = B$ for all $k \geq 1$, then we abbreviate

$$\left[A_0; \frac{B_k}{A_k}\right]_{k=1}^{+\infty} = \left[A_0; \frac{B}{A}\right]_1^{+\infty}.$$

We note that the evaluation of n^{th} convergent according to Definition 2.1 is not practical because we have to repeatedly invert matrices. The following proposition gives an adequate method to calculate $K\left(B_n/A_n\right)$.

Proposition 2.2. *([8]) For the continued fraction $K\left(B_n/A_n\right)$, define*

$$\left\{\begin{array}{l} P_{-1} = I,\ P_0 = A_0 \\ Q_{-1} = 0,\ Q_0 = I \end{array}\right. \quad \textit{and} \quad \left\{\begin{array}{l} P_n = A_n\ P_{n-1} + B_n P_{n-2} \\ Q_n = A_n\ Q_{n-1} + B_n Q_{n-2} \end{array}\right. \quad n \geq 1. \tag{2.1}$$

Then the matrix P_n/Q_n is the n^{th} convergent of $K\left(B_n/A_n\right)$.

Proof. The proof of the next proposition is elementary and we leave it to the reader. □

Proposition 2.3. *For any two matrices C and D with C invertible, we have*

$$C\left[A_0; \frac{B_k}{A_k}\right]_{k=1}^{n} D = \left[CA_0D; \frac{B_1D}{A_1C^{-1}}, \frac{B_2C^{-1}}{A_2}, \frac{B_k}{A_k}\right]_{k=3}^{n}.$$

Definition 2.4. Let $\{A_n\}, \{B_n\}$ $\{C_n\}$ and $\{D_n\}$ be four sequences of matrices. We say that the continued fractions $K(B_n/A_n)$ and $K(D_n/C_n)$ are equivalent if we have $F_n = G_n$ for all $n \geq 1$, where F_n and G_n are the n^{th} convergents of $K(B_n/A_n)$ and $K(D_n/C_n)$ respectively.

In order to simplify the statements on some partial quotients of continued fractions with matrix arguments, we need the following proposition which is an example of equivalent continued fractions.

Proposition 2.5. *([10]) Let b $\left[A_0; \dfrac{B_k}{A_k}\right]_{k=1}^{+\infty}$ be a given continued fraction. Then*

$$\frac{P_n}{Q_n} := \left[A_0; \frac{B_k}{A_k}\right]_{k=1}^{n} = \left[A_0; \frac{X_kB_kX_{k-2}^{-1}}{X_kA_kX_{k-1}^{-1}}\right]_{k=1}^{n},$$

where $X_{-1} = X_0 = I$ and $X_1, X_2, ..., X_n$ are arbitrary invertible matrices.

Now, we end this section by giving a continued fraction expansion of the matrix A^{α}.

Lemma 2.6. *([9]) Let $A \in \mathcal{M}_m$ be a positive definite matrix and α a positive real number. Then, A^{α} can be written as follows*

$$A^{\alpha} = \left[I; \frac{2\alpha\Phi(A)}{-I - \alpha\Phi(A)}, \frac{(\alpha^2 - k^2)(\Phi(A))^2}{-(2k+1)I}\right]_{k=1}^{+\infty}, \tag{2.2}$$

where, by definition, we put $\Phi(A) = \dfrac{I - A}{I + A}$.

3 MAIN RESULTS

This section is devoted to give a continued fraction expansion of the quasi-arithmetic power mean with parameter (p, α) of two postive matrices A and B given by $f_{p,\alpha}(A, B) = A^{1/2}((1 - \alpha)I + \alpha(A^{-1/2}BA^{-1/2})^p)^{1/p}A^{1/2}$.
We can easily verify that $f_{1,\alpha}(A, B) = (1 - \alpha)A + \alpha B$. Then, we restrict ourselves to the case where $p \neq 1$

Lemma 3.1. *Let $A, B \in \mathcal{M}_m$ be a positive definite matrices, α a real number such that $0 \leq \alpha \leq 1$ and $p \neq 1$ be a strictly positive integer. Then*

$$((1 - \alpha)I + \alpha(A^{-1/2}BA^{-1/2})^p)^{1/p} = \left[I; \frac{B_n}{A_n}\right]_{n=1}^{+\infty},$$

$$\begin{cases} B_1 = 2\dfrac{\alpha}{p}A^{1/2}\dfrac{L}{K}A^{-1/2}, \\ \\ A_1 = -I - \dfrac{\alpha}{p}A^{1/2}\dfrac{L}{K}A^{-1/2}, \end{cases} \tag{3.1}$$

and

$$\begin{cases} B_2 = \alpha^2(\frac{1}{p^2} - 1)A^{1/2}\left(\dfrac{L}{K}\right)^2 A^{-1/2},, \\ \\ B_n = \alpha^2(\frac{1}{p^2} - (n-1)^2)A^{1/2}\left(\dfrac{L}{K}\right)^2 A^{-1/2}, \quad \textit{for all } \ n \geq 3, \\ \\ A_n = -(2n-1)I, \quad \textit{for all } \ n \geq 2. \end{cases} \tag{3.2}$$

where $L = A - B(A^{-1}B)^{p-1}$ and $K = (2 - \alpha)A + \alpha B(A^{-1}B)^{p-1}$.

Proof. According to Lemma 2.6, we have

$$((1-\alpha)I+\alpha(A^{-1/2}BA^{-1/2})^p)^{1/p} = \left[I; \frac{2\Phi(C)}{-pI - \Phi(C)}, \frac{(1/p - p)(\Phi(C))^2}{-3I}, \frac{(1/p^2 - k^2)(\Phi(C))^2}{-(2k+1)I}\right]_{k=2}^{+\infty}, \tag{3.3}$$

where $C = (1 - \alpha)I + \alpha(A^{-1/2}BA^{-1/2})^p)$ and $\Phi(C) = \frac{I-C}{I+C}$.
But

$$\begin{aligned} \Phi(C) = & \quad \frac{\alpha(I-(A^{-1/2}BA^{-1/2})^p)}{(2-\alpha)I+(A^{-1/2}BA^{-1/2})^p} \\ = & \quad \frac{\alpha A^{-1/2}(A-B(A^{-1}B)^{p-1})A^{-1/2}}{A^{-1/2}((2-\alpha)A+B(A^{-1}B)^{p-1})A^{-1/2}} \\ = & \quad \alpha A^{1/2}\tfrac{L}{K}A^{-1/2}, \end{aligned}$$

where $L = A - B(A^{-1}B)^{p-1}$ and $K = (2 - \alpha)A + B(A^{-1}B)^{p-1}$.
This completes the proof. □

Theorem 3.2. *Let $A, B \in \mathcal{M}_m$ be two positive definite matrices, α be a real number such that $0 \leq \alpha \leq 1$ and p be a strictly positive integer. Then a continued fraction expansion of $f_{\alpha,p}(A, B)$ is given by:*

$$f_{p,\alpha}(A, B) = \left[A; \frac{B_n}{A_n}\right]_{n=1}^{+\infty},$$

where we set

$$\begin{cases} B_1 = 2\dfrac{\alpha}{p}\dfrac{L}{K}, \\ \\ A_1 = -A^{-1} - \dfrac{\alpha}{p}\dfrac{L}{K}A^{-1}, \end{cases} \tag{3.4}$$

and

$$\begin{cases} B_2 = \alpha^2(1/p^2-1)\left(\frac{L}{K}\right)^2 A^{-1}, \\ B_n = \alpha^2(1/p^2-(n-1)^2)\left(\frac{L}{K}\right)^2, \quad \textit{for all } \ n\geq 3, \\ A_n = -(2n-1)I, \quad \textit{for every } \ n\geq 2, \end{cases} \tag{3.5}$$

where $L = A - B(A^{-1}B)^{p-1}$ *and* $K = (2-\alpha)A + \alpha B(A^{-1}B)^{p-1}$.

Proof. We have

$$A^{-1/2} f_{p,\alpha}(A,B) A^{-1/2} = ((1-\alpha)I + \alpha(A^{-1/2}BA^{-1/2})^p)^{1/p}$$

Defining the sequences $(\widetilde{P_n})_{n\geq -1}$ and $(\widetilde{Q_n})_{n\geq -1}$ as follows

$$\begin{cases} \widetilde{P}_0 = I,\ \widetilde{P}_{-1} = I,\ \widetilde{P}_1 = -I + \frac{\alpha}{p}\frac{I-(A^{-1/2}BA^{-1/2})^p}{(2-\alpha)I+\alpha(A^{-1/2}BA^{-1/2})^p} \\ \widetilde{P}_n = -(2n-1)\widetilde{P}_{n-1} + \alpha^2(1/p^2-(n-1)^2)\left(\frac{I-(A^{-1/2}BA^{-1/2})^p}{(2-\alpha)I+\alpha(A^{-1/2}BA^{-1/2})^p}\right)^2 \widetilde{P}_{n-2},\ \text{for }\ n\geq 2, \end{cases} \tag{3.6}$$

and

$$\begin{cases} \widetilde{Q}_0 = I,\ \widetilde{Q}_{-1} = 0,\ \widetilde{Q}_1 = -I - \frac{\alpha}{p}\frac{I-(A^{-1/2}BA^{-1/2})^p}{(2-\alpha)I+\alpha(A^{-1/2}BA^{-1/2})^p} \\ \widetilde{Q}_n = -(2n-1)\widetilde{Q}_{n-1} + \alpha^2(1/p^2-(n-1)^2)\left(\frac{I-(A^{-1/2}BA^{-1/2})^p}{(2-\alpha)I+\alpha(A^{-1/2}BA^{-1/2})^p}\right)^2 \widetilde{Q}_{n-2},\ \text{for }\ n\geq 2. \end{cases} \tag{3.7}$$

A slight modification in (3.6) and (3.7) gives

$$\begin{cases} \widetilde{P}_0 = I,\ \widetilde{P}_{-1} = I,\ \widetilde{P}_1 = -I + \frac{\alpha}{p}A^{1/2}\frac{L}{K}A^{-1/2} \\ \widetilde{P}_n = -(2n-1)\widetilde{P}_{n-1} + \alpha^2(1/p^2-(n-1)^2)A^{1/2}\left(\frac{L}{K}\right)^2 A^{-1/2}\widetilde{P}_{n-2},\ \text{for }\ n\geq 2. \end{cases} \tag{3.8}$$

and

$$\begin{cases} \widetilde{Q}_0 = I,\ \widetilde{Q}_{-1} = 0,\ \widetilde{Q}_1 = -I - \frac{\alpha}{p}A^{1/2}\frac{L}{K}A^{-1/2} \\ \widetilde{Q}_n = -(2n-1)\widetilde{Q}_{n-1} + \alpha^2(\frac{1}{p^2}-(n-1)^2)A^{1/2}\left(\frac{L}{K}\right)^2 A^{-1/2}\widetilde{Q}_{n-2},\ \text{for }\ n\geq 2, \end{cases} \tag{3.9}$$

respectively, where $L = A - B(A^{-1}B)^{p-1}$ and $K = (2-\alpha)A + \alpha B(A^{-1}B)^{p-1}$.

Proposition 2.1 gives that the ratio $\dfrac{\widetilde{P}_n}{\widetilde{Q}_n}$ is the n^{th} convergent of the continued fraction

$$\left[I; \frac{\frac{2\alpha}{p}\Delta}{-I - \frac{\alpha}{p}\Delta}, \frac{\alpha^2(1/p^2 - n^2)\Delta^2}{-(2n+1)I} \right]_{n=1}^{+\infty}$$

where $\Delta = A^{1/2}\dfrac{L}{K}A^{-1/2}$ and $\dfrac{\widetilde{P}_n}{\widetilde{Q}_n}$ converges to $((1-\alpha)I + \alpha(A^{-1/2}BA^{-1/2})^p)^{1/p}$.

Considering now the following sequences $(P_n)_{n\geq -1}$ and $(Q_n)_{n\geq -1}$ such that

$$\begin{cases} P_{-1} = 0,\ P_0 = A,\ P_n = \widehat{A}_n P_{n-1} + \widehat{B}_n P_{n-2}, \\ Q_{-1} = 0,\ Q_0 = I,\ Q_n = \widehat{A}_n Q_{n-1} + \widehat{B}_n Q_{n-2}, \end{cases}$$

where

$$\begin{cases} \widehat{A}_0 = A,\ \widehat{A}_1 = -A^{-1/2} - \frac{1}{p}A^{1/2}DA^{-1}, \\ \widehat{A}_n = -(2n-1)I, \quad \text{for each } n \geq 2 \\ \widehat{B}_1 = \frac{2}{p}A^{1/2}D,\ \widehat{B}_2 = (\frac{1}{p^2} - 1)A^{1/2}D^2A^{-1}, \\ \widehat{B}_n = (\frac{1}{p^2} - (n-1)^2)A^{1/2}D^2A^{-1/2}, \quad \text{for all } n \geq 3 \\ D = \alpha\dfrac{L}{K}. \end{cases}$$

Using Proposition 2.2 together with (3.5) and (3.6), we conclude that $\dfrac{P_n}{Q_n}$ is the n^{th} convergent of

$$A^{1/2}\left[I; \frac{\frac{2\alpha}{p}\Delta}{-I - \frac{\alpha}{p}\Delta}, \frac{\alpha^2(1/p^2 - n^2)\Delta^2}{-(2n+1)I} \right]_{n=1}^{+\infty} A^{1/2}$$

and for all $n \geq 0$, we have

$$\frac{P_n}{Q_n} = A^{1/2}\ \frac{\widetilde{P}_n}{\widetilde{Q}_n}\ A^{1/2}.$$

Hence, $\left(\dfrac{P_n}{Q_n}\right)$ converges to $A^{1/2}((1-\alpha)I + \alpha(A^{-1/2}BA^{-1/2})^p)^{1/p}A^{1/2} = f_{p,\alpha}(A,B)$.

Let us take $\begin{cases} X_{-1} = X_0 = I, \\ X_n = A^{-1/2}, \quad \text{for every } n \geq 1. \end{cases}$

Then we obtain

$$\frac{X_1\widehat{B}_1X_{-1}^{-1}}{X_1\widehat{A}_1X_0^{-1}} = \frac{\frac{2}{p}D}{-A^{-1} - \frac{1}{p}DA^{-1}}, \quad \frac{X_2\widehat{B}_2X_0^{-1}}{X_2\widehat{A}_2X_1^{-1}} = \frac{(\frac{1}{p^2} - 1)D^2A^{-1}}{-3I},$$

$$\frac{X_n\widehat{B}_nX_{n-2}^{-1}}{X_n\widehat{A}_nX_{n-1}^{-1}} = \frac{(\frac{1}{p^2} - (n-1)^2)D^2}{-(2n-1)I} \quad \text{for all } n \geq 3.$$

Using Proposition 2.3, we conclude that

$$\frac{P_n}{Q_n} = \left[\widehat{A}_0, \frac{\widehat{B}_k}{\widehat{A}_k}\right]_{k=1}^{n} = \left[A, \frac{B_k}{A_k}\right]_{k=1}^{n}.$$

□

Remark 3.3. Let $A, B \in \mathcal{M}_m$ be two positive definite matrices, α a real number such that $0 \le \alpha \le 1$ and $p \ne 1$ be a strictly positive integer.

1) If $p = 2$ we have

$$f_{2,\alpha}(A,B) = A^{1/2}(A^{-1/2}((1-\alpha)A + \alpha BA^{-1}B)A^{-1/2})^{1/2}A^{1/2}.$$

Accordingly

$$f_{2,\alpha}(A,B) = A^{1/2}(A^{-1/2}C_1A^{-1/2})^{1/2}A^{1/2},$$

where $C_1 = (1-\alpha)A + \alpha BA^{-1}B$. It follows that $f_{2,\alpha}(A,B) = g_2(A, C_1)$ is the geometric matrix mean of the matrices A and C_1. Hence $f_{2,\alpha}(A,B)$ is the unique solution of the matrix equation

$$XA^{-1}X = (1-\alpha)A + \alpha BA^{-1}B. \tag{3.10}$$

2) If $p = 3$ we have

$$f_{3,\alpha}(A,B) = A^{1/2}(A^{-1/2}((1-\alpha)A + \alpha B(A^{-1}B)^2)A^{-1/2})^{1/3}A^{1/2}.$$

Accordingly

$$f_{3,\alpha}(A,B) = A^{1/2}(A^{-1/2}C_2A^{-1/2})^{1/3}A^{1/2} = g_3(A, C_2),$$

where $C_2 = (1-\alpha)A + \alpha B(A^{-1}B)^2$.

By virtue of Theorem 3.2, we conclude that $f_{3,\alpha}(A,B)$ is the unique positive definite solution of

$$XA^{-1}XA^{-1}X = (1-\alpha)A + \alpha B(A^{-1}B)^2. \tag{3.11}$$

3) If A and B are two positive definite matrices such that $AB = BA$, then

$$f_{p,\alpha}(A,B) = ((1-\alpha)A^p + \alpha B^p)^{\frac{1}{p}}.$$

Therefore

$$f_{p,\alpha}(A,A) = A$$

4 NUMERICAL APPLICATIONS

In this section, we present some numerical experiments of our theoretical results. We will deal with two cases:

1) The real case. Let us consider the function $f(x) = f_{2,1/2}(x,1) = \sqrt{\frac{1}{2}x^2 + \frac{1}{2}}$. Using Theorem 3.2 in real case, the first convergents of $f(x)$ are given by:

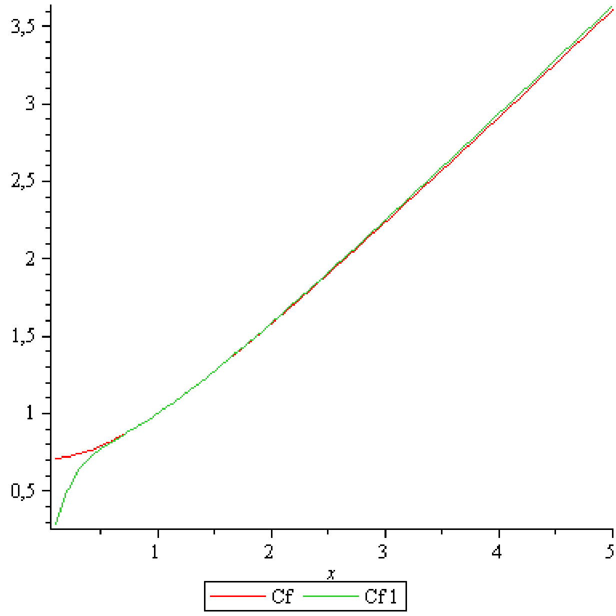
$$f_1(x) = \frac{P_1}{Q_1}(x) = \frac{(5x^2+3)x}{7x^2+1}$$

$$f_2(x) = \frac{P_2}{Q_2}(x) = \frac{(29x^4+30x^2+5)x}{41x^4+22x^2+1}$$

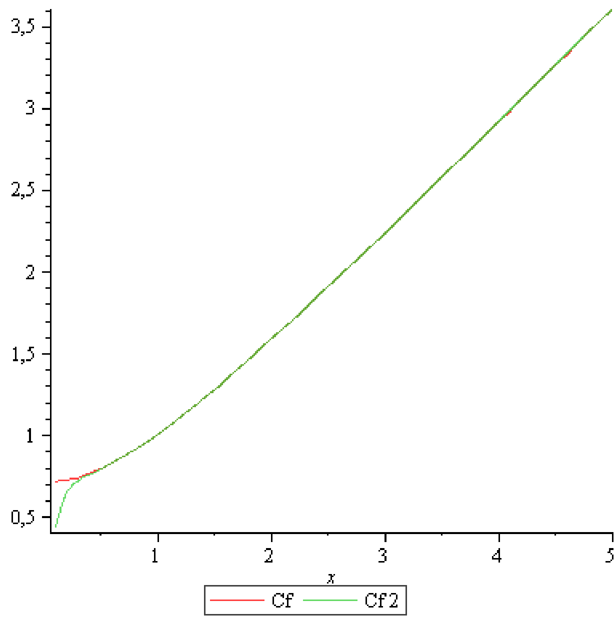
$$f_3(x) = \frac{P_3}{Q_3}(x) = \frac{(169x^6+245x^4+91x^2+7)x}{232x^6+227x^4+45x^2+1}.$$

The following graphics illustrate the approximation of $f(x)$ in terms of continued fractions.

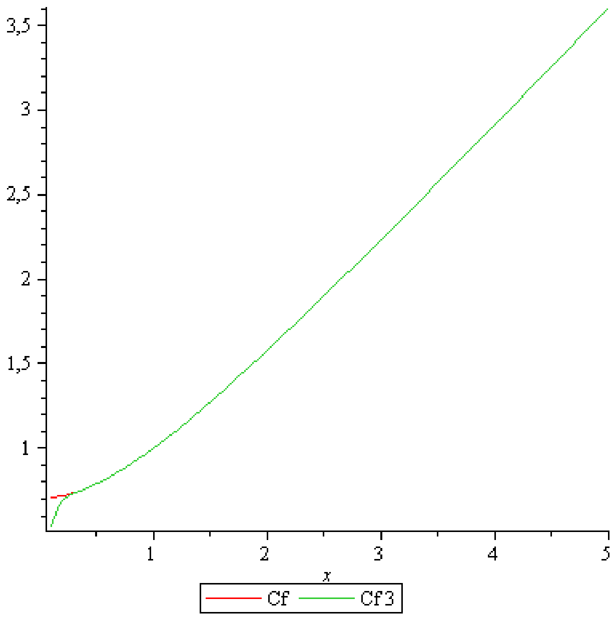
Iteration 1:



Iteration 2:



Iteration 3:



The graphics $C(f_i)$ represent $\frac{P_i}{Q_i}(x)$. It can be seen that the curves $C(f)$ and (Cf_i) are close to each other from the first values of x. Then we deduce that good approximations of $f(x)$ are obtained from the first iterations.

2)The matrix case.

Let $A = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{pmatrix}$

The solution of the equation

$$XA^{-1}X = \frac{1}{2}(A + BA^{-1}B)$$

is $X = \begin{pmatrix} \frac{1}{6}\sqrt{61}\sqrt{2} + \frac{1}{3}\sqrt{13}\sqrt{2} & \frac{1}{6}\sqrt{61}\sqrt{2} - \frac{1}{6}\sqrt{13}\sqrt{2} & \frac{1}{6}\sqrt{61}\sqrt{2} - \frac{1}{6}\sqrt{13}\sqrt{2} \\ \frac{1}{6}\sqrt{61}\sqrt{2} - \frac{1}{6}\sqrt{13}\sqrt{2} & \frac{1}{6}\sqrt{61}\sqrt{2} + \frac{1}{3}\sqrt{13}\sqrt{2} & \frac{1}{6}\sqrt{61}\sqrt{2} - \frac{1}{6}\sqrt{13}\sqrt{2} \\ \frac{1}{6}\sqrt{61}\sqrt{2} - \frac{1}{6}\sqrt{13}\sqrt{2} & \frac{1}{6}\sqrt{61}\sqrt{2} - \frac{1}{6}\sqrt{13}\sqrt{2} & \frac{1}{6}\sqrt{61}\sqrt{2} + \frac{1}{3}\sqrt{13}\sqrt{2} \end{pmatrix}$,

and thus

$$X = \begin{pmatrix} 3.54056667400999991 & 0.991056918009999954 & 0.991056918029999956 \\ 0.991056918009999954 & 3.54056667400999991 & 0.991056918029999956 \\ 0.991056917999999953 & 0.991056917999999953 & 3.54056667399999991 \end{pmatrix}$$

According to remark 3.4, it is clear that $X = f_{2,1/2}(A, B)$
Using Theorem 3.2, we can obtain the following approximations of this solution

$$F_1 = \begin{pmatrix} 3.53413603176636304 & 0.993595491225823247 & 0.993595491225823247 \\ 0.993595491225822913 & 3.53413603176636393 & 0.993595491225823024 \\ 0.993595491225823024 & 0.993595491225823135 & 3.53413603176636393 \end{pmatrix}$$

$$F_2 = \begin{pmatrix} 3.54047817549171473 & 0.991099496616634212 & 0.99109949661663410 \\ 0.991099496616633657 & 3.54047817549171562 & 0.991099496616633990 \\ 0.991099496616633768 & 0.991099496616633879 & 3.54047817549171562 \end{pmatrix}$$

$$F_3 = \begin{pmatrix} 3.54056539643065093 & 0.991057551961279093 & 0.991057551961279204 \\ 0.991057551961278760 & 3.54056539643065138 & 0.991057551961278982 \\ 0.991057551961278871 & 0.991057551961278871 & 3.54056539643065138 \end{pmatrix}$$

$$F_4 = \begin{pmatrix} 3.54056665545622806 & 0.991056926558535056 & 0.991056926558534945 \\ 0.991056926558534501 & 3.54056665545622895 & 0.991056926558534834 \\ 0.991056926558534834 & 0.991056926558534945 & 3.54056665545622939 \end{pmatrix}$$

$$F_5 = \begin{pmatrix} 3.54056667379078149 & 0.991056917401399673 & 0.991056917401399784 \\ 0.991056917401399451 & 3.54056667379078149 & 0.991056917401399673 \\ 0.991056917401399562 & 0.991056917401399562 & 3.54056667379078149 \end{pmatrix}$$

We can see that the speed of convergence is quick from the first iterations.

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